Homotopy theory, and change of base for groupoids and multiple groupoids[*†]

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Abstract

This survey article shows how the notion of "change of base", used in some applications to homotopy theory of the fundamental groupoid, has surprising higher dimensional analogues, through the use of certain higher homotopy groupoids with values in forms of multiple groupoids.

Introduction

A major problem in homotopy theory is to compute how high dimensional homotopy invariants of a space are affected by low dimensional changes to the space. These changes can be substantial. For example, the 2-sphere S^2 is obtained from the 2-disk, D^2, by identifying the bounding circle S^1 of D^2 to a point. However, D^2 is contractible, i.e. of trivial homotopy type, while the homotopy groups of the 2-sphere S^2 are non trivial in an infinite number of dimensions. Such changes, and those obtained in forming more general complexes, seems unable to be modelled by traditional methods, such as the Mayer-Vietoris sequence, in which only neighbouring dimensions can interact. Thus any new methods of obtaining some new information are of interest.

A possible way towards obtaining new information was conjectured in 1967 in [2], suggested by the new use for computations of the fundamental groupoid. The successes of the fundamental groupoid, as shown in [2, 4], seemed to stem from the fact that it had structure in two dimensions, 0 and 1. This raised the prospect of reflecting more of the way homotopy theory worked by using algebraic objects with structure which is in a range of dimensions, which becomes more complex as dimension increases, but which still has analogies to that of groups.

The purpose of this survey is to show how one particular method, that of "change of base", has been exploited in this programme, and has yielded new information. Here the "base" is, in

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the case of spaces, a lower dimensional part of a space, and in the case of the algebraic objects, is also their "lower dimensional part". Part of the interest is the way this notion links with many traditional uses of "change of base", for example with induced representations, and with change of base for slice categories, i.e. categories $\mathcal{C}/A$ where A is an object of the category $\mathcal{C}$ and the slice category has objects the morphisms to A (see [50, 42]).

The algebraic structures we consider are forms of multiple groupoid, and the corresponding "higher homotopy groupoids" give generalisations of the fundamental group. Thus the following extensions seem to be significant and have particular uses for the study of certain local-to-global problems:

$$\text{(groups)} \subset \text{(groupoids)} \subset \text{(multiple groupoids)}.$$

The study of multiple groupoids and their applications could be thought of as a higher dimensional, or many variable, theory, which would be expected to yield:

- a range of new algebraic structures, with new applications
- better understanding, from a higher dimensional viewpoint, of some phenomena in group theory
- new computations with these objects, and hence also in the areas in which they apply
- new algebraic understanding of the structure of certain geometric situations.

One reason for studying groupoids rather than groups for local-to-global problems is the technique of using an "algebraic inverse to subdivision". If you mark points on a base pointed loop drawn in the plane, then the natural algebra to describe this is not a group but a groupoid. Analogous situations occur widely, for example in holonomy and monodromy.

Another reason for studying groupoids rather than groups is the classical argument, dating back to the 1930's, that higher dimensional group theory cannot exist. That is, if a set has two group structures each of which is a morphism for the other, then the two structures coincide and are abelian. (In this argument, it is enough to have monoids, and even associativity comes for free.) This, in essence, is why higher homotopy groups are abelian groups. This shows that the higher homotopy groups do not generalise the fundamental group, a fact which led to an initial disappointment with Čech's proposal of these groups at the 1932 ICM in Zurich [24]. Later work on homotopy groups continues to move away from group theoretic methods.

It is worth explaining how groupoids arose. The notion of groupoid dates from Brandt's attempts to extend to quaternary forms Gauss' work on the composition of binary quadratic forms, which has a strong place in *Disquitiones Arithmeticae.* It is of interest here that Bourbaki [1], p.153, cites this composition as an influential early example of a composition law which arose not from numbers, even taken in a broad sense, but from distant analogues[1]. Brandt found that

[1] C'est vers cette même époque que, pour le premier fois en Algèbre, la notion de loi de composition s'étend, dans deux directions différents, à des élements qui ne présentent plus avec les <<nombres>> (au sens le plus large donné jusque-là à ce mot) que des analogies lointaines. La première de ces extensions est due à C.F.Gauss, à l'occasion de ses recherches arithmétiques sur les formes quadratiques ...

each quaternary quadratic form had a left unit and a right unit, and that two forms were composable if and only if the left unit of one was the right unit of the other. This led to his 1926 paper on groupoids. (A modern account of the work on composition of forms is given by Kneser *et al.* [46].) Groupoids were then used in the theory of orders of algebras. Curiously, groupoids did not form an example in Eilenberg and Mac Lane's basic 1945 paper on category theory. Groupoids appear in Reidemeister's 1932 book on topology [52], as the edge path groupoid, and for handling isomorphisms of a family of structures. The fundamental groupoid of a space was well known by the 1950's. The fundamental groupoid on a set of base points is used in [2].

Grothendieck writes in 1985 [34]:

> The idea of making systematic use of groupoids (notably fundamental groupoids of spaces, based on a given set of base points), however evident as it may look today, is to be seen as a significant conceptual advance, which has spread into the most manifold areas of mathematics. ... In my own work in algebraic geometry, I have made extensive use of groupoids - the first one being the theory of the passage to quotient by a "pre-equivalence relation" (which may be viewed as being no more, no less than a groupoid in the category one is working in, the category of schemes say), which at once led me to the notion (nowadays quite popular) of the nerve of a category. The last time has been in my work on the Teichmüller tower, where working with a "Teichmüller groupoid" (rather than a "Teichmüller group") is a must, and part of the very crux of the matter ....

In a groupoid, an object x is equivalent to an object y if there is in the groupoid an arrow $x \to y$. The arrows $x \to y$ may be thought of as "proofs" that x is equivalent to y. The different paths in the groupoid from x to y may also carry extra information. This perhaps explains why groupoids occur for example in formalisations of holonomy, which is intuitively described as pairs (position, value) moving continuously or smoothly through a path and returning to the initial position but not the initial value.

Groupoids form a central feature of Ehresmann's foundational work on differential geometry and topology [28]. In the 1960s, G W Mackey found groupoids useful for work in ergodic theory, to give an analogue for an ergodic action of the way a group of stability of a transitive action determines that action up to equivalence. Mackey's work stimulated Connes' research on non commutative integration, and the convolution algebras of groupoids continue to be a main tool in Connes' work on non commutative geometry.

Connes remarks [26] that Heisenberg discovered quantum mechanics by considering the algebra of observables for the groupoid of quantum transitions, rather than the traditional group of symmetry. Here, if I is a set, the groupoid $I \times I$, which has multiplication $(i,j)(j,k) = (i,k)$, could be called the *transition groupoid.* The convolution operation for the complex valued functions of finite support on this groupoid has the form

$$(f * g)(i,k) = \sum_j f(i,j)g(j,k).$$

When Born pointed out that this was the well known matrix algebra, the groupoid viewpoint disappeared from sight.

We can now see that one value of groupoids is that they give a common algebraic framework for considering sets, groups, group actions and equivalence relations, since all of these are, or give rise to, special kinds of groupoids. This common framework can be even more useful for groupoids with structure, such as one or more of the structures of topological space, differential manifold, Borel space (as in Mackey's case), algebraic variety, scheme, and so on, especially as groupoids can carry a wider range of non trivial structures than can groups. For a general survey of groupoids, with 160 references, including many omitted above, see [3].

It is also interesting to note the relation of ordered groupoids to inverse semigroups, and so to notions of partial symmetry. In this area, Lawson has developed some of Ehresmann's foundational work in [47, 48].

A groupoid has a set of objects, which we call the *base* of the groupoid. We are concerned with *change of base*, a notion which has no meaning for groups. The idea is that for a given set X there is a category Gpd/X of groupoids with base X, and morphisms the groupoid morphisms which are the identity on X. If $\sigma : X \to Y$ is a function, then there is a pullback functor

$$\sigma^* : Gpd/Y \to Gpd/X,$$

which could be called *change of base*. In fact we are interested in the whole situation of possible left and right adjoints of σ^*, and the analogies with induced modules in representation theory. We discuss briefly right adjoints to pullbacks, which do not always exist in algebraic categories, in the penultimate section.

1 Change of base for groupoids

Higgins' 1964 paper [38] applied groupoids to group theory, independently of earlier work of Hasse [35]. One main tool was the notion of covering morphism of groupoids, motivated by the notion of covering map of spaces. The theory of covering maps may be summarised [4] by saying that for spaces with good local properties, the fundamental groupoid gives an equivalence of categories

$$\pi_1 : (\text{covering maps of } X) \simeq (\text{covering morphisms of } \pi_1 X),$$

where $\pi_1 X$ is the fundamental groupoid of X, while for any groupoid G there are equivalences of categories

$$\begin{aligned} (\text{covering morphisms of } G) &\simeq (\text{actions of } G \text{ on sets}) \\ &\simeq (\text{functors from } G \text{ to sets }). \end{aligned}$$

In particular, a subgroup H of a group G determines a covering morphism $Tr(G, H) \to G$, derived from the action of G on the set of right cosets of H in G, such that $Tr(G, H)$ has a vertex group mapped isomorphically to the subgroup H.

The second main tool was his notion of *universal morphism.* This gives a groupoid morphism $\bar{\sigma} : G \to U_\sigma G$ for any function $\sigma : Ob(G) \to Y$, and which is universal for morphisms f of G to a groupoid such that $Ob(f)$ factors through σ. Alternatively, the functor $U_\sigma : Gpd/(Ob(G)) \to Gpd/Y$ is left adjoint to σ^*. Thus these universal morphisms come well under the notion of change of base. The existence of these morphisms is equivalent to having for each groupoid G and function $\sigma : Ob(G) \to Y$ the following pushout diagram of groupoids, in which the objects of a groupoid G are regarded as a subgroupoid of G:

$$\begin{tikzcd} Ob(G) \arrow[r, "\sigma"] \arrow[d] & Y \arrow[d] \\ G \arrow[r, "\bar{\sigma}"] & U_\sigma G \end{tikzcd} \tag{1}$$

This notion includes as special cases free products of groupoids, and free groupoids on a graph. The construction of U_σ given in [38, 40] is in terms of equivalence classes of certain words of arrows of G.

A key technical result is the solution of the word problem for $U_\sigma G$, that is the uniqueness of reduced forms of these words. This is used to prove that the pullback of a universal morphism by a covering morphism of groupoids is also universal. This enables presentations of a group as a free group or as a free product to be lifted to corresponding presentations of a covering groupoid, so that information on the vertex groups of this covering groupoid, may be derived. Since these vertex groups are isomorphic to subgroups of the original groupoid, classical subgroup theorems may be obtained by this method, in some cases giving stronger versions [39]. These methods also allow for topological versions of the main subgroup theorems [7]. Section 6 refers to other methods of proving the required lifting of universal morphisms without using the solution of the word problem.

A major reason for passing from the fundamental group to the fundamental groupoid is that the latter gives a version of the Van Kampen Theorem for non connected spaces, and this allows for wider computations, even of the fundamental group. This can be illustrated with universal morphisms, i.e. for change of base, as follows.

Let A be a "well embedded" discrete subset A of a space X, and let $\sigma : A \to B$ be a function to a discrete space B. Then we can form the diagrams

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d] & B \arrow[d] \\ X \arrow[r] & B \cup_\sigma X \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d] & B \arrow[d] \\ \pi_1(X, A) \arrow[r, "\bar{\sigma}"] & \pi_1(B \cup_\sigma X, B) \end{tikzcd}$$

The first square is a pushout of spaces defining the adjunction space $B \cup_\sigma X$. The second square is a pushout of groupoids, by the Van Kampen theorem [4]. Hence there is an isomorphism

$$\pi_1(B \cup_\sigma X, B) \cong U_\sigma \pi_1(X, A).$$

As a particular example, if $\mathcal{I}$ is the transition, or indiscrete groupoid, on the set $\{0,1\}$, and $\sigma : \{0,1\} \to \{0\}$ is the unique function, then $U_\sigma \mathcal{I}$ is the group $\mathbb{Z}$ of integers. This gives an 'explanation' of the determination of the fundamental group of the circle, since this space is obtained from the unit interval $[0,1]$ by identifying 0 and 1. That is, we have the analogous pushouts

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ {[0,1]} \arrow[r] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \mathcal{I} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

As a more general example, a directed graph is obtained from a disjoint union of its edges by identifying the vertices appropriately. The corresponding free groupoid is obtained from the disjoint union of copies of the groupoid $\mathcal{I}$, one copy for each edge, by making the same identification of the vertices, this time in the category of groupoids, that is by applying an appropriate U_σ.

Note how a general theorem (the Van Kampen Theorem for the fundamental groupoid) gives a range of specific calculations, in this case of the fundamental groups of spaces, with the only specific example required to be calculated being a trivial one, the fundamental groupoid of a convex set relative to some subset.

These are basic examples in homotopy theory of how a low dimensional change may influence higher dimensional homotopy structures. As indicated earlier, groupoids are successful in the calculations because their range of structure in dimensions 0 and 1 has the capacity to model analogous geometric constructions on spaces. Another example of such success, given in [4], is the theory of the fundamental group(oid) of an orbit space under a discontinuous action of a group.

A feature of these calculations is that, quite apart from giving non abelian information, they also in some cases give more information than can reasonably be expected. It is traditional in algebraic topology and homological algebra to obtain interactions of invariants of different dimensions by means of exact sequences and spectral sequences. These determine the invariant only up to extension. By contrast, the Van Kampen Theorem for the fundamental groupoid $\pi_1(X,A)$ determines the various vertex groups $\pi_1(X,a)$ completely.

It would be interesting to know how widely this philosophy of using objects with a broad structure in various dimensions, and replacing exact sequences by colimit theorems, can be extended in algebraic topology and homological algebra, and in the applications of these subjects.

One of the fascinations of the use of groupoids, in comparison with groups, is that the groupoid $\mathcal{I}$ is a generator for the category of groupoids, so that all groupoids may be constructed as colimits of diagrams of copies of $\mathcal{I}$. However, this groupoid is a finite groupoid and all its properties are easy to verify. By contrast, the category of groups has as generator the infinite cyclic group $\mathbb{Z}$, and to prove anything about $\mathbb{Z}$ requires induction. In view of other talks at this conference, it is interesting to speculate whether there are analogues of these properties for other toposes than that of sets, possibly with some replacement of the Boolean object $\{0,1\}$.

We mention another use of these universal groupoids in connection with classifying spaces. There is a classifying space functor B : (groupoids) $\to$ (spaces) with the property that

$$\pi_i(BG, e) = \begin{cases} G(e) & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i > 1 \end{cases}$$

It is not true that B maps a pushout of groupoids to a homotopy pushout. However, this is true for pushouts of the form (1). This is proved by Zisman in [58], who also gives a generalisation to topological categories and groupoids. Further generalisations of these results are proved in the doctoral thesis of Wolf [57]. He calls a functor $F : \mathcal{C} \to \mathcal{D}$ of categories or groupoids *flat* if the left Kan extension $F_* : \mathcal{AB}^{\mathcal{C}} \to \mathcal{AB}^{\mathcal{D}}$ is exact. He requires certain flatness and other conditions on the morphisms of a pushout square of categories or groupoids, for the classifying space functor to yield a homotopy pushout of spaces. In particular, a universal morphism of groupoids is flat.

2 Double groupoids

The starting point of this investigation was the aim of obtaining higher dimensional versions of the results and methods for groupoids. One aspect is simply that of curiosity: if groupoids are successful in dimension 1, then it is of interest to know if this success does, or does not, transfer to higher dimensions. One reason for supposing that groupoids might work well for higher dimensional local-to-global problems, is that the higher dimensional versions seem to satisfy a useful intuitive criterion for applicability to such problems, namely that of giving a model for describing "algebraic inverses to subdivision".

A *double groupoid* is a set with two groupoid structures each of which is a morphism for the other. This notion is due to Ehresmann [27]. It can also be thought of as comprising an underlying diagram of sets

$$\begin{tikzcd} S \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & V \arrow[d, shift left] \arrow[d, shift right] \\ H \arrow[r, shift left] \arrow[r, shift right] & P \end{tikzcd} \tag{2}$$

such that H and V are groupoids over P, and there are two groupoid structures on S, one over H, one over V, and all satisfying various compatibility conditions which I will not write down here.

The argument mentioned above for double groups being abelian groups can still be applied to double groupoids. All it shows is that a double groupoid contains a family of abelian groups. In fact, double groupoids are in some sense "more non abelian" than groups (see section 3), which is what one expects from a 2-dimensional theory.

The structure of general double groupoids is a bit mysterious. Perhaps they should be regarded as basic objects in mathematics, like other combinations of structures, for example rings. However, some types of double groupoids may be described in other terms. The most general known results are given in [18].

The category of double groupoids contains a category of 2-*groupoids*, namely the double groupoids as above in which the groupoid V over P is the discrete groupoid, with only identities. This category is equivalent to at least three other categories, of which two are those of *crossed modules over groupoids*, and of *double groupoids with connection* [11]. (The case of these equivalences when P is a singleton is due to Brown and Spencer [19, 20].) Each of these forms of double groupoids have their own particular value, and circumstances when they are most appropriate.

Crossed modules over groupoids [10]

These objects are more obviously related to classical tools, namely (i) groupoids, (ii) modules over groups, (iii) second relative homotopy groups, (iv) chain complexes [13]. For these reasons, it is natural to attempt to compute with these objects rather than with the other forms. There is a useful monoidal closed structure on this category, see [14].

Double groupoids with connection [19, 10]

These are an essential tool in one proof of the Van Kampen Theorem for the fundamental crossed module, because they nicely handle subdivision and the homotopy addition lemma [9]. They also have a monoidal closed structure, related to a notion of homotopy, and which may be derived from that for the case of all dimensions given in [14].

2-groupoids [45, 12]

These are nearer to the well used 2-categories. On the other hand, their monoidal closed structure, which follows from the equivalence with the previous example, seems more difficult to describe than in either of the previous examples. The corresponding case of 2-categories is dealt with by Gray in [32, 33]. The equivalence of 2-categories with a form of double categories with connection is due to Spencer [53, 54], but seems not to have been otherwise exploited, except in recent work of Verity on cubical nerves of ω-categories. The work of [54] shows the value of the connections and the corresponding 'thin' structure for computing with multiple compositions. Analogous computations in 2-categories require the notion of 'pasting'.

Simplicial groupoids with constant object sets and with Moore complexes of length 1 [49, 51]

Simplicial groups have an extensive use in algebraic topology, homological algebra, and algebraic K-theory, which gives an advantage to this category. It is crucial (at least for the group case) in the use of cat^n-groups [49, 16]. The notion of homotopy has been written down in this case [51], but not the monoidal closed structure.

One of the comforting features of work on double and multiple groupoids has been the links with existing subjects which have appeared, as shown above. In particular, double groupoids were found to link with crossed modules, which had been in the literature since 1946 [55], and crossed modules themselves link with the standard second relative homotopy groups of homotopy theory, and with a long tradition of homological algebra, including chains of syzygies, and, for the non abelian cases, identities among relations [15]. Such links seem non accidental.

Now let us go back to the double groupoid of diagram (2), a more general structure than those just considered. The idea is that the groupoid structures associated to H, V, P form the lower dimensional part of this double groupoid. Taking this part as the notion of "base", we are interested in how to construct and use double groupoids obtained from the left adjoint to pullback.

In this generality, we do not have any results. However, for crossed modules over groups, we have presentations for this left adjoint given in [9]. We explain some recent specific calculations in the next section, again, for simplicity, restricting to crossed modules over groups.

3 Crossed modules

It is known that crossed modules have analogues for other categories than that of groups. Indeed, their use represents a long tradition in homological algebra (Rinehart, Gerstenhaber, Frohlich, Lue). The reason, as pointed out some years ago by Tim Porter, is that, roughly speaking, in categories in which congruences may be described by kernels, crossed modules are equivalent to internal groupoid objects, and groupoids of course generalise equivalence relations. Thus crossed modules give a more refined analysis of the quotienting operation. This possibly explains their use in identities among relations.

Recall that a *crossed module* is a morphism of groups $\mu : M \to P$ together with an action $(m, p) \mapsto m^p$ of P on M satisfying the two axioms

- CM1) $\mu(m^p) = p^{-1}(\mu m)p$
- CM2) $n^{-1}mn = m^{\mu n}$

for all $m, n \in M, p \in P$.

The category $\mathcal{XM}$ of crossed modules has objects all crossed modules with morphisms the commutative diagrams

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "g"'] & P \arrow[d, "f"] \\ N \arrow[r, "\nu"'] & Q \end{tikzcd}$$

in which the horizontal maps are crossed modules, and the pair g, f preserves the action in the sense that for all $m \in M, p \in P$ we have $g(m^p) = (gm)^{fp}$. If P is a group, then the category $\mathcal{XM}/P$ of crossed P-modules is the subcategory of $\mathcal{XM}$ whose objects are the crossed P-modules and in which a *morphism* $g : M \to N$ of crossed P-modules is a morphism of groups such that g preserves the action ($g(m^p) = (gm)^p$, for all $m \in M,\ p \in P$), and $\nu g = \mu$.

Standard algebraic examples of crossed modules are:

1. an inclusion of a normal subgroup, with action given by conjugation;
2. the inner automorphism map $\chi : M \to \mathrm{Aut}\ M$, in which χm is the automorphism $n \mapsto m^{-1}nm$;

3. the zero map $M \to P$ where M is a P-module;

4. an epimorphism $M \to P$ with kernel contained in the centre of M.

All these yield examples of finite crossed modules. Other finite examples may be constructed from those above, the induced crossed modules described below, and coproducts [4] and tensor products [16, 29] of crossed P-modules. We discuss later the important *free crossed modules.*

The geometric example of a crossed module is the boundary map

$$\partial : \pi_2(X, A) \to \pi_1(A)$$

of the second relative homotopy group of a pointed pair of spaces, together with the standard action on it of the fundamental group $\pi_1(A)$. This is the *fundamental crossed module* of the based pair (X, A). It is due to Whitehead [56]. Because of this example, in a crossed module $\mu : M \to P$ we regard P as the 1-dimensional part and M as the 2-dimensional part.

Whitehead's major result was the algebraic description of the second relative homotopy group $\pi_2(A \cup \{e^2_\lambda\}_{\lambda\in\Lambda}, A)$ as the *free crossed* $\pi_1(A)$*-module* on the 2-cells [56]. This was an important influence on the work on the 2-dimensional Van Kampen theorem. On the one hand, it showed that universal properties were available in 2-dimensional homotopy theory, and so raised confidence in the possibility of wider universal properties being found. On the other hand, it was a crucial test: a 2-dimensional Van Kampen Theorem should, to be any good, have Whitehead's theorem as a corollary. Since Whitehead's theorem is about relative homotopy groups, it seemed reasonable to define a *homotopy double groupoid* in a relative situation (X, A). The simple and natural definition is to take homotopy classes of maps of squares $I \times I$ into X whose edges go into A and whose vertices go to the base point. This was the key step [9]. Much of the theory then directly generalised the 1-dimensional case, with some new ideas.

In the applications, induced crossed modules play an important role. They are defined in a manner analogous to the change of base for groupoids. A morphism $\iota : P \to Q$ of groups determines a pullback functor $\iota^* : \mathcal{XM}/Q \to \mathcal{XM}/P$. The left adjoint ι_* to pullback gives the induced crossed modules. The latter are also given by pushouts of crossed modules of the form

$$\begin{tikzcd}
(1, P) \arrow[r, "{(1,\iota)}"] \arrow[d] & (1, Q) \arrow[d] \\
(M, P) \arrow[r] & (\iota_* M, Q)
\end{tikzcd}$$

in which we abbreviate terms such as $\mu : M \to P$ to (M, P).

In particular, let $\mu : M \to P$ be a crossed P-module. The elements of the induced crossed Q-module $\partial : \iota_* M \to Q$ may be regarded as formal products

$$m_1^{q_1} m_2^{q_2} \dots m_r^{q_r}$$

for $r \geq 0,\ m_i \in M,\ q_i \in Q$ subject to the rules $(m^p)^q = m^{(\iota p)q},\ m^q m_1^q = (mm_1)^q,$ as well as the rule CM2 for crossed modules, where $\partial(m^q) = q^{-1}(\iota\mu m)q$.

An intriguing feature follows from this description. The image of the boundary $\partial : \iota_* M \to Q$ is the normal closure of $\iota\mu M$ in Q. In particular, let $M = P, \mu = 1$. Then the normal closure of ιP in Q is replaced by a morphism $\partial : \iota_* P \to Q$, that is by a larger group with a universal property, but still depending only on ι. There is clearly interest in Ker (∂). For this module over Coker ι we know of no algebraic construction analogous, say, to constructions in homology. However, we shall see later that this module has a topological interpretation as

$$\pi_2(BQ \cup_{B\iota} \Gamma BP), \tag{3}$$

the second homotopy module of the mapping cone of the map of classifying spaces induced by ι.

The free crossed modules $\partial : C(R) \to Q$ are the special case of induced crossed modules in which μ is the identity and $M = P$ is a free group on a set R, say. Let $N(\iota R)$ be the normal closure of ιR in Q. If Q is a free group then the pair of morphisms

$$C(R) \xrightarrow{\partial} Q \to \text{Coker } \iota$$

may be regarded as the beginning of a non abelian free crossed resolution of Coker ι, and the kernel of the boundary ∂ gives the *identities among the relations*. There is considerable study of this notion in the case of groups [15, 43].

Other presentations of induced crossed modules are given in [9]. In particular, if $\iota : P \to Q$ is an epimorphism, and $\mu : M \to P$ is a crossed module, then $\iota_* M$ is the quotient of M by the group $[M, P]$ generated by the elements $m^{-1}m^k$ for all $m \in M,\ k \in$ Ker ι. So the interest is in the case when P is a subgroup of Q.

Motivated by the applications, Chris Wensley and I have given recently a range of new calculations of induced crossed modules [21, 22, 23]. One of our results is that if M and the index of P in Q are finite, then the induced crossed module $\iota_* M$ is finite [21]. This gives further point to finding explicit calculations. One aim described in [23] is to produce a computer program to which the user can input a finite group, and the program will produce a table of $\iota_* M$ for each subgroup P of Q and each normal subgroup M of P. Even though the full version is not yet complete, some results which have appeared are not easy to explain. For example, if P and Q are the symmetric groups S_3 and S_4 respectively, then $\iota_* P$ is the group $GL(2, 3)$, the general linear group of 2×2 matrices over the field with three elements, while if $P = D_4$, then $\iota_* P$ is the semidirect product $(C_2)^3 \rtimes S_3$.

Here is another example, which can be done in a page by hand calculation [21]. Let D_n be the dihedral group of order $2n$ generated by x, y with relations $x^n = y^2 = xyxy = 1$. Let D'_n be another copy, generated by u, v with relations $u^n = v^2 = uvuv = 1$. Let $\partial : D'_n \to D_n$ be defined by $u \mapsto x^2, v \mapsto y$, and let D_n act on D'_n by $v^x = vu$ (other actions of generators on generators being trivial). This gives a crossed module, and in fact $D'_n = \iota_* C_2$, where C_2 is generated by y. The second rule for a crossed module follows from

$$v^{\partial u} = v^{x^2} = (vu)^x = vuu = u^{-1}vu.$$

The computer calculations give many other surprising examples of finite crossed modules, and so, by the above equivalences of categories, give also non trivial examples of finite double groupoids and of finite 2-groupoids.

We also show in [21] that the methods of free crossed resolutions, which combine free crossed modules with free resolutions in the usual sense, are a useful computational tool for determining the cohomology class represented by a crossed module. We calculate for example that if C_n denotes the cyclic group of order n, then the non trivial element of $H^3(C_2, C_2)$ is represented by a finite crossed module of the form $\delta : C_2 \times C_2 \to C_4$. It does not seem to be known if any element of $H^3(G, A)$, where G is a finite group and A is a finite G-module, can be represented by a finite crossed module.

The homotopical description of Ker ∂ given in (3) is not the most powerful result possible. In fact, the first Postnikov invariant, and so the homotopy 2-type itself, of this mapping cone is represented by the induced crossed module $\partial : \iota_* P \to Q$. This result follows from a special case of the Van Kampen Theorem for the fundamental crossed module, as follows:

Theorem 3.1 ([9], Theorem D) *Let (W, V) be a cofibred pair of spaces, let $f : V \to A$ be a map, and let $X = A \cup_f W$. Suppose that A, W, V are path-connected, and the pair (W, V) is 1-connected. Then the pair (X, A) is 1-connected and the diagram*

$$\begin{array}{ccc} \pi_2(W,V) & \xrightarrow{\delta} & \pi_1(V) \\ {\scriptstyle\epsilon}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \pi_2(X,A) & \xrightarrow[\delta']{} & \pi_1(A) \end{array}$$

presents $\pi_2(X, A)$ as the crossed $\pi_1(A)$-module $\lambda_*(\pi_2(W, V))$ induced from the crossed $\pi_1(V)$-module $\pi_2(W, V)$ by the group morphism $\lambda : \pi_1(V) \to \pi_1(A)$ induced by f.

Note that this theorem deals with non abelian objects, and so is unlikely to be obtained by the usual abelian homological methods. The proof in [9] uses methods of double groupoids, and in particular the *fundamental double groupoid* of a pair of spaces to prove a general Van Kampen type theorem. (A completely different proof is given in [16].) The result cited on classifying spaces follows by taking $(W, V) = (\Gamma BP, BP)$, $A = BQ$.

Theorem 3.1 also implies the relative Hurewicz Theorem in this dimension. We explain the higher dimensional version of this theorem in the next section.

4 Modules over groups

Let P be a group. Then there is a category $\mathcal{MOD}/P$ of modules over P. If $\iota : P \to Q$ is a morphism of groups, then there is a functor

$$\iota^* : \mathcal{MOD}/Q \to \mathcal{MOD}/P,$$

where P acts on a Q-module via ι. This functor has a left adjoint

$$\iota_* : \mathcal{MOD}/P \to \mathcal{MOD}/Q,$$

giving the well known *induced* module.

Brown and Higgins give in [11] an application of induced modules to algebraic topology, namely the following generalisation of the relative Hurewicz theorem.

Theorem 4.1 [11] *Let $n > 2$. Let (W, V) be a pair of spaces, let $f : V \to A$ be a map, and let $X = A \cup_f W$. Suppose that one at least of f or $V \to W$ is a closed cofibration, that A, W, V are path-connected, and that the pair (W, V) is $(n-1)$-connected. Then the pair (X, A) is $(n-1)$-connected and the morphism*

$$\epsilon : \pi_n(W, V) \to \pi_n(X, A)$$

presents $\pi_n(X, A)$ as the $\pi_1(A)$-module $\lambda_*(\pi_n(W, V))$ induced from the $\pi_1(V)$-module $\pi_n(W, V)$ by the group morphism $\lambda : \pi_1(V) \to \pi_1(A)$ induced by f.

This theorem is a corollary of the Generalised Van Kampen Theorem for crossed complexes [11] (Theorem B), which implies in particular that under the assumptions given, the following diagram is a pushout in the category of modules over groups:

$$\begin{tikzcd}
(0, \pi_1(V))) \arrow[r] \arrow[d] & (0, \pi_1(A)) \arrow[d] \\
(\pi_n(W, V), \pi_1(V)) \arrow[r, "\epsilon"'] & (\pi_n(X, A), \pi_1(A))
\end{tikzcd} \qquad (4)$$

This pushout describes also the induced module.

The relative Hurewicz Theorem is the special case of Theorem 4.1 in which $A = \Gamma V$, the cone on V, so that the induced module is simply that obtained by factoring the action of $\pi_1(V)$. Further the relative homotopy group $\pi_n(W \cup \Gamma V, \Gamma V)$ is isomorphic to the absolute group $\pi_n(W \cup \Gamma V)$. Thus we obtain a version of the relative Hurewicz theorem which does not mention homology! Of course, the usual theorem is now easily obtained by applying the absolute Hurewicz theorem to the space $W \cup \Gamma V$. Actually, C.T.C. Wall has explained how Theorem 4.1 may be obtained from the relative Hurewicz Theorem and covering space arguments. The more general pushout and even coequaliser theorems of [11] (Theorem C) have not been obtained in this way. Their proof in [11] uses methods of multiple groupoids, and in particular the *fundamental ω-groupoid* of a filtered space.

However, these results do not allow one to obtain the fact that $\pi_3(S^2) \cong \mathbb{Z}$, and in particular to exhibit the Hopf map $S^3 \to S^2$ in an algebraic form. This may be obtained using the Van Kampen Theorem for crossed squares of groups [16].

5 Cat^n-groups, and crossed n-cubes of groups

More general forms of higher homotopy groupoids were introduced by Loday in [49]. These are defined on n-cubes of pointed spaces, rather than on filtered spaces, as for ω-groupoids and crossed complexes. His cat^n-groups are essentially $(n+1)$-fold groupoids in which one of the structures is a group. Loday proves the extraordinary theorem that these objects model all connected, pointed homotopy $(n+1)$-types of CW-spaces. Indeed, he obtains an equivalence of homotopy categories. This shows that multiple groupoids are complicated objects, since homotopy theory is known to be complicated. In fact cat^n-groups are shown by Ellis and Steiner in [31] to be equivalent to what they call *crossed n-cubes of groups.* It is intriguing that the theory of the latter objects includes a lot of commutator theory, so that once again multiple groupoids link with standard methods.

The notion of change of base is also allowable for multiple groupoids, and is used for cat^n-groups in [17]. The "base" of say an n-fold groupoid has to be defined, and there are lots of choices. An n-fold groupoid G gives rise to an n-cube of n-fold groupoids, where the object G_A at a vertex A of the cube, for A a subset of $\langle n\rangle = \{1, 2, \dots, n\}$, is the sub n-fold groupoid of G whose elements are identities for the groupoid structures in directions $i \notin A$. Thus each of the G_A for $A \neq \langle n\rangle$ is a kind of degenerate n-fold groupoid. These G_A, and their inclusions, form the "lower dimensional part" of G, and one can study how changes to this lower dimensional part, or to some substructure of it, affect G. It is shown in [17] how one can formulate induced constructions as a kind of generalized excision, using $(n+1)$-pushouts of n-cubes. This yields a notion of induced cat^n-groups, which uses a complex form of "change of base" involving n-cubes of $(n-1)$-cubes.

The Generalized Van Kampen theorem of [16] is used in [17] to give a topological application of these induced cat^n-groups, and, as a special case, an n-cubical form of the Hurewicz theorem, previously known only in the classical relative case, $n = 1$. It was Loday's conjecturing of such a theorem for $n = 2$ which started the generalisation of the work of Brown and Higgins, and so gave a kind of culminating point of the route from the Van Kampen Theorem described in section 1, for the fundamental groupoid.

This general Hurewicz theorem is used by Brown and Ellis in [6] to give a Hopf formula for the $(n+1)$-st homology of a group G, in terms of n normal subgroups $R_1, R_2, \dots, R_n$ of a free group F such that $F/(R_1 \dots R_n)$ is isomorphic to G and various other quotients of F are free. This result has stimulated a lot of other work, including other proofs of the Hopf formula. However the special case of even the triadic Hurewicz theorem, described in [5], has no other proof to date.

Results from [17] on induced crossed squares are used by Ellis in [30] to define what he calls "free crossed squares" and to give applications to 3-dimensional combinatorial homotopy theory. Thus applications of multiple groupoids reach parts of homotopy theory not accessible at present to other methods.

6 Right adjoints to pullback

Here we sketch an area which is part of, or related to, change of base, and seems ready for wider exploitation. The general problem is the consideration of right adjoints to pullback $f^* : \mathcal{C}_Y \to \mathcal{C}_X$ when $f : X \to Y$ is a morphism of the category $\mathcal{C}$, and $\mathcal{C}_Y$ denotes the slice category of objects of $\mathcal{C}$ over Y.

For $\mathcal{C}$ the category of small categories, this right adjoint was considered by F. Conduché in [25]. He concludes that the right adjoint exists if f is a fibration. Following this lead, an analogous result was proved by Jim Howie in [44] for crossed complexes, and in particular for groupoids. It follows if $f : G \to H$ is a fibration of groupoids, then the pullback functor f^* preserves colimits. For more on fibrations of groupoids, see [41].

Philip Higgins has observed that, since a covering morphism of groupoids is a special case of a fibration, the result described earlier, that the pullback of a universal morphisms of groupoids by a covering morphism is again universal, follows without requiring a solution of the word problem. Brown and Heath in [8] observe that an epimorphism of groups is a fibration of groupoids, and deduce that a pullback of a colimit of a connected diagram of groups by an epimorphism of groups is also a colimit of the pullback diagram. The connected diagrams are exactly those whose colimits are preserved by the inclusion of categories (groups) $\subset$ (groupoids). The construction of these right adjoints, and the applications to pullbacks of colimits, are carefully analysed in [36]. Some of the colimit results are shown in [37] to be derivable by other means. The fact that a covering crossed complex of a free crossed complex is again free may be proved by these methods. This result has planned uses in the context of free crossed resolutions and contracting homotopies.

7 Conclusion

One of the original aims of Whitehead can be seen to be that of exploring the extension to higher dimensional homotopy theory of the notions of combinatorial group theory which were deeply studied in the 1930's. The tools that he developed, particularly crossed modules, form an essential part of the version of this programme which uses higher homotopy groupoids.

As in any algebraic theory, it is useful to have a notion of "free" objects. These are special cases of, and may usefully be defined in terms of, the more general notion of "induced structure". These "induced" structures are useful for theoretical studies, may sometimes be calculated explicitly, and are related to other kinds of induced construction and to "change of base".

The general aims of the use of multiple groupoids are wider than homotopy theory, and are to give an algebraic and calculable setting for a general use of whatever might be conceived to be "higher dimensional group(oid) theory". It is possible that the broad aspects of this aim may require different structures, for example with lax rather than strict axioms. The investigation of, and calculation with, induced structures would then be more complicated.

The results to date, even for just the technique of induced structures, demonstrate through published papers that the expectations for a theory of higher dimensional group(oid)s listed in the Introduction have already been satisfied. Of course the problem with which we started, the determination of the homotopy groups of the 2-sphere S^2, is as far off as ever. Nonetheless, some basic results in homotopy theory, such as an n-adic Hurewicz theorem, which had not previously even be formulated, and for which no other proof is available, are obtained. There are also new calculations of homotopy invariants of spaces, and even of homotopy types. So these new methods have been tested in a hard area and shown to work. This testing is one of the aims of the applications in homotopy theory.

It would be interesting to see if and how these new tools for local-to-global problems may be applied to other areas, and so give, as might be hoped, an extraordinary widening of the notion and application of the basic intuitions of abstract group theory, in a manner analogous to the extension from one variable to many variable calculus. This is a possibility well worth putting to the test.

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