Some Mathematical Issues in Correspondence with Alexander Grothendieck 1982-1991

Ronald Brown

zoom meeting
in honor of Grothendieck
organised by
John Alexander Cruz Morales and Colin McLarty
this talk: August 28 2020

Aim of talk

1. To give an idea of some of AG's modes of thought.
2. To note a basic algebraic problem in homotopy theory: how do identifications in low dimensions produce high dimensional homotopy invariants.
3. To see at least one sample of AG's methods, cofibrations of categories, as relevant to a basic method in homotopy theory: homotopical excision
4. In memoriam

1958:Edinburgh ICM

1. I overheard Raoul Bott saying: “Grothendieck was remarkable. He could play with concepts. And was prepared to work very hard to make something almost tautological.”
2. I saw with amazement Alexander Grothendieck laying down the law to J.-P.Serre!

My interests

1. 1961 DPhil on k-invariants of function spaces X^Y in terms of those of X (from which, "convenient categories")
2. 1968 Elements of Modern Topology: which included an exposition, and significant use, of groupoids
3. since 1965, higher dimensional Van Kampen Theorems and the associated "higher dimensional algebra"
4. How to model algebraically that low dimensional identifications in topology can produce changes in high dimensional homotopy invariants?
5. Use of modified cubical higher dimensional categories, for expressing "higher dimensional algebra".
6. Popularisation of maths

Contact with Grothendieck

In 1982 I read a paper of Duskin which said AG was interested in n-categories. I was planning to go to a conference in Marseilles-Luminy so I wrote to him, with, separately, offprints/preprints. He mentioned some letters of Breen to him. I expressed interest. I took them to the conference.
This started the correspondence, which he later termed "a baton rompu", and maybe drew him back into public writing of maths, till 1991.
In 1983 I sent him a largely failed research proposal from myself and Tim Porter which drew referees' reports varying from marvellous to speculative and rubbish! I suspect he thought he could do better! In 1984 I visited him for 2 nights at his home, on my way to Toulouse.

Expressing Mathematics

None of this would have happened without my trying to write a text on topology, somewhat geometric and categorical, and, in the end, proving a new version of the Van Kampen Theorem.

I am glad to see this question of “expression” and “writing” is one theme of this conference.

The serious question is whether these are themes of undergraduate teaching?

Usual Van Kampen Theorem

Anomaly: The usual VKT does not compute the fundamental group of the circle S^1, **the basic exampl**e in topology.
Philip Higgins 1963 "Presentations of groupoids with applications to groups" Used pushouts of groupoids.
1965 I proved a VKT for the fundamental groupoid $\pi_1(X)$.
It still did not compute $\pi_1(S^1, 0)$!
Goldilocks situation: one base point was too small. the whole space was too big but a set of two base points was exactly right.
The general answer was $\pi_1(X, S)$ where S is a **set of base points**.
MORAL Dim 0 should not be ignored! Groupoids have structure in dimensions $0, 1$. Are there strict algebraic structures in dimensions $0, 1, \ldots, n$.?

A basic feature of the algebra of groupoids

Let G be a groupoid, and $f : Ob(G) \to Y$ be a function. $D(Y)$ the discrete groupoid on the set Y. Then there is a pushout diagram of groupoids

$$\begin{array}{ccc} D(Ob(G)) & \xrightarrow{D(f)} & D(Y) \\ \downarrow & & \downarrow \\ G & \xrightarrow[U(f)]{} & U_f(G). \end{array}$$

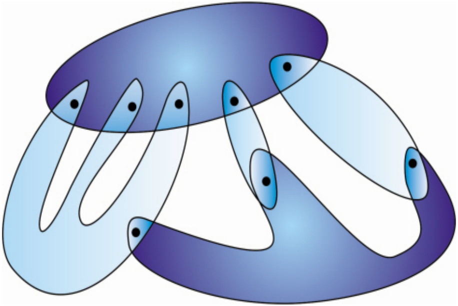
$U(f)$ is called a *universal morphism*, It allows identification in dimension 0.

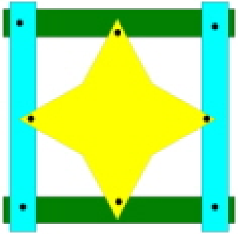
Corollaries of construction:

Free groups, free products of groups, free groupoids.

Some consequences of the many pointed VKT

Getting fundamental group information on:





Many base points in the second example captures the symmetry.
Example: if the space X is the union of open simply connected subspaces, then $\pi_1(X, S)$ is a free groupoid.

Why be tethered to a single base point?
What is wrong with needing say 2,000 base points (in these days of large data!) ?
Have a look at the Conway groupoid!

The difficulty of bringing concepts out of the dark: Grothendieck on groupoids

"People are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base points chosen with care – or equivalently working in the algebraic context of groupoids, rather than groups."

“Choosing paths for connecting the base points natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won’t be of any use whatever, and just confuse the picture rather than clarify it. ”

Continuation

" I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids. Still, standing habits of thought are very strong, and during the long march through Galois theory, two years ago, it took me weeks and months trying to formulate everything in terms of groups or 'exterior groups' (i.e. groups 'up to inner automorphism'), and finally learning the lesson and letting myself be convinced progressively, not to say reluctantly, that groupoids only would fit nicely."

Comment: It seems to me interesting to see AG crash-testing his concepts!

This “many base points” theme was, or seems to me, the main area of complete agreement with AG on methodology and aims! And seems largely to reflect an opposition from “mainstream” algebraic topology.
The divergence from AG partially reflects my age, initiation to homotopy theory under Henry Whitehead and Michael Barratt, and my lack of knowledge of algebraic geometry! And the eventual success of the cubical methods.

Homotopy groups at the 1932: ICM Zürich

Why am I considering this ancient meeting? Surely we have advanced since then? And the basic ideas have surely long been totally sorted? Alexander Grothendieck has shown us that basic ideas can be looked at again and in some cases appropriately renewed.

At this ICM, Eduard Čech gave a seminar introducing homotopy groups $\pi_n(X, x)$, and proving they were abelian for $n > 1$.

At the time there was strong interest among senior topologists (Hopf, Alexandrov, Dehn,...) in:

(*) : finding higher dimensional versions of the fundamental group.

Čech was persuaded to withdraw his paper: Alexandrov is reported to have said: "But my dear Čech, how can they be anything but homology groups?" The relation between homotopy and homology turned out to be a good question for Hurewicz! But we now say: *a group object in the category of group;s is an abelian group* –"Higher dimensional group theory" does not exist"

Einstein wrote on science and truth "'What goal will be reached by the science to which I am dedicating myself? What is essential and what is based only on the accidents of development? ... Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. ... "

Note that homotopy groups are defined only for spaces with one base point.

Group objects in the category of groupoids are NOT necessarily abelian, but are equivalent to Henry Whitehead's crossed modules.

After later work on homotopy groups, the question

(*) : finding higher dimensional versions of the fundamental group.

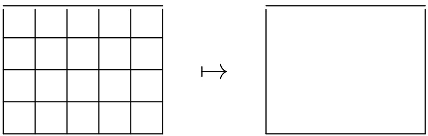
disappeared from sight!

AG to RB, 1984: "one cannot confuse 'space' and 'space with base point' without causing serious trouble".

AG's ideas were to look at groupoidal ideas involving homotopies and higher homotopies.

Higher homotopy groupoids

My search in this area began in 1965. Basic intuition:



(multcomp)

From right to left gives subdivision.From left to right should give composition. as an algebraic inverse to subdivision, to be applied in “local-to-global ” problems.

The 1-dimensional VKT required also for its proof: any composition of commutative squares in a groupoid is commutative. The dim 2 result needed the analogue for commutative cubes. This turned out to require enriching cubical theory with new degeneracies based on

$$\max, \min : \{0,1\} \to \{0,1\},$$

which Chris Spencer and I called "connections",
This turned out in due course to resolve the major problems thought to be inseparable from cubical theory! (But too late to be considered by AG!)

A strict homotopy double groupoid

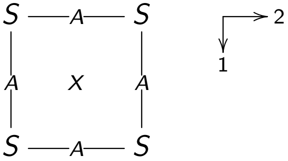
The idea came from Henry Whitehead's free crossed modules. 1949, CHII : which arise as

$$\pi_2(A \cup_i \{e_i^2\}.A, x) \to \pi_1(A, x).$$

This freeness was a nonabelian dim 2 universal property in homotopy theory, and suggested to PJH and to me we look at a relative situation (X, A, x). And it all worked!
cf: R. Brown, "Modelling and Computing Homotopy Types: I" Indag. Math. 2018. (Special Issue: LEJ. Brouwer) for an Introduction to a 1981 complete solution to the problem (*).

What was the simplest way, since time was near ending for PJH's visit, .of using squares, composition, homotopy classes, and a relative situation?
Look at maps $I^2 \to X$ which take the boundary to A and the vertices to a set $S \subseteq A$ and take homotopy classes of these rel vertices to give

$$\rho_2(X, A, S).$$



This allowed for the childish idea of gluing two squares if the right side of one is the same as the left side of the other or the bttom side of one is the same as the top side of the other. These give partial algebraic operations defined under geometric condition.
Cubical notions are very good for compositionality!

This $\rho_2(X, A, S)$ is part of a strict double groupoid over $\pi_1(A, S)$.with connections, and satisfying a VKT, (RB-PJH, 1974 published 1978, in the teeth of opposition from notables.)
It is the connections which have revitalised cubical theory, and allow it to express and so do things not possible or clear simplicially or globularly.

Fibred and Cofibred Categories

This area due to AG has been found to be related to homotopical excision : if $X = A \cup B$, study homotopy invariants of the inclusion

$$\varepsilon : (B, A \cap B) \to (X, A),$$

e.g $\varepsilon : \pi_n(B, A \cap B, x) \to \pi_n(X, A, x)$ Under some openness and connectivity conditions, this morphism is determined by $\pi_1(A \cap B, x) \to \pi_1(A, x)$.

This is a general setting which AG introduced for taking a more global view of some well known concepts such as induced representations, induced and restrictions of modules, Mackey functors, Higgins' universal morphisms of groupoids, and many others. People could also consider it in the light of AG's six operations (do a web search).

It also gives a setting for given a functor $\Phi : \mathsf{X} \to \mathsf{B}$ for showing how to compute colimits of functors $T : \mathsf{C} \to \mathsf{X}$ in terms of colimits of $\Phi T : \mathsf{C} \to \mathsf{B}$ provided Φ is a fibration, and has a left adjoint: These are mild and not unusual conditions. For more info see the EMS Tract 15 "Nonabelian Algebraic Topology" Appendix B. (The book is basically a rewrite of singular homology theory including relative homotopical excision.)

Modules over groupoids

Let $\mathcal{I}$ be the groupoid with two objects $0, 1$ and exactly one arrow $\iota : 0 \to 1$; it is realised topologically as

$$\pi_1([0,1], \{0,1\}).$$

This is the basic transition. It is a generator for the category of groupoids. If you identify $0, 1$ in the category of groupoids you get a generator for the category of groups, i.e. the integers $\mathbb{Z}$.

Lets move to modules, Let $X = S^n \vee [0,1], n > 1$, with 0 identified with N, the North pole of S^n, Then

$$\pi_n(X, \{0,1\})$$

is clearly a free module over $\mathcal{I} = \pi_1([0,1], \{0,1\})$.
Now identify $0, 1$. With the right excision theorem, you get $\pi_n(S^n \vee S^1, x)$ as a free $\mathbb{Z}$-module.

Calculating Colimits of Groupoids

People do ask: How to calculate colimits of groupoids? Let's look at coequalisers..

$$A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B \xrightarrow{\gamma} C.$$

If α, β are the identity on objects, the solution is just like that for groups. You factor out by the normal subgroupoid generated ny the obvious relations:

$$\alpha(a)\beta(a)^{-1} : a \in A.$$

Otherwise you first have to coequalise α, β on objects. This gives a set Y and a function $f : Ob(B) \to Y$. So you form the groupoid morphism $U(f) : B \to U_f(B)$ and work on $U_f(B)$.
This gives a general method for working on particular algebraic data $\mathcal{A}$ with levels $\mathcal{A}^{(n)}$ when the levels are provided with, say, a bifibration pair $\mathcal{A}^{(n)} \to \mathcal{A}^{(n-1)}$.

Conclusion

In Pursuing Stacks, AG took an entirely separate line from mine, disappointed by the fact that strict ω- groupoids as defined by Brown and Higgins modelled only a limited range of homotopy types. He developed a range .of simplicial and globular methods, and then at times left mathematics, till he finally isolated himself in 1991. My last communication from him was in 1991, in response to a post card sent to him from the Isle of Iona. A total of 35 + 34 letter will be published by the Société Math France, to say nothing of "Pursuing Stacks",

"written in English in response to a correspondence in English." In 1983 I told him in detail of a failed research proposal from Brown and Porter. I like to think that was suggestive of "Esquisse"!

Final Letter
in response to a postcard from Iona Island

Les Aumettes April 9, 1991

Dear Ronnie,

As always, it was a pleasure to get a lifesign from you, and the beautiful picture with the severe and serene landscape around Iona Abbey. I am glad, too, all seems to be well with you and Margaret and the tribe.

As for me, health and spirit in best shape. One news is that for the last five months, I've taken up some maths again, which I had dropped totally during four full years. I've set out to develop the program of "topological algebra" (as I like to call it) which I told you about here and there, and which I had started on with Pursuing Stacks – but without outlining there the overall program, except in a scattered way by bribes and bits. (As I thought I would do the work by myself.) I made pretty fast headway and things are steadily taking shape. But the work still ahead appears more extensive as work progresses.

And I doubt there will be time enough left for me to get much further than now, as I expect that events (this time not unforeseen) will put an end soon to my mathematical (and badly needed!) vacations. If not here, I trust I'll carry it on beyond!
Affectionately as ever to you and Margaret,
your
Alexander

As a final guide from AG, for whom this conference is in honor, let me quote from a letter dated 12/04/83:

The question you raise "how can such a formulation lead to computations" doesn't bother me in the least! Throughout my whole life as a mathematician, the possibility of making explicit, elegant computations has always come out by itself, as a byproduct of a thorough conceptual understanding of what was going on. Thus I never bothered about whether what would come out would be suitable for this or that, but just tried to understand – and it always turned out that understanding was all that mattered.

An edited version of this and other correspondence of AG will be published by the Société Mathématique de France.
I feel AG had special views, of use to the general, of ways of doing mathematics.

Immortality
Robert A.Hefner III Collection
Aspen, Australia
Height: 6ft2in
Passing on the Torch of Life
John Robinson, Sculptor

Research Methodology?

a la Karl Popper? Look for the wildest ideas, evaluate them, and then argue against them if they seem to be too good to be valid!
`http://www.groupoids.org.uk/publar.html`
Good luck!
The composer Ravel somewhere said: Copy! If you have originality it will show. If not, never mind!
My comment: Maybe you need to copy 4 times before some originality peeps through!