Colimit Theorems for Relative Homotopy Groups*

Ronald Brown†
School of Mathematics,
University of Wales, Bangor,
Dean St.,
Bangor, Gwynedd LL57 1UT,
UK

Philip J. Higgins,
Department of Mathematics,
Durham University,
South Rd.,
Durham DH1 3LE
UK

December 14, 2012

Introduction

This is the second of two papers whose main purpose is to prove a generalisation to all dimensions of the Seifert-Van Kampen theorem on the fundamental group of a union of spaces.

The first paper [10] (whose results were announced in [8]) developed the necessary 'algebra of cubes'. Categories $\mathcal{G}$ of ω-groupoids and $\mathcal{C}$ of crossed complexes were defined, and the principal result of [10] was an equivalence of categories $\gamma : \mathcal{G} \to \mathcal{C}$. Also established were a version of the homotopy addition lemma, and properties of 'thin' elements, in an ω-groupoid. In particular it was proved that an ω-groupoid is a special kind of Kan cubical complex, in that every box has a unique thin filler. All these results will be used here.

Throughout this paper we consider filtered spaces

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

and associate with such an X_* a cubical complex RX_* which in dimension n is the set of filtered maps $I^n_* \to X_*$, where I^n_* is the standard n-cube with its filtration by skeletons. Then RX_* has defined

*This is a version of the paper with this title published in the J. Pure Appl. Algebra, 22 (1981) 11-41, revised initially by the first author in May, 1999, and later, to take into account later views of both authors, and to make minor clarifications. The main change is to avoid the J_0 condition on filtered space which was used in the published version – this is done by defining higher homotopy groupoids using homotopy classes *rel vertices* of I^n. This makes the theory nearer to standard homotopy theory, and is also essential for later work in defining the homotopy crossed complexes for filtered function spaces, when the J_0 condition is unlikely to be fulfilled. It is hoped that this version will be useful to readers. We also change here to a notation used in later works, and replacing $\mathbf{X}$ by X_*, for a filtered space, and π by Π for the fundamental crossed complex of a filtered space. We also change the term 'homotopy full' to 'connected', and the term 'filtered homotopy' to 'thin homotopy', to agree with terminology in [25] in the additional bibliography.

†r.brown@bangor.ac.uk

on it, in a natural, geometric way, structures of connections Γ_j and compositions $+_i$, satisfying rules given in [10, Section 1]. In particular, the n compositions $+_1, \cdots, +_n$ defined on $R_n X_*$ correspond to gluing n-cubes together in n different directions.

We now factor RX_* by the relation of homotopy rel end points through filtered maps rel vertices to obtain a quotient map $p : RX_* \to \varrho X_*$ where ϱX_* also is a cubical complex with connections. The first main result (Theorem A of Section 2) is that the $+_i$ are inherited by ϱX_*, which becomes an ω-groupoid.

Our promised generalisation of the Seifert-Van Kampen theorem to all dimension is Theorem B of Section 4, which takes the form of a colimit theorem for ϱX_*. Its proof follows closely the structure of some proofs of the one-dimensional theorem (as in [11], for example) but makes crucial use of properties of thin elements in ϱX_*. For the applications, this colimit theorem is recast in terms of the closely related invariant ΠX_*, the fundamental crossed complex of X_* (studied under other names in [3] and [23]). We show in Section 5 that $\gamma \varrho X_*$ is naturally isomorphic to ΠX_*, and hence obtain colimit theorems for ΠX_* (Theorems C and D of Section 5). In the proofs of all these results, one of the key ingredients is the deformation theorem of Section 3 which says, essentially, that $p : RX_* \to \varrho X_*$ is a fibration in the sense of Kan. This allows a characterisation of thin elements in ϱX_* and also helps to establish the connection between ϱX_* and ΠX_*.

In Section 6 we show how to construct colimits of crossed complexes, making particular use of induced modules, and induced crossed modules, over groupoids. In Section 7 we show that Theorem C contains as very special cases not only the Seifert-Van Kampen theorem (in its groupoid version), but also the fact that $\pi_n S^n \cong \mathbb{Z}$, that $\pi_n(U \cup \{e^n_\lambda\}, U)$ is a free $\pi_1 U$-module on the n-cells for $n > 2$ (free crossed $\pi_1 U$-module if $n = 2$), and that if (V, W) is an $(n-1)$-connected pair, then $\pi_i(V \cup CW)$ is 0 for $i < n$ and is $\pi_n(V, W)$ factored by the action of $\pi_1 W$ if $i = n$.

At this stage, we have not used homology at all. However, the last mentioned result, together with the absolute Hurewicz theorem, is easily seen to imply the relative Hurewicz theorem; in Section 8 we give a proof of the absolute theorem in the present context, and relate the homotopy exact sequence of the fibration $p : RX_* \to \varrho X_*$ to work of Blakers [3] and Whitehead [22, 24]. In Section 9 we establish that an ω-groupoid is isomorphic to some ϱX_*, and that any crossed complex is isomorphic to some ΠY_*; hence these constructions generalise constructions of Eilenberg-Mac Lane complexes. Finally, we prove that ϱI^n_* is the free ω-groupoid on one generator of dimension n.

1 Thin homotopies

By a *filtered space* X_* is meant a space X and a sequence $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ of subspaces of X. By a *filtered map* $f : X_* \to Y_*$ of filtered spaces is meant a map $f : X \to Y$ of spaces such that $f(X_s) \subseteq Y_s, s = 0, 1, 2, \ldots$. A standard example of a filtered space is a CW-complex with its skeletal filtration. A cellular map of CW-complexes is then a filtered map of the associated filtered spaces.

Let I^n be the standard n-cube with its standard cell structure as a product of n copies of $I = [0, 1]$. Then the filtered space consisting of I^n with its skeletal filtration $I^n_0 \subseteq I^n_1 \subseteq I^n_2 \subseteq \cdots$ will be written I^n_*. We also write $\boldsymbol{\partial}\,(I^n)$ for the boundary of I^n, i.e. the subcomplex I^n_{n-1}. The filtered space associated with the skeletal filtration of a subcomplex B of I^n will be written B_*.

Two filtered maps $f_0, f_1 : X_* \to Y_*$ of filtered spaces will be called *thin homotopic* if there is a homotopy $f : X \times I \to Y$ from f_0 to f_1 such that $f(X_s \times I) \subseteq Y_s, s = 0, 1, 2, \cdots$; such an f is called a *thin homotopy*, and we write $f : f_0 \equiv f_1$.

Let X_* be a filtered space. Then $R_n X_*$ will denote the set of filtered maps $I^n_* \to X_*$. These sets for all $n \geqslant 0$, together with the standard face and degeneracy maps, and with the connections and compositions defined in [10], form a cubical complex with connections and compositions, which we write RX_*.

The set of classes of elements of $R_n X_*$ under thin homotopy rel vertices of I^n is written $\varrho_n X_*$, the class of $\alpha \in R_n X$ is written $\bar{\alpha}$, and the quotient map is written $p : R_n X_* \to \varrho_n X_*$. It is easy to check that the connections and the face and degeneracy maps of RX_* are inherited by ϱX_*, giving it the structure of cubical complex with connections. We will prove in Section 2 that the compositions also are inherited. This and later proofs require techniques for constructing thin homotopies, and for this we use methods of collapsing.

Let B, C be subcomplexes of I^n such that $C \subseteq B$. Recall that C is an *elementary collapse* of B, written $B \searrow^e C$, if for some $s \geqslant 1$ there is an s-cell a of B and $(s-1)$-face b of a such that

$$B = C \cup a, \qquad C \cap a = \boldsymbol{\partial}\,(a) \setminus b$$

(where $\boldsymbol{\partial}\,(a)$ denotes the union of the proper faces of a). If there is a sequence

$$B_1 \searrow^e B_2 \searrow^e \cdots \searrow^e B_r$$

of elementary collapses, then we write $B_1 \searrow B_r$ and say B_1 *collapses* to B_r.

It is well known that if C is a subcomplex of B then $B \times I$ collapses to $B \times \{0\} \cup C \times I$ (this is proved by induction on the dimension of $B \setminus C$), and that I^n collapses to any one of its vertices (this may be proved by induction on n using the first example.)

Let B be a subcomplex of I^n, let $m \geqslant 2$, and let $B \times I^m$ be given the product cell structure, so that the skeletal filtration gives a filtered space $B_* \times I^m_*$. Let $h : B \times I^m \to X$ be a map. Fixing the ith coordinate of I^m at the value t, where $0 \leqslant t \leqslant 1$, we obtain a map $\partial^t_i h : B \times I^{m-1} \to X$. If X_* is a filtered space, and $\partial^t_i h : B_* \times I^{m-1}_* \to X_*$ is a filtered map for each $0 \leqslant t \leqslant 1$, we say h is a *thin homotopy in the* ith *direction of* I^m. (A similar definition applies to a map $h : I^m \to X$.) In such case we write $h : \alpha \equiv_i \beta$ where $\alpha = \partial^0_i h, \beta = \partial^1_i h$. It is easy to see that the relation $\equiv_i$ defined on filtered maps $B \times I^{m-1} \to X$ by the existence of such an h is an equivalence relation independent of $i, 1 \leqslant i \leqslant m$.

A map $h : B_* \times I^2_* \to X_*$ is call a *thin double homotopy* if it is a thin homotopy in each of the two directions of I^2; this is equivalent to $h(B_s \times I^2) \subseteq X_{s+1}, h(B_s \times \boldsymbol{\partial}\, I^2) \subseteq X_s, s = 0, 1, 2, \cdots$. If K is a *proper* subcomplex of I^2, and $k : B \times K \to X$ satisfies $k(B_s \times K) \subseteq X_s, s = 0, 1, 2, \cdots$, then by an abuse of language we call k also a thin double homotopy.

Consider now a filtered space X_*.

Proposition 1.1 *Let B, C be subcomplexes of I^n such that $B \searrow C$. Let*

$$f : B \times \boldsymbol{\partial}\, I^2 \to X, \qquad g : C \times I^2 \to X$$

be thin double homotopies which agree on $C \times \partial I^2$. Then $f \cup g$ extends to a thin double homotopy $h : B \times I^2 \to X$.

Proof It is sufficient to consider the case of an elementary collapse $B \searrow^{e} C$. Suppose then $B = C \cup a, C \cap a = \partial a \setminus b$, where a is an s-cell and b is an $(s-1)$-face of a.

Let $r : a \times I^2 \to (a \times \partial I^2) \cup ((\partial a \setminus b) \times I^2)$ be a retraction. Then r defines an extension $h : B \times I^2 \to X$ of $f \cup g$. Since f is a thin double homotopy,

$$h(a \times I^2) = f(a \times \partial I^2) \subseteq X_s,$$

and since g is a thin double homotopy

$$h((\partial a \setminus b) \times I^2) = g((\partial a \setminus b) \times I^2) \subseteq X_s.$$

Hence $h(\alpha \times I^2) \subseteq X_s$, and in particular $h(b \times I^2) \subseteq X_s$. These conditions, with those of $f \cup g$, imply that h is a thin double homotopy. □

Corollary 1.2 *Let X_* be a filtered space and let B be a subcomplex of I^n such that B collapses to one of its vertices. Then any thin double homotopy rel vertices $f : B_* \times \partial I^2_* \to X_*$ extends to a thin double homotopy rel vertices $h : B_* \times I^2_* \to X_*$.*

Proof Let ν be a vertex of B such that $B \searrow \{\nu\}$. Now $f(\{\nu\} \times \partial I^2) \subseteq X_0$. Since the homotopies are rel vertices, $f \mid \{\nu\} \times \partial I^2$ extends to a constant map $g : \{\nu\} \times I^2 \to X$ with image in X_0. Thus g is a thin double homotopy. By Proposition 1.1, $f \cup g$ extends to a thin double homotopy $h : B \times I^2 \to X$. □

The following result is joint work with N. Ashley.

Proposition 1.3 *Let B, A be subcomplexes of I^n such that $B \subseteq A$ and B collapses to one of its vertices. Let X_* be a filtered space. Let $\alpha, \beta : A_* \to X_*$ be filtered maps and let $\psi : \alpha \equiv \beta, \phi : \alpha \mid B \equiv \beta \mid B$ be filter homotopies rel vertices. Then there is a thin double homotopy $H : A \times I^2 \to X$ such that H is a homotopy rel end maps of ψ to a thin homotopy $\alpha \equiv \beta$ extending ϕ.*

Proof Let $L = (I \times \{0\}) \cup (I \times I)$. Define

$$l : (A \times L) \cup (B \times I \times \{1\}) \to X$$

by $l(x, t, 0) = \psi(x, t), l(x, 0, t) = \alpha(x), l(x, 1, t) = \beta(x), l(y, t, 1) = \phi(y, t), x \in A, y \in B, t \in I$. Then $f = l \mid B \times \partial I^2$ and $k = l \mid A \times L$ are thin double homotopies.

By Corollary 1.2, f extends to a thin double homotopy $h : B \times I^2 \to X$.

We extend $k \cup h : (A \times L) \cup (B \times I^2) \to X$ to a thin double homotopy $H : A \times I^2 \to X$ by induction on the dimension of $A \setminus B$.

Suppose that H_s is a thin double homotopy defined on $(A \times L) \cup ((A_s \times B) \times I^2)$, extending $H_{-1} = k \cup h$. For each $(s+1)$-cell a of $A \setminus B$, choose a retraction

$$r_a : a \times I^2 \to (\partial a \times L) \cup (a \times I^2).$$

These retractions extend H_s to H_{s+1} defined also on $A_{s+1} \times I^2$. Since $r_a(a \times I^2) \subseteq X_{s+1}$, it follows that H_{s+1} is also a thin double homotopy.

Clearly $H = H_n$ is a thin double homotopy as required. $\square$

Corollary 1.4 *Let B, A, X_* be as in Proposition 1.3. If $\alpha, \beta : A_* \to X_*$ are maps which are thin homotopic rel vertices, then any thin homotopy rel vertices $\alpha \mid B \equiv \beta \mid B$ extends to a thin homotopy $\alpha \equiv \beta$.* $\square$

If $f : Y_* \to X_*$ is a filtered map, where Y_* is a CW-complex with its skeletal filtration, we say that f is *deficient on a cell a of Y* if $\dim a = s$ but $f(a) \subseteq X_{s-1}$.

Proposition 1.5 (thin homotopy extension property). *Let B, A be subcomplexes of I^n such that $B \subseteq A$. Let $f : A \times \{0\} \cup B \times I \to X$ be a map such that $f \mid A \times \{0\}$ is a filtered map and $f \mid B \times I$ is a thin homotopy rel vertices. Then f extends to a thin homotopy $h : A \times I \to X$. Further, h can be chosen so that if f is deficient on a cell $a \times \{0\}$ of $(A \setminus B) \times \{0\}$, then h is deficient on $a \times \{1\}$.* $\square$

The proof of this proposition is an easy induction on the dimension of the cells of $A \setminus B$, using retractions $a \times I \to a \times \{0\} \cup \boldsymbol{\partial} a \times I$ for each cell a of $A \setminus B$.

2 ϱX_* is an ω-groupoid

We now show that the compositions in RX_* are inherited by ϱX_*. This gives us a definition of a *higher homotopy groupoid*.

Theorem A. *If X_* is a filtered space, then the compositions on RX_* induce compositions on ϱX_* which, together with the induced face and degeneracy maps and connections, give ϱX_* the structure of ω-groupoid.*

Proof We need some notation for multiple compositions in $R_n X_*$.

Let $(m) = (m_1, \ldots, m_n)$ be an n-tuple of positive integers. Let

$$\phi_{(m)} : I^n \to [0, m_1] \times \cdots \times [0, m_n]$$

be the map $(x_1, \ldots, x_n) \mapsto (m_1 x_1, \ldots, m_n x_n)$. Then a *subdivision of type* (m) of a map $\alpha : I^n \to X$ is a factorisation $\alpha = \alpha' \circ \phi_{(m)}$; its *parts* are the cubes $\alpha_{(r)}$ where $(r) = (r_1, \ldots, r_n)$ is an n-tuple of integers with $1 \leqslant r_i \leqslant m_i, i = 1, \ldots, n$, and where $\alpha_{(r)} : I^n \to X$ is given by

$$(x_1, \ldots, x_n) \mapsto \alpha'(x_1 + r_1 - 1, \ldots, x_n + r_n - 1).$$

We then say that α is the *composite* of the cubes $\alpha_{(r)}$ and write $\alpha = [\alpha_{(r)}]$. The *domain* of $\alpha_{(r)}$ is then the set $\{(x_1, \ldots, x_n) \in I^n : r_i - 1 \leqslant_i x_i \leqslant r_i, 1 \leqslant i \leqslant n\}$.

The composite is *in direction* j if m_j is the only $m_i > 1$, and we then write $\alpha = [\alpha_1, \ldots, \alpha_{mj}]_j$; the composite is *in the directions* $j, k (j \neq k)$ if m_j, m_k are the only $m_i > 1$, and we then write

$$\alpha = [\alpha_{rs}]_{j,k}$$

$(r = 1, \cdots, m_j; s = 1, \cdots, m_k)$.

A composition $+_i$ on $\varrho_n X_*$ is defined as follows.

Let $\bar{\alpha}, \bar{\beta} \in \varrho_n X_*$ satisfy $\partial_i^1 \bar{\alpha} = \partial_i^0 \bar{\beta}$. Then $\partial_i^1 \alpha \equiv \partial_i^0 \beta$, so we may choose $h : I^n \to X$, a thin homotopy in the ith direction, so that $\gamma = [\alpha, h, \beta]_i$ is defined in $R_n X_*$. We let $\bar{\alpha} +_i \bar{\beta} = \bar{\gamma}$ and prove this composition well-defined.

For this it is sufficient, by symmetry, to suppose $i = n$. The following picture illustrates the proof.

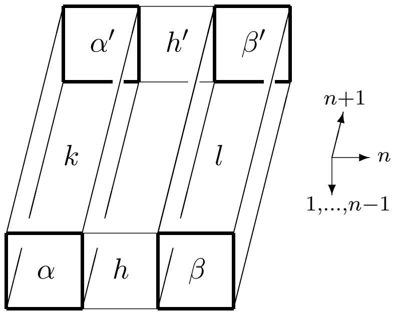

Figure 1

Let $\gamma' = [\alpha', h', \beta']_n$ be alternative choices. Then there exist thin homotopies $k : \alpha \equiv \alpha', l : \beta \equiv \beta'$ (in the $(n+1)$st direction). We view I^{n+1} as a product $I^{n-1} \times I^2$ and define a thin double homotopy rel vertices $f : I^{n-1} \times \boldsymbol{\partial} I^2 \to X$ by $f(x, t, 0) = h(x, t), f(x, t, 1) = h'(x, t), f(x, 0, t) = k(x, 1, t), f(x, 1, t) = l(x, 0, t)$, where $x \in I^{n-1}$ and $t \in I$. By Corollary 1.2, f extends to a thin double homotopy $H : I^{n-1} \times I^2 \to X$. Then $[k, H, l]_n$ is defined and is a thin homotopy $\gamma \equiv \gamma'$. This completes the proof that $+_n$, and by symmetry $+_i$, is well defined.

Suppose now that $\alpha +_i \beta$ is defined in $R_n X_*$. Let $h : \partial_i^1 \alpha \equiv_i \partial_i^0 \beta$ be the constant thin homotopy in the ith direction. Then $\alpha +_i \beta$ is a thin homotopic to $[\alpha, h, \beta]_i$ and so $\overline{\alpha +_i \beta} = \bar{\alpha} +_i \bar{\beta}$. Thus the operations $+_i$ on $\varrho_n X_*$ are induced by those on $R_n X_*$ in the usual algebraic sense.

Further, if $\bar{\alpha} +_i \bar{\beta}$ is defined in $\varrho_n X_*$, then we may choose representatives α', β' of $\bar{\alpha}, \bar{\beta}$ such that $\alpha' +_i \beta'$ is defined and represents $\bar{\alpha} +_i \bar{\beta}$ (for example we may take $\alpha' = \alpha, \beta' = h +_i \beta'$ where $h : \partial_i^1 \alpha \equiv_i \partial_i^0 \beta$).

Defining $-_i(\bar{\alpha}) = (\overline{-_i \alpha})$, one easily checks that $+_i$ and $-_i$ make $\varrho_n X_*$ a groupoid with initial, final and identity maps $\partial_i^0, \partial_i^1$ and ε_i.

The laws for $\varepsilon_j, \partial_j^\tau, \Gamma_j$ of a composite $\bar{\alpha} +_i \bar{\beta}$ follow from the laws in $R_n X_*$ by choosing the representatives α, β so that $\alpha +_i \beta$ is defined.

Finally, we must verify the interchange law for $+_i, +_j (i \neq j)$. By symmetry, it is sufficient to assume $i = n-1, j = n$.

Suppose that $\bar{\alpha} +_{n-1} \bar{\beta}, \bar{\gamma} +_{n-1} \bar{\delta}, \bar{\alpha} +_n \bar{\gamma}, \bar{\beta} +_n \bar{\delta}$ are defined in $\varrho_n X_*$. We choose the representatives $\alpha, \beta, \gamma, \delta$ and construct in $R_n X_*$ a composite

$$\begin{bmatrix} \alpha & k & \gamma \\ h & H & h' \\ \beta & k' & \delta \end{bmatrix}_{n-1,n} \tag{2.1}$$

in which the thin homotopies h, h' in the $(n-1)$st direction and the thin homotopies k, k' in the nth direction exist, because the appropriate composites are defined. To construct H, we define a thin double homotopy $f : I^{n-2} \times I^2 \to X$ by $f(x, 0, t) = k(x, 1, t), f(x, 1, t) = k'(x, 0, t), f(x, t, 0) = h(x, t, 1), f(x, t, 1) = h'(x, t, 0)$ where $x \in I^{n-2}$, and $t \in I$. By Corollary 1.2, f extends to a thin double homotopy $H : I^{n-2} \times I^2 \to X$. Then the composite (2.1) is defined in $R_n X$ and the interchange law

$$(\bar{\alpha} +_{n-1} \bar{\beta}) +_n (\bar{\gamma} +_{n-1} \bar{\delta}) = (\bar{\alpha} +_n \bar{\gamma}) +_{n-1} (\bar{\beta} +_n \bar{\delta})$$

is readily deduced by evaluating (2.1) in two ways.

This completes the proof that ϱX_* is an ω-groupoid. □

We call ϱX_* the *homotopy ω-groupoid* of the filtered space X_*.

A filtered map $f : X_* \to Y_*$ of filtered spaces clearly defines a map $Rf : RX_* \to RY_*$ of cubical complexes with connections and compositions, and a map $\varrho f : \varrho X_* \to \varrho X_*$ of ω-groupoids. So we have a functor

$$\varrho : (\text{filtered spaces}) \to (\omega\text{-groupoids}).$$

The question of the behaviour of ϱ with regard to homotopies of filtered maps will be considered in later papers[1]. At this stage we can use standard results in homotopy theory to prove:

Proposition 2.2 *Let $f : X_* \to Y_*$ be a filtered map of filtered spaces such that each $f_n : X_n \to Y_n$ is a homotopy equivalence. Then $\varrho f : \varrho X_* \to \varrho Y_*$ is an isomorphism of ω-groupoids.*

Proof This is immediate from [13, (10.11)]. □

3 The fibration and deformation theorems

The following result is an easy and memorable consequence of the deformation theorem (Theorem 3.2) below.

Theorem 3.1 (the fibration theorem). *Let X_* be a filtered space. Then the quotient map $p : RX_* \to \varrho X_*$ is a Kan fibration.*

The deformation theorem is a more explicit and slightly stronger form of this result; it is needed as a technical tool in later proofs.

[1] The full account of such a notion in which a homotopy f_t of filtered maps f_0, f_1 should satisfy $f_t(X_n) \subseteq Y_{n+1}$, in analogy with cellular homotopies, was given in references [26,27], and also in [25].

First let C be an r-cell in the n-cube I^n. Two $(r-1)$-faces of C are called *opposite* if they do not meet. A *partial box* in C is a subcomplex B of C generated by one $(r-1)$-face b of C (called a *base* of B) and a number, possibly zero, of other $(r-1)$-faces of C none of which is opposite to b. The partial box is a *box* if its $(r-1)$-cells consist of all but one of the $(r-1)$-faces of C.

Theorem 3.2 (the deformation theorem). *Let X_* be a filtered space, and let $\alpha \in R_n X_*$. Let B be a partial box in $I^n, \gamma : B_* \to X_*$ a filtered map, and suppose that for each $(n-1)$-face a of B, the maps $\alpha \mid a, \gamma \mid a$ are thin homotopic rel vertices. Then α is a thin homotopic to a map $\beta : I^n \to X$ extending γ. Further, if α is deficient (i.e. $\alpha(I^n) \subseteq X_{n-1}$), then β may be chosen to be deficient.*

The proof requires the following lemma.

Lemma 3.3 *Let B, B' be partial boxes in an r-cell C of I^n such that $B' \subseteq B$. Then there is a chain*

$$B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1 = B'$$

such that

(i) *each B_i is a partial box in C;*

(ii) *$B_{i+1} = B_i \cup a_i$ where a_i is an $(r-1)$-cell of C not in B_i;*

(iii) *$a_i \cap B_i$ is a partial box in a_i.*

Proof We first show that there is a chain $B = B_s \supset B_{s-1} \supset \cdots \supset B_1 = B'$ of partial boxes and a set of $(r-1)$-cells $a_1, a_2, \cdots, a_{s-1}$ such that $B_{i+1} = B_i \cup a_i, a_i \subseteq B_i$. If B and B' have a common base this is clear, since we may adjoin to B' the $(r-1)$-cells of $B \setminus B'$ one at a time in any order. If B and B' have no common base, choose a base b for B and let b^* be its opposite face in C. Then neither b nor b^* is in B'. Hence $B_2 = B' \cup b$ is a partial box with base b and we are reduced to the first case.

Now consider the partial box $B_{i+1} = B_i \cup a_i, a \subseteq B_i$. We claim that $a_i \cap B_i$ is a partial box in a_i. To see this, choose a base b for B_{i+1} with $b \neq a_i$; this is possible because if a_i were the only base for B_{i+1}, then B_i would consist of a number of pairs of opposite faces of C and would not be a partial box. We now have $a_i \neq b, a_i \neq b^*$, so $a_i \cap b$ is an $(r-2)$-face of a_i. Its opposite face in a_i is $a_i \cap b^*$ and this is not in B_i because the only $(r-1)$-faces of C which contain it are a_i and b^*. Hence $a_i \cap B_i$ is a partial box with base $a_i \cap b$.

The proof is now completed by induction on the dimension r of C. If $r = 1$, the lemma is trivial. If $r > 1$, choose B_i, a_i as above. Since $a_i \cap B_i$ is a partial box in a_i, there is a box J in a_i containing it. The elementary collapse $a_i \overset{e}{\searrow} J$ gives $B_{i+1} \overset{e}{\searrow} B_i \cup J$. But by the induction hypothesis, J can be collapsed to the partial box $a_i \cap B_i$ in a_i, and this implies $B_{i+1} \searrow B_i$. □

Proof of Theorem 3.2. Let B_1 be any $(n-1)$-cell contained in B. We choose a chain $B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1$ of partial boxes and $(n-1)$-cells $a_1, a_2, \cdots, a_{s-1}$ as in the lemma.

We construct thin homotopies $\phi_i : \alpha \mid B_i \equiv \gamma \mid B_i$ by induction on i, starting with ϕ_1 any thin homotopy $\alpha \mid B_1 \equiv \gamma \mid B_1$. Suppose ϕ_i has been constructed and extends ϕ_{i-1}. Then $\phi_i \mid a_i \cap B_i)$ is defined. Since $a_i \cap B_i$ is a partial box, the lemma implies that $a_i \cap B_i$ collapses to any of its vertices. Since $\alpha \mid a_i \equiv \gamma \mid a_i$, the homotopy $\phi_i \mid (a_i \cap B_i)$ extends, by Corollary 1.4, to a thin homotopy $\alpha \mid a_i \equiv \gamma \mid a_i$; this, with ϕ_i, defines ϕ_{i+1}.

Finally, we apply the thin homotopy extension property (Proposition 1.5) to extend $\phi_s : \alpha \mid B \equiv \gamma$ to a thin homotopy $\alpha \equiv \beta$, for some β extending γ. The last part of Proposition 1.5 gives the final part of Theorem 3.2. □

For some applications of the deformation theorem, it is convenient to work in the category of cubical complexes. To this end, we write I^n not only for the geometric n-cube, but also for its model as a cubical complex, namely the free cubical complex on one generator c^n of dimension n. Then an element γ of dimension n of a cubical complex C determines a unique cubical map $\hat{\gamma} : I^n \to C$ such that $\hat{\gamma}(c^n) = \gamma$. In particular, a filtered map $\gamma : I^n_* \to X_*$ determines a unique cubical map $\hat{\gamma} : I^n \to RX_*$ such that $\hat{\gamma}(c^n) = \gamma$. Also, if P is a subcomplex of the geometric n-cube P then P determines a subcomplex, also written P, of the cubical complex I^n , and a filtered map $\gamma : P_* \to X_*$ determines uniquely a cubical map $\hat{\gamma} : P \to RX_*$.

We can now rewrite the deformation theorem as follows.

Corollary 3.4 *Let B be a box in I^n and let $i : B \to I^n$ be the inclusion. Let X_* be a filtered space, and suppose given a commutative diagram of cubical maps*

$$\begin{tikzcd} B \arrow[r, "\gamma"] \arrow[d, "i"'] & RX_* \arrow[d, "p"] \\ I^n \arrow[r, "\bar{\alpha}"'] & \varrho X_* \end{tikzcd}$$

Then there is a map $\beta : I^n \to RX_*$ such that $\beta i = \gamma, p\beta = \bar{\alpha}$. Further, if $\bar{\alpha}(c^n)$ has a deficient representative, then β may be chosen so that $\beta(c^n)$ is deficient. □

The fibration theorem (Theorem 3.1) is immediate from the first part of Corollary 3.4.

One application of Corollary 3.4 is to the lifting of subdivisions from $\varrho_n X_*$ to $R_n X_*$. For the proof of this, and of the union theorem in the next section, we require the following construction.

Let $(m) = (m_1, \cdots, m_n)$ be an n-tuple of positive integers. The subdivision of I^n with small n-cubes $c_{(r)}, (r) = (r_1, \cdots, r_n), 1 \leqslant r_i \leqslant m_1$, where $c_{(r)}$ lies between the hyperplanes $x_i = (r_i - 1)/m_i$ and $x_i = r_i/m_i$ for $i = 1, \cdots, n$, is called the subdivision of I^n of *type* (m).

Proposition 3.5 *Let X_* be a filtered space and $\bar{\alpha} = [\bar{\alpha}_{(r)}]$ a subdivision of an element $\bar{\alpha}$ of $\varrho_n X_*$. Then there is an element β of $R_n X_*$ and a subdivision $\beta = [\beta_{(r)}]$ of β, where all $\beta_{(r)}$ lie in $R_n X_*$ such that $\bar{\beta} = \bar{\alpha}$ and $\bar{\beta}_{(r)} = \bar{\alpha}_{(r)}$ for all (r). Further, if each $\bar{\alpha}_{(r)}$ has a deficient representative, then the $\beta_{(r)}$, and hence also β, may be chosen to be deficient.*

Proof Let K be the cell complex of the subdivision of I^n of the same type as the given subdivision of $\bar{\alpha}$. Then K collapses to a vertex, so that there is a chain

$$K = A_s \searrow A_{s-1} \searrow \cdots \searrow A_1 = \{\nu\}$$

of elementary collapses, where $A_{i+1} = A_i \cup a_i$ for some cell a_i of K, and $A_i \cap a_i$ is a box in a_i.

We now work in terms of the corresponding cubical complexes $K = A_s, A_{s-1}, \ldots, A_1$, where K has unique nondegenerate elements $c_{(r)}$ of dimension n. The subdivision of $\bar{\alpha}$ determines a unique cubical map $g : K \to \varrho X_*$ such that $g(c_{(r)}) = \bar{\alpha}_{(r)}$. We construct inductively maps $f_i : A_i \to RX_*, i = 1, \cdots, s$, such that f_i extends $f_{i-1}, pf_i = g \mid A_i$, and $f_{i+1}(a_i)$ is deficient if $g(a_i)$ has a deficient representative. The induction is started by choosing $f_1(\nu)$ to be any element such that $pf_1(\nu) = g(\nu)$. The inductive step is given by Corollary 3.4.

Let $f = f_s : K \to RX_*$, and let $\beta_{(r)} = f(c_{(r)})$ for all (r). The the $\beta_{(r)}$ compose in $R_n X_*$ to give an element $\beta = [\beta_{(r)}]$ as required. $\square$

In any ω-groupoid G, an element $x \in G_n$ is *thin* if it can be written as a composite $x = [x_{(r)}]$ with each entry of the form $\varepsilon_j y$ or of the form a repeated negative of $\Gamma_j y$ [10, Definition (4.11)]. The following characterisation of thin elements of $\varrho_n X_*$ is essential for later work.

Theorem 3.6 *Let X_* be a filtered space and let $n \geqslant 1$. Then an element of $\varrho_n X_*$ is thin if and only if it has a deficient representative.*

Proof The case $n = 1$ is trivial, so we suppose $n \geqslant 2$.

First suppose that α in $R_n X_*$ is deficient. Define $\Psi_i \alpha \in R_n X_*$ by

$$\Psi_i \alpha = [-\varepsilon_i \partial_i^1 \alpha, -\Gamma_i \partial_{i+1}^0 \alpha, \alpha, \Gamma_i \partial_{i+1}^1 \alpha]_{i+1}$$

where $-$ denotes $-_{i+1}$. Let $\Psi\alpha = \Psi_1 \cdots \Psi_{n-1}\alpha$; then $\Psi\alpha$ also is deficient.

Recall that a 'folding operation' Φ is defined for any ω-groupoid, and hence also for $\varrho_n X_*$, in [10, Section 4], and that the formula for Ψ is the same as that for Φ. It follows that $p\Psi = \Phi p$, where $p : RX_* \to \varrho X_*$ is the quotient map.

Now $\partial_1^\tau \Phi p(\alpha) = \varepsilon_1^{n-1} \bar{\nu}$ for some $\bar{\nu} \in \varrho_0 X = \pi_0 X_0$, if $(\tau, j) \neq (0, 1)$ (by [10, Proposition (4.5)]). Thus if B is the box in I^n with base $\partial_1^1 I^n$, then for each $(n-1)$-cell a of $B, \Psi\alpha \mid a$ is a thin homotopic to the constant map at ν. By the deformation theorem (Theorem 3.2), $\Psi\alpha$ is thin homotopic to an element β such that $\beta(B) = \{\nu\}$, and such that β is deficient. Therefore, the homotopy of β to the constant map at ν, defined by a strong deformation retraction of I^n onto B, is a thin homotopy. Therefore $p\Psi\alpha = p\beta = 0$. So $\Phi p\alpha = 0$. By [10, (4.12)], $\bar{\alpha} = p\alpha$ is thin.

For the other implication, suppose that $\bar{\alpha}$ is thin. Then $\bar{\alpha}$ has a subdivision $\bar{\alpha} = [\bar{\alpha}_{(r)}]$ in which each $\alpha_{(r)}$ is deficient. By Proposition 3.5, $\bar{\alpha}$ has a deficient representative. $\square$

4 The union theorem for ω-groupoids

The groupoid version of the Van Kampen theorem [4, 8.4.2], gives useful results for nonconnected spaces, but still requires a 'representativity' condition in dimension 0. The union theorem of [7], which computes second relative homotopy groups, requires conditions in dimension 0 and 1. It is thus not surprising that our general union theorem requires conditions in all dimensions.

A filtered space X_* is said to be *connected* if the following conditions $\phi(X_*, m)$ hold for each $m \geqslant 0$:
$\phi(X_*, 0)$: If $j > 0$, the map $\pi_0 X_0 \to \pi_0 X_j$, induced by inclusion, is surjective.
$\phi(X_*, m)(m \geqslant 1)$: If $j > m$ and $\nu \in X_0$, then the map

$$\pi_m(X_m, X_{m-1}, \nu) \to \pi_m(X_j, X_{m-1}, \nu)$$

induced by inclusion, is surjective.

A standard example of a connected filtered space is a CW-complex X with its skeletal filtration.

Suppose for the rest of this section that X_* is a filtered space. We suppose given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\zeta \in \Lambda^n$ we set $U^\zeta = U^{\zeta_1} \cap \cdots \cap U^{\zeta_n}, U_i^\zeta = U^\zeta \cap X_i$. Then $U_0^\zeta \subseteq U_1^\zeta \subseteq \cdots$ is called the *induced filtration* U_*^ζ of U^ζ. So the homotopy ω-groupoids in the following *ϱ-diagram* of the cover are well defined:

$$\bigsqcup_{\zeta \in \Lambda^2} \varrho U_*^\zeta \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \varrho U_*^\lambda \xrightarrow{c} \varrho X_*$$

Here $\bigsqcup$ denotes disjoint union (which is the same as coproduct in the category of ω-groupoids); a, b are determined by the inclusions $a_\zeta : U^\lambda \cap U^\mu \to U^\lambda, b_\zeta : U^\lambda \cap U^\mu \to U^\mu$ for each $\zeta = (\lambda, \mu) \in \Lambda^2$; and c is determined by the inclusions $c_\lambda : U^\lambda \to X$.

Theorem B (the union theorem). *Suppose that for every finite intersection U^ζ of elements of $\mathcal{U}$, the induced filtration U_*^ζ is connected. Then*

(C) *X_* is connected;*

(I) *in the above ϱ-diagram c is the coequaliser of a, b in the category of ω-groupoids.*

Proof Suppose we are given a morphism

$$f' : \bigsqcup_{\lambda \in \Lambda} \varrho U_*^\lambda \to G \tag{4.1}$$

of ω-groupoids such that $f' \circ a = f' \circ b$. We have to show there is a unique morphism $f : \varrho X_* \to G$ of ω-groupoids such that $f \circ c = f'$.

Let i_λ be the inclusion of ϱU_*^λ into the disjoint union in (4.1). Let $p_\lambda : RU_*^\lambda \to \varrho U_*^\lambda$ be the quotient map, and let $F_\lambda = f' i_\lambda p_\lambda : RU_*^\lambda \to G$. We use these F_λ to construct $F\theta$ for certain θ in $R_n X_*$.

Suppose that θ in $R_n X_*$ is such that θ lies in some set U^λ of $\mathcal{U}$. Then θ determines uniquely an element θ^λ of $R_n U_*^\lambda$, and the rule $f' \circ a = f' \circ b$ implies that an element of G_n

$$F\theta = F_\lambda \theta^\lambda$$

is determined by θ.

Suppose given a subdivision $[\theta_{(r)}]$ of an element θ of $R_n X_*$ such that each $\theta_{(r)}$ is in $R_n X_*$ and also lies in some $U^{\lambda(r)}$ of $\mathcal{U}$. Since the composite $\theta = [\theta_{(r)}]$ is defined, it is easy to check, again using $f' \circ a = f' \circ b$, that the elements $F\theta_{(r)}$ compose in G_n to give an element $g = [F\theta_{(r)}]$ of G_n. We write this g as $F\theta$, although *a priori* it depends on the subdivision chosen.

Suppose now that α is an arbitrary element of $R_n X_*$. The construction from α of an element g in G_n and the proof that g depends only on the class of α in $\varrho_n X_*$ depend on the following lemma.

Lemma 4.2 *Let $\alpha : I^n \to X$ and let $\alpha = [\alpha_{(r)}]$ be a subdivision of α such that each $\alpha_{(r)}$ lies in some set $U^{\lambda(r)}$ of $\mathcal{U}$. Then there is a homotopy $h : \alpha \simeq \theta$ with $\theta \in R_n X_*$ such that in the subdivision $h = [h_{(r)}]$ determined by that of α, each homotopy $h_{(r)} : \alpha_{(r)} \simeq \theta_{(r)}$ satisfies:*

(i) *$h_{(r)}$ lies in $U^{\lambda(r)}$;*

(ii) *$\theta_{(r)}$ belongs to $R_n X_*$;*

(iii) *if some m-dimensional face of $\alpha_{(r)}$ lies in X_j, so also do the corresponding faces of $h_{(r)}$ and $\theta_{(r)}$;*

(iv) *if ν is a vertex of I^n and $\alpha(\nu) \in X_0$ then h is the constant homotopy on ν.*

Proof Let K be the cell-structure on I^n determined by the subdivision $\alpha = [\alpha_{(r)}]$. Let $L_m = K^m \times I \cup K \times \{0\}$. We construct maps $h_m : L_m \to X$ for $m = 0, \ldots, n$ such that h_m extends h_{m-1}, where $h_{-1} = \alpha$. Further we construct h_m to satisfy the following conditions, for each m-cell σ of K:

(i)$_m$ if σ is contained in the domain of $\alpha_{(r)}$, then $h_m(\sigma \times I) \subseteq U^{\lambda(r)}$;

(ii)$_m$ $h_m \mid \sigma \times \{1\}$ is an element of $R_m(X_*)$;

(iii)$_m$ if α maps σ into X_j, then $h_m(\sigma \times I) \subseteq X_j$;

(iv)$_m$ if $\alpha|\sigma : \sigma \to X$ is a filtered map, then h is constant on σ.

For an m-cell σ of K, let j be the smallest integer such that α maps σ into X_j. Let U^σ be the intersection of all the sets $U^{\lambda(s)}$ such that σ is contained in the domain of $\alpha_{(s)}$.

Let $h_m|K \times 0$ be given by α, and for those cells σ of K such that $\alpha|\sigma$ is filtered, let h_m be the constant homotopy on $\sigma \times I$.

Let σ be a 0-cell of K. If $\alpha(\sigma)$ does not lie in X_0, then by $\phi(U^\sigma_*, 0)$ we can define h_0 on $\sigma \times I$ to be a path in U^σ joining σ to a point of X_0.

Let $m \geqslant 1$. The construction of h_m from h_{m-1} is as follows on those m-cells σ such that $\alpha|\sigma$ is not filtered. If $j \leqslant m$, then h_{m-1} can be extended to h_m on $\sigma \times I$ by means of a retraction $\alpha \times I \to \sigma \times \{0\} \cup \boldsymbol{\partial} \sigma \times I$. If $j > m$ the restriction of h_{m-1} to the pair $(\sigma \times \{0\} \cup \boldsymbol{\partial} \sigma \times I, \boldsymbol{\partial} \sigma \times I)$ determines an element of $\pi_m(U^\sigma_j, U^\sigma_{m-1})$. By $\phi(X_*, m)$, h_{m-1} extends to h_m on $\sigma \times I$ mapping into U^σ_j and such that $\sigma \times \{1\}$ is mapped into U^σ_m. $\square$

Corollary 4.3 *Let $\alpha \in R_n X_*$. Then there is a thin homotopy rel vertices $h : \alpha \equiv \theta$ such that $F\theta$ is defined in G_n.*

Proof Choose a subdivision $\alpha = [\alpha_{(r)}]$ such that $\alpha_{(r)}$ lies in some set $U^{\lambda(r)}$ of $\mathcal{U}$. Lemma 4.2 gives a thin homotopy $h : \alpha \equiv \theta$ and subdivision $\theta = [\theta_{(r)}]$ as required. □

We will show in Lemma 4.5 below that this element $F\theta$ depends only on the class of α in $\varrho_n X_*$.

Proof of (C)

We can now prove that X_* is connected.

The condition $\phi(X_*, 0)$ is clear since each point of X_j belongs to some U^λ and so may be joined in U^λ to a point of X_0.

Let $J^{m-1} = I \times \boldsymbol{\partial} I^{m-1} \cup \{1\} \times I^{m-1}$. Let $j > m > 0$, $\nu \in X_0$ and let $\bar{\alpha} \in \pi_m(X_j, X_{m-1}, \nu)$, so that $\alpha : (I^m, \{0\} \times I^{m-1}, J^{m-1}) \to (X_j, X_{m-1}, \nu)$. By Lemma 4.2, α is deformable as a map of triples into X_m.

This proves X_* is connected.

Remark. Up to this stage, our proof of the union theorem is very like the proof for the 2-dimensional case given in [7]. We now diverge from that proof for two reasons. First, the form of the homotopy addition lemma given in [7] is not so easily stated in higher dimensions. So we employ thin elements, since these are elements with 'commuting boundary'. Second, we can now arrange that the proof is nearer in structure to the 1-dimensional case, for example the proof of the Van Kampen theorem given in [11].

Two facts about ω-groupoids which made the proof work are that composites of thin elements are thin, and the following proposition.

Proposition 4.4 *Let G be an ω-groupoid and x a thin element of G_{n+1}. Suppose that for $m = 1, \cdots, n$ and each face operator $d : G_{n+1} \to G_m$ not involving[2] ∂^0_{n+1} or ∂^1_{n+1}, the element dx is thin. Then $x = \varepsilon_{n+1}\partial^0_{n+1}x$ and hence*

$$\partial^0_{n+1}x = \partial^1_{n+1}x.$$

Proof The proof is by induction on n, the case $n = 0$ being trivial since a thin element in G_1 is degenerate.

The inductive assumption thus implies that every face $\partial^\tau_i x$ with $i \neq n+1$ is of the form $\varepsilon_n \partial^0_n \partial^\tau_i x$. So the box consisting of all faces of x except $\partial^1_{n+1}x$ is filled not only by x but also by $\varepsilon_{n+1}\partial^0_{n+1}x$. Since a box in G has a unique thin filler [10, Proposition (7.2)], it follows that $x = \varepsilon_{n+1}\partial^0_{n+1}x$ □

Suppose now that $h' : \alpha \equiv \alpha'$ is a thin homotopy between elements of $R_n X_*$, and $h : \alpha \equiv \theta, h" : \alpha' \equiv \theta'$ are thin homotopies constructed as in Corollary 4.3, so that $F\theta, F\theta'$ are defined. From the given thin homotopies we can obtain a thin homotopy $H : \theta \equiv \theta'$. So to prove $F\theta = F\theta'$ it is sufficient

[2] A cubical face operator d is simply a product of various ∂^τ_js. This product may be empty, so that we allow $d = 1$. We say d does not involve ∂^τ_{n+1}, if d cannot be written as $d'\partial^\tau_{n+1}$.

to prove the following key lemma. In fact, the previous machinery has been developed in order to give expression to this proof.

Lemma 4.5 *Let $\theta, \theta' \in R_n X_*$ and let $H : \theta \equiv \theta'$ be a thin homotopy. Suppose $\theta = [\theta_{(r)}], \theta' = \theta'_{(s)}]$ are subdivisions into elements of $R_n X_*$ each of which lies in some set of $\mathcal{U}$. Then in G_n*

$$[F\theta_{(r)}] = [F\theta'_{(s)}].$$

Proof Suppose $\theta_{(r)}$ lies in $U^{\lambda(r)} \in \mathcal{U}, \theta'_{(s)}$ lies in $U^{\lambda'(s)} \in \mathcal{U}$, for all $(r), (s)$. Now $\theta = H(-,0) = \partial^0_{n+1} H, \theta' = H(-,1) = \partial^1_{n+1} H$. We choose a subdivision $H = [H_{(t)}]$ such that each $H_{(t)}$ lies in some set $V^{(t)}$ of $\mathcal{U}$ and so that on $\partial^0_{n+1} H$ and $\partial^1_{n+1} H$ it induces refinements of the given subdivisions of θ and θ' respectively. Further, this subdivision can be chosen fine enough so that $\partial^0_{n+1} H_{(t)}$, if it is a part of $\theta_{(r)}$, lies in $U^{\lambda'(s)}$. So we can and do choose $V^{(t)} = U^{\lambda(r)}$ in the first instance, $V^{(t)} = U^{\lambda'(s)}$ in the second instance (and avoid both cases holding together by choosing, if necessary, a finer subdivision).

We now apply Lemma 4.2 with the substitution of $n+1$ for n, H for α, K for θ, and (t) for (r), to obtain in $R_{n+2} X_*$ a thin homotopy $h : H \equiv K$ such that in the subdivision $h = [h_{(t)}]$ determined by that of H, each homotopy $h_{(t)} : H_{(t)} \simeq K_{(t)}$ satisfies

(i) $h_{(t)}$ lies in $V^{(t)}$

(ii) $K_{(t)}$ belongs to $R_{n+1} X_*$,

(iii) if some m-dimensional face of $H_{(t)}$ lies in X_j, so also do the corresponding faces of $h_{(t)}$ and $K_{(t)}$.

Now $k = \partial^0_{n+1} h, k' = \partial^1_{n+1} h$ are thin homotopies $k : \theta \equiv \phi, k' : \theta' \equiv \phi'$, say. Further, the previous choices ensure that in the subdivision $k = [k_{(r)}]$ induced by that of $\theta, k_{(r)}$ is a thin homotopy $\theta_{(r)} \equiv \phi_{(r)}$ (by (iii)) and lies in $U^{\lambda(r)}$ (by (i)). It follows that $F\theta_{(r)} = F\phi_{(r)}$ in G_n and hence $F\theta = F\phi$. Similarly $F\theta' = F\phi'$, so it is sufficient to prove $F\phi = F\phi'$.

We have a thin homotopy $K : \phi \equiv \phi'$ and a subdivision $K = [K_{(t)}]$ such that each $K_{(t)}$ belongs to $R_{n+1} X_*$ and lies in some $V^{(t)}$ of $\mathcal{U}$. Thus $FK = [FK_{(t)}]$ is defined in G_{n+1}. Further, the induced subdivisions of $\partial^0_{n+1} FK, \partial^1_{n+1} FK$ refine the subdivisions $[F\phi_{(r)}], [F\phi'_{(s)}]$ respectively. Hence $\partial^0_{n+1} FK = F\phi, \partial^1_{n+1} FK = F\phi'$, and it is sufficient to prove $\partial^0_{n+1} FK = \partial^1_{n+1} FK$. For this we apply Proposition 4.4.

Let d be a face operator from dimension $n+1$ to dimension m, and not involving ∂^0_{n+1} or ∂^1_{n+1}. Let $\sigma = d(H), \tau = d(K)$. Then σ is deficient (since H is a filter homotopy) and so by the choice of h in accordance with (iii), τ is deficient. In the subdivision $\tau = [\tau_{(u)}]$ induced by the subdivision $K = [K_{(t)}], \tau_{(u)} \in R_m X_*$ and is deficient. By Theorem 3.6, the $F\tau_{(u)} \in G_m$ are thin, and hence their composite $F\tau \in G_m$ is thin. But $FK = [FK_{(t)}]$ has, by its construction, the property that $dFK = F\tau$. So dFK is thin. By Proposition 4.4, $\partial^0_{n+1} FK = \partial^1_{n+1} FK$ □

This completes the proof that there is a well-defined function $f : \varrho_n X_* \to G_n$ given by $f(\bar{\alpha}) = F(\theta)$, where θ is constructed as in Corollary 4.3. These maps $f : \varrho_n X_* \to G_n, n \geqslant 0$, determine a morphism $f : \varrho X_* \to G$ of ω-groupoids. By its construction, f satisfies $f \circ c = f'$ and is the only such morphism. Thus the proof of Theorem B is complete. □

Remark. There is a simplicial version Theorem $B^{\triangle}$ of Theorem B. The statement of Theorem $B^{\triangle}$ is as for Theorem B but with ϱX_* replaced by $\varrho^{\triangle} X_*$, say, which denotes the simplicial homotopy T-complex of the filtered space X_*, as defined and constructed in [2]. However, the proof of Theorem $B^{\triangle}$ involves noting that we have equivalences of categories

$$(\text{simplicial}T\text{-complexes}) \left(\xrightarrow{N}\right) (\text{crossed complexes}) \left(\xleftarrow{\lambda}\right) (\omega\text{-groupoids})$$

of which the first is given in [2] and the second in [8, 10]. Further it is proved in [2] that $N\varrho^{\triangle}X_* = \pi X_*$, and we prove in Section 5, as announced in [9], that $\lambda\varrho X_* = \Pi X_*$. Thus Theorem $B^{\triangle}$ follows from these facts and Theorem B, and at the time of writing no other proof of Theorem $B^{\triangle}$ is known.

5 The union theorem for crossed complexes

In order to interpret the union theorem (Theorem B), we relate the ω-groupoid ϱX_* to familiar structures in homotopy theory.

For a filtered space X_* the fundamental groupoid $C_1 = \pi_1(X_1, X_0)$ is defined as the set of homotopy classes of maps $(I, \boldsymbol{\partial} I) \to (X_1, X_0)$. For $n \geqslant 2$ and $\nu \in X_0$, let $C_n(\nu) = \pi_n(X_n, X_{n-1}, \nu)$, the usual relative homotopy group at ν of $(X_n, X_n - 1)$. There are boundary maps $\delta : C_n(\nu) \to C_{n-1}(\nu)(n \geqslant 2$, where $C_1(\nu) = \pi_1(X_1, \nu))$ and an operation of the groupoid C_1 on C_n so that the family C of all the C_n has the structure of crossed complex over a groupoid as given in [10]. This crossed complex, written ΠX_*, is the basic example of such a structure. We call ΠX_* the *fundamental crossed complex* of the filtered space X_*. (It is sometimes called the *fundamental crossed complex* of X_*.)

In [10] we defined a functor $\lambda : \mathcal{G} \to \mathcal{C}$ from the category of ω-groupoids to the category of crossed complexes, such that if G is an ω-groupoid and $D = \gamma G$, then $D_0 = G_0, D_1 = G_1$, and for $n \geqslant 2, D_n(\nu) = \{x \in G_n : \partial_i^{\tau} x = \varepsilon_1^{n-1}\nu, \text{ all } (\tau, i) \neq (0,1)\}$, where $\nu \in G_0$.

Theorem 5.1 *If X_* is a filtered space then $\gamma\varrho X_*$ is naturally isomorphic to ΠX_*.*

Proof Let $C = \Pi X_*$ and $D = \gamma\varrho X_*$. Then by definition and the fact that $\pi_0 X_0 = X_0$, we have $C_0 = D_0, C_1 = D_1$.

Let $n \geqslant 2$, and $\nu \in C_0$. We construct an isomorphism $\theta_n : C_n(\nu) \to D_n(\nu)$. The elements of $C_n(\nu)$ are homotopy classes of maps of triples $\alpha : (I^n, \partial_1^0 I^n, B) \to (X_n, X_{n-1}, \nu)$, where B is the box in I^n with base $\partial_1^1 I^n$. Such a map α defines a filtered map $\theta'\alpha : I^n \to X_*$ with the same values as α, and $\theta'\alpha$ is constant on B. If α is homotopic to β (as maps of triples), then $\theta'\alpha$ is thin homotopic to $\theta'\beta$, and so θ' induces a map $\theta_n : C_n(\nu) \to D_n(\nu)$. But addition in the relative homotopy group $C_n(\nu)$ is defined using any $+_i, i \geqslant 2$. So θ_n is a morphism of groups.

Suppose α represents in $C_n(\nu)$ an element mapped to 0 by θ_n. Then $\theta'\alpha$ is thin homotopic to $\nu*$, the constant map at ν. But $\alpha \mid B$ is constant. By Proposition 1.4 and since B collapses to a vertex (by Lemma 3.3), the constant thin homotopy $\theta'\alpha \mid B \equiv \nu* \mid B$ extends to a thin homotopy $\theta'\alpha \equiv \nu*$. This thin homotopy defines a homotopy $\alpha \simeq \nu*$. So θ_n is injective. (This proof is due to N. Ashley.)

We now prove θ_n surjective. Let $\bar{\gamma} \in D_n(\nu)$. Then for each $(n-1)$-face a of $B, \gamma \mid a$ is thin homotopic to $\tilde{\nu} \mid a$ (where $\tilde{\nu}$ is the constant map $B \to X_*$ at ν). By the deformation theorem (Theorem 3.2), γ is thin homotopic to a map $\gamma' : I^n \to X_*$ extending $\tilde{\nu}$. Hence θ_n is surjective.

We thus have an isomorphism $\theta : C \to D$ of graded groupoids which also preserves the boundary maps δ. To complete the proof, we show that θ preserves the action of C_1 on C.

Let α represent an element of $C_n(\nu)$, and let ξ represent an element of $C_1(\nu, w)$. A standard method of constructing $\beta = \alpha^\xi$ representing an element of $C_n(w)$ is to use the homotopy extension property as follows. Let $\xi' : B \times I \to X_*$ be $(x,t) \mapsto \xi(t)$. Then ξ' is a homotopy of $\alpha \mid B$ which extends to a homotopy $h : \alpha \simeq \beta$, and we set $\alpha^\xi = \beta$. In fact, h is constructed by extending ξ' over $\partial_1^0 I^n \times I$ using a retraction of $\partial_1^0 I^n \times I$ to its box with base $\partial_1^0 I^n \times \{0\}$, and then extending again using a retraction of $I^n \times I$ to its box with base $I^n \times \{0\}$. Thus h is a filtered map $I^{n+1} \to X_*$ with h and $\partial_i^\tau h (i \neq n+1)$ deficient; hence $\bar{h}$ and $\partial_i^\tau \bar{h} (i \neq +1)$ are thin (Theorem 3.6). Therefore the folding map $\Phi : \varrho_n X_* \to \varrho_n X_*$ [10, Section 4] vanishes on these elements [10, Proposition (4.12)] and so the homotopy addition lemma [10, (7.1)] reduces to

$$\Phi \partial_{n+1}^1 \bar{h} = (\Phi \partial_{n+1}^0 \bar{h})^{u_{n+1} \bar{h}}.$$

By [10, (4.6)], Φ is the identity on D_n, to which belong both $\partial_{n+1}^1 \bar{h} = \theta_n \bar{\beta}$ and $\partial_{n+1}^0 \bar{h} = \theta_n \bar{\alpha}$. Further $u_{n+1} \bar{h} = \bar{\xi}$. So

$$\theta_n \bar{\beta} = (\theta_n \bar{\alpha})^{\bar{\xi}}.$$

Thus θ preserves the operations.

Finally, the naturality of θ is clear. □

Since the functor $\gamma : \mathcal{G} \to \mathcal{C}$ is an equivalence of categories, we obtain immediately from Theorem B and the previous definitions the main result of this paper.

Theorem C. *Under the same assumptions as Theorem B, there is a coequaliser diagram of crossed complexes over groupoids*

$$\bigsqcup_{\zeta \in \Lambda^2} \Pi U_*^\zeta \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \Pi U_*^\lambda \xrightarrow{c} \Pi X_*$$ □

The above diagram is called the Π-*diagram* of the cover $\mathcal{U}$.

A particularly useful application of this result is to CW-complexes.

Corollary 5.2 *Let X_* be the skeletal filtered space of a CW-complex X, and let $\mathcal{X} = \{X^\lambda\}_{\lambda \in \Lambda}$ be a cover of X by subcomplexes. Then the Π-diagram*

$$\bigsqcup_{\zeta \in \Lambda^2} \Pi X_*^\zeta \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \Pi X_*^\lambda \xrightarrow{c} \Pi X_*$$

of the cover $\mathcal{X}$ is a coequaliser diagram of crossed complexes.

Proof It is well known that the skeletal filtration of any CW-complex is connected.

There is a standard method of assigning to each subcomplex Y of X a neighbourhood U_Y of Y in X and a retraction $r_Y : U_Y \to Y$ such that
(i) Y is a strong deformation retract of U_Y ;
(ii) if $Y \subseteq Z$ are subcomplexes of X, then $U_Y \subseteq U_Z$ and $r_Z \mid U_Y = r_Y$;
(iii) if $Y_1, \cdots, Y_n$ are subcomplexes of X, then $U_{Y_1 \cap \cdots \cap Y_n} = U_{Y_1} \cap \cdots \cap U_{Y_n}$.
The method of constructing the U_Y, r_Y is by induction on the dimension of the cells of $X \setminus Y$ which meet Y.

We now set $U^\lambda = U_{X^\lambda}, \lambda \in \Lambda$. Then $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ is a family whose interiors cover X and for which the induced filtration U^ς of each finite intersection of its elements is connected. However, the map induced by inclusion of the Π-diagram of $\mathcal{X}$ to the Π-diagram of $\mathcal{U}$ is an isomorphism (by Proposition 2.2). Since the Π-diagram of $\mathcal{U}$ is a coequaliser, by Theorem C, so also is the Π-diagram of $\mathcal{X}$ $\square$

As in [7, Section 3], we can also obtain results for adjunction spaces in the form of push-outs, rather then coequalisers, of crossed complexes.

Theorem D. *Suppose that the commutative diagram of filtered spaces*

$$\begin{tikzcd} W_* \arrow[r, "f"] \arrow[d, "i"'] & U_* \arrow[d, "\bar{\imath}"] \\ V_* \arrow[r, "\bar{f}"'] & X_* \end{tikzcd}$$

satisfies one of the following hypotheses:

Hypothesis $\mathcal{A}$: *The maps $i, f, \bar{\imath}, \bar{f}$ are inclusions of subspaces; $W = U \cap V$; X is the union of the interiors of the sets U, V; and $W_n = W \cap X_n, V_n = V \cap X_n, U_n = U \cap X_n, n \geqslant 0$.*

Hypothesis $\mathcal{B}$: *For $n \geqslant 0$, the maps $i_n : W_n \to V_n$ are closed cofibrations, $W_n = W \cap V_n$, and X_n is the adjunction space $U_n \cup_{f_n} V_n$.*

Suppose also that the filtrations U_*, V_*, W_* are connected. Then the induced diagram

$$\begin{tikzcd} \Pi W_* \arrow[r] \arrow[d] & \Pi U_* \arrow[d] \\ \Pi V_* \arrow[r] & \Pi X_* \end{tikzcd}$$

is a pushout of crossed complexes.

Proof This is a deduction of standard kind from Theorem C. $\square$

We end this section with a useful condition for a filtered space to be connected.

Proposition 5.3 *A filtered space X_* is connected if and only if for all $n > 0$ the induced map $\pi_0 X_0 \to \pi_0 X_n$ is surjective and for all $r > n > 0$ and $\nu \in X_0, \pi_n(X_r, X_n, \nu) = 0$.*

Proof Let $r > n > 0$. Part of the homotopy exact sequence of the triple (X_r, X_n, X_{n-1}) based at $\nu \in X_0$ is

$$\cdots \to \pi_n(X_n, X_{n-1}, \nu) \xrightarrow{i^r_n} \pi_n(X_r, X_{n-1}, \nu) \xrightarrow{j^r_n} \pi_n(X_r, X_n, \nu) \qquad (*)$$

(where for $n = 1$ this is an exact sequence of based sets). Hence $\pi_n(X_r, X_n, \nu) = 0$ implies i^r_n surjective, as required for connectedness.

Suppose conversely that X_* is connected. Then $\pi_0 X_0 \to \pi_0 X_n$ is surjective for $n > 0$. Let $r > 1$. Then i^r_1 is surjective and so $j^r_1 = 0$. But if $\lambda : (I, 0, 1) \to (X_r, X_1, \nu)$ is a map, then by choosing a path joining $\lambda(0)$ to a point of X_0 we may deform λ to a path μ with $\mu(0) \in X_0$. Hence j^r_1 is surjective, and so $\pi_1(X_r, X_1, \nu) = 0$ for $r > 1$.

If $r > n > 1$, the exact sequence $(*)$ may be extended to the right by

$$\delta^r_n : \pi_n(X_r, X_n, \nu) \to \pi_{n-1}(X_n, X_{n-1}, \nu).$$

So i^r_n surjective implies δ^r_n (this is still true for $n = 2$ since we have the rule $\delta^r_2 a = \delta^r_2 b$ if and only if $ab^{-1} \in \mathrm{Im}\ j^r_2$ (see [5]). Hence the composite

$$\delta^3_2 \delta^4_3 \ldots \delta^n_{n-1} : \pi_n(X_r, X_n, \nu) \to \pi_1(X_2, X_1, \nu)$$

is injective. Therefore $\pi_n(X_r, X_n, \nu) = 0$. □

6 Colimits of crossed complexes

The usefulness of Theorems C and D depends on the ability to describe colimits in the category $\mathcal{C}$ of crossed complexes in more familiar terms. To this end, we first show that the determination of colimits in $\mathcal{C}$ can be reduced to the determination of colimits in (i) the category $\mathcal{CU}$ of crossed modules (over groupoids), and (ii) the category $\mathcal{U}$ of modules (over groupoids). In the special cases of modules or crossed modules over groups, these colimits are relatively easy to describe; and even in the very special cases of induced modules or induced crossed modules over groups they have applications which give some classical theorems of algebraic topology, as we see in Section 7.

For $n \geqslant 0$, let $\mathcal{C}_n$ denote the category of *n-truncated crossed complexes* in which all structure above dimension n is ignored. Then $\mathcal{C}_1$ is the category $\mathcal{G}$ of groupoids and $\mathcal{C}_2$ is the category $\mathcal{GU}$ of crossed modules over groupoids. There is a forgetful functor $tr^n : \mathcal{C} \to \mathcal{C}_n$ sending C to $(C_n, C_{n-1}, \cdots, C_0)$.

The category $\mathcal{U}$ of modules over groupoids is defined as follows. An object of $\mathcal{U}$ is a pair (M, G), where G is a groupoid with set of vertices G_0 and $M = \{M_p\}_{p \in G_0}$ is a family of abelian groups on which G acts (so that $x \in G_{(p,q)}$ induces an isomorphism $m \mapsto m^x$ from M_p to M_q). A morphism $(M, G) \to (M', G')$ in $\mathcal{U}$ is a pair (θ, ϕ), where $\phi : G \to G'$ is a morphism of groupoids and $\theta = \{\theta_p\}_{p \in G_0}$ is a family of group morphisms $\theta_p : M_p \to M'_{\phi(p)}$ satisfying $\theta_q(m^x) = (\theta_p m)^{\phi(x)}, (x \in G_{(p,q)}, m \in M_p)$.

Proposition 6.1 *Let $C = \mathrm{colim} C^\lambda$ be a colimit in the category $\mathcal{C}$ of crossed complexes. Then*

(i) *the groupoid $G = (C_1, C_0)$ is $\mathrm{colim} G^\lambda$, the colimit in $\mathcal{G}$ of the groupoids $G^\lambda = (C^\lambda_1, C^\lambda_0)$;*

(ii) *the crossed complex $tr^2 C$ (over the groupoid G of (i) is $\mathrm{colim} tr^2 C^\lambda$, the colimit in $\mathcal{CU}$ of the crossed modules $tr^2 C^\lambda$;*

(iii) *if* $n \geqslant 3$ *and* $\bar{G} = (C_1/\delta C_2, C_0), \bar{G}^\lambda = (C_1^\lambda/\delta C_2^\lambda, C_0^\lambda)$*, then the module* $(C_n, \bar{G})$ *is* $\mathrm{colim}(C_n^\lambda, \bar{G}^\lambda)$*, the colimit in the category* $\mathcal{U}$ *of modules over groupoids.*

Proof (i),(ii) These follow from the fact that, for $n \geqslant 0$, the truncation functor $tr^n : \mathcal{C} \to \mathcal{C}_n$ has a right adjoint the coskeleton functor $cosk^n : \mathcal{C}_n \to \mathcal{C}$ given by $cosk^n(A_n, A_{n-1}, \cdots, A_0) = (\cdots, 0, 0, \cdots, 0, K_n, A_n, A_{n-1}, \cdots, A_0)$, where 0 denotes the discrete groupoid over $A_0, K_0 = 0, K_1$ is the family of all vertex groups of $A_1, K_n (n \geqslant 2)$ is the kernel of $\delta : A_n \to A_{n-1}$, and the map $\delta : K_n \to A_n$ is inclusion (cf. [10, Section 5). (The truncation functor tr^n also has a left adjoint sk^n : $\mathcal{C}_n \to \mathcal{C}$, the skeleton functor, given by $sk^n(A_n, A_{n-1}, \cdots, A_0) = (\cdots, 0, 0, \cdots, 0, A_n, A_{n-1}, \cdots, A_0)$.)

(iii). In any crossed complex C , the image of C_2 under δ is a totally disconnected, normal subgroupoid of C_1 , so the quotient $C_1/\delta C_2$ is a groupoid $\bar{G}$ with vertex set C_0 . Furthermore, if $n \geqslant 3$, then δC_2 acts trivially on C_n , so C_n can be viewed as a $\bar{G}$-module. Let $F_n : \mathcal{C} \to \mathcal{U}$ be the functor sending C to the module $(C_n, G), (n \geqslant 3)$. Then F_n has a right adjoint $E_n : \mathcal{U} \to \mathcal{C}$ which sends the module (M, H) to the crossed complex $(\cdots, 0, 0, \cdots, 0, M, M, 0, \cdots, 0, H_1, H_0)$ where the two copies of M occur in dimensions $n, n+1$, and $\delta : M \to M$ is the identity. Hence F_n preserves colimits, as claimed. □

Note that, from this description of $tr^2 C$ and C_n for $n \geqslant 3$, the boundary maps $\delta : C_n \to C_{n-1}$ can be recovered as induced by the maps $\delta^\lambda : C_n^\lambda \to C_{n-1}^\lambda$, for all λ.

Colimits of groupoids are easily described by generators and relations and are as readily computed as colimits of groups (see [15, 16, 17]). Colimits in $\mathcal{U}$ and $\mathcal{C}, \mathcal{U}$ are less transparent and we analyse their structure further by the use of induced modules and induced crossed modules (over groupoids).

Given a module (M, H) and a morphism of groupoids $\alpha : H \to G$, the *induced* G*-module* $\alpha_* M$ is defined by the pushout diagram

$$\begin{tikzcd} (0,H) \arrow[r, "{(0,\alpha)}"] \arrow[d, "{(0,\mathrm{id})}"'] & (0,G) \arrow[d] \\ (M,H) \arrow[r] & (\alpha_* M, G) \end{tikzcd} \tag{6.2}$$

in $\mathcal{M}$. If $\mathcal{M}_G$ denotes the category of modules over the fixed groupoid G (with morphisms inducing the identity on G), one obtains, for each $\alpha : H \to G$, a functor $\alpha_* : \mathcal{M}_H \to \mathcal{M}_G$ which preserves colimits. Similarly, let (M, H) be a crossed module over H (where now M is non-abelian and we omit mention of the boundary map $\delta : M \to H$ as well as the action of H). For any morphism of groupoids $\alpha : H \to G$ we define the induced crossed module $\alpha_* M$ over G by the pushout diagram (6.2), but now a pushout in $\mathcal{CM}$. This gives a functor $\alpha_* : \mathcal{CM}_H \to \mathcal{CM}_G$ which also preserves colimits. More generally, we have the following.

Proposition 6.3 *Let* $(M, H) = \mathrm{colim}(M^\lambda, H^\lambda)$ *be a colimit in* $\mathcal{M}$ *(resp.* $\mathcal{CM}$ *) with canonical morphisms* $(\theta^\lambda, \alpha^\lambda) : (M^\lambda, H^\lambda) \to (M, H)$*. For each* λ*, let* $N^\lambda = \alpha_*^\lambda M^\lambda$ *be the induced* H*-module (resp. the induced crossed module over* H *). Then* $M = \mathrm{colim} N^\lambda$*, a colimit in* $\mathcal{M}_H$*(resp.* $\mathcal{CM}_H$*).* □

Propositions 6.1 and 6.3 give a recipe for computing a colimit $C = \mathrm{colim} C^\lambda$ of crossed complexes:

(i) compute the groupoid $G = (C_1, C_0)$ as $\mathrm{colim} G^\lambda$ in $\mathcal{G}$, where $G^\lambda = (C_1^\lambda, C_0^\lambda)$;

(ii) find the induced crossed G-modules $D_2^\lambda = \alpha_*^\lambda C_2^\lambda$, where $\alpha^\lambda : G^\lambda \to G$ are the canonical morphisms, and obtain C_2 as $\mathrm{colim} D_2^\lambda$ in $\mathcal{CU}_G$;

(iii) find the induced $\bar{G}$-modules $D_n^\lambda = \beta_* \alpha_*^\lambda C_n^\lambda (n \geqslant 3)$, where $\beta : G \to \bar{G} = (C_1/\delta C_2, C_0)$ is the quotient morphism, and obtain C_n as $\mathrm{colim} D_n^\lambda$ in $\mathcal{M}_{\bar{G}}$, viewing C_n as a G-module via the morphism $\beta : G \to \bar{G}$. Alternatively, C_n can be obtained from $\mathrm{colim}\alpha_*^\lambda C_n^\lambda$ in $\mathcal{M}_G$ by killing the action of δC_2.

Induced modules over groupoids afford some interesting constructions and we hope to discuss them in detail elsewhere. For the applications in Section 7 we mainly need colimits $\mathrm{colim} C^\lambda$ as above in which case C_0^λ is a singleton (i.e. each G^λ is a group), and the colimit is taken over a connected diagram. Then $G = \mathrm{colim} G^\lambda$ is also a group and this colimit may be taken in the category of groups. Thus C itself is the colimit $\mathrm{colim} C^\lambda$ in the category of crossed complexes over groups and C can be completely described in terms of (a) colimits of groups, induced modules over groups, and colimits of modules over groups, and colimits of crossed modules over a fixed group, all of which are familiar construction; and (b) induced crossed modules over groups, and colimits of crossed modules over a fixed group. A presentation for induced crossed modules was given in Proposition 8 of [7], and a presentation for pushouts of crossed modules over a fixed group G was given in Proposition 11 of [7]. The extension of the latter to colimits $M = \mathrm{colim} M^\lambda$ in $\mathcal{CM}_G$ is easy: let B be the colimit of the M^λ in the category of groups, equipped with the induced morphism $\partial : B \to G$ and the induced action of G ; then $M = B/S$ where S is the normal closure in B of the elements $b^{-1}c^{-1}bc^{\partial b}$ for $b, c \in B$, and the boundary map $M \to G$ is induced by ∂.

Note also that in this easy we obtain a description of the coproduct $C = *_\lambda C^\lambda$ in the category of crossed complexes over groups; we call C the *free product* of crossed complexes over groups.

7 Application and examples

We illustrate the use of Theorem D and Section 6 for determining relative homotopy groups in some cases in which the computations are straightforward.

A filtered space X_* is *based* if X_0 consists of a single point; the element of X_0 is taken as base point of each $X_n, n \geqslant 0$, and the relative homotopy groups of X_* are abbreviated to $\pi_n(X_n, X_{n-1})$. The base point in X_0 is *nondegenerate* if each inclusion $X_0 \to X_n, n \geqslant 1$, is a closed cofibration.

Theorem 7.1 *Let $X_*^\lambda, \lambda \in \Lambda$, be a family of based, filtered spaces each with non-degenerate base-point. Let $X_* = \bigvee_\lambda X_*^\lambda$ be the wedge of all the X_*^λ , with filtration $X_n = \bigvee_\lambda X_n^\lambda$. Suppose each X^λ is homotopy full. Then ΠX_* is isomorphic to $*_\lambda \Pi X_*^\lambda$, the free product of crossed complexes over groups.*

Proof Let $V_* = \bigsqcup_\lambda X_*^\lambda$ be the disjoint union of the X_*^λ and let V_* have the induced filtration $V_n = \bigsqcup_\lambda X_n^\lambda$. Let W_* be the filtered space with $W_n = \bigsqcup_\lambda X_0^\lambda$ for all $n \neq 0$. Let U_* be the filtered space with $U_n = \{*\}$ for all $n \geqslant 0$. Then we have a diagram of maps of filtered spaces as in Theorem D, and

Hypothesis $\mathcal{B}$ of that theorem is satisfied. Hence we have a pushout of crossed complexes

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ \bigsqcup_\lambda \Pi X_*^\lambda \arrow[r] & \Pi X_* \end{tikzcd}$$

where W_0, U_0 denote the crossed complexes which in dimension $n \geqslant 1$ are the discrete groupoids on $\bigsqcup_\lambda X_0^\lambda, \{*\}$ respectively. This pushout diagram determines ΠX_* as the required free product. □

The methods of Section 6 enable us to deduce from Theorem 7.1, under the given assumptions, a formula for the relative homotopy groups of a wedge. A particular example is the following.

Corollary 7.2 *Let $(V^\lambda, W^\lambda), \lambda \in \Lambda$, be a family of based pairs each with non-degenerate base-point, and let (V, W) be the based pair $\bigvee_{\lambda\in\Lambda}(V^\lambda, W^\lambda)$. Let $G = \pi_1 W = *_\lambda \pi_1 W^\lambda$, let $n \geqslant 3$, and suppose $\pi_i(V^\lambda, W^\lambda) = 0, 1 \leqslant i < n, \lambda \in \Lambda$. Then $\pi_i(V, W) = 0, 1 \leqslant i < n$, and the G-module $\pi_n(V, W)$ is the direct sum of the G-modules induced from the $\pi_1 W^\lambda$-module $\pi_n(V^\lambda, W^\lambda)$ by $\pi_1 W^\lambda \to G$. The same holds for $n = 2$ with 'module' replaced by 'crossed module', and 'direct sum' replaced by 'coproduct in $\mathcal{CU}_G$ '.*

Proof The description of $\pi_n(V, W)$ follows from Section 6 and Theorem 7.1 on taking X^λ to be the filtered space with $X_0^\lambda - *, X_1^\lambda = W^\lambda(1 \leqslant i < n), X_1^\lambda = V^\lambda(i \geqslant n)$, since the condition that X^λ be connected is then equivalent to the given connectivity condition on (V^λ, W^λ). The connectivity of (V, W) now follows since the direct sum, or coproduct of zero objects is zero. □

The following is an immediate application of Theorem D and Section 6.

Theorem 7.3 *Let V_* be a filtered space, let $W \subseteq V, X = V/W$, let $W_n = V_n \cap W, X_n = V_n/W_n, n \geqslant 0$. Assume that each $W_n \to V_n$ is a closed cofibration, and that each of W_*, V_* is connected. Then we have a pushout of crossed complexes*

$$\begin{tikzcd} \Pi W_* \arrow[r] \arrow[d, "i_*"'] & 0 \arrow[d] \\ \Pi V_* \arrow[r] & \Pi X_* \end{tikzcd} .$$

Hence if also V_* is based, and $n \geqslant 3$, then $\pi_n(X_n, X_{n-1}) = \pi_n(V_n, V_{n-1})/N$ where N is the $\pi_1 V^1$-submodule generated by $i_*\pi_n(W_n, W_{n-1})$ and all elements $u - u^a$ where $u \in \pi_n(V_n, V_{n-1}), a \in i_*\pi_i W^1$. □

Our remaining examples will all be deduced from the following application of Theorem D.

Theorem E. *Suppose that the commutative square of based spaces*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "i"'] & U \arrow[d, "\bar{\imath}"] \\ V \arrow[r, "\bar{f}"'] & X \end{tikzcd}$$

satisfies one of the two hypotheses:

Hypothesis $\mathcal{A}$: *The maps* $i, f, \bar{\imath}, \bar{f}$ *are inclusions of subspaces,* $W = U \cap V$ *and* X *is the union of the interiors of* U *and* V*.*

Hypothesis $\mathcal{B}$: *The map* i *is a closed cofibration and* X *is the adjunction space* $U \cup_f V$*.*

Suppose that U, V, W *are path-connected and* (V, W) *is* $(n-1)$*-connected. Let* $\lambda = f_* : \pi_1 W \to \pi_1 U$*. Then for* $n > 2$ *the* $\pi_1 U$*-module* $\pi_n(X, U)$ *is* $\lambda_* \pi_n(V, W)$*, the module induced from the* $\pi_1 W$*-module* $\pi_n(V, W)$ *by* λ*. The same holds for* $n = 2$ *with 'module' replaced by 'crossed module'.*

Proof Under these conditions we may take filtrations

$$X_i = \begin{cases} * & i = 0, \\ U & 1 \leqslant i < n, \\ X & n \leqslant i, \end{cases} \qquad V_i = \begin{cases} * & i = 0, \\ W & 1 \leqslant i < n, \\ V & n \leqslant i, \end{cases}$$

where $U_i = U \cap X_1$, $W_i = W \cap V_1$ in Theorem D. The associated pushout of crossed complexes gives the result. (See [7] for a discussion of the case $n = 2$). □

The following examples justify our claim in the introduction to [10] that the union theorem (Theorem B) includes as a special case a number of classical theorems of algebraic topology.

Example 1. Let A, B, U be path-connected, based spaces. Let $X = U \cup_j (CA \times B)$ where CA is the (unreduced) cone on A and f is a map $A \times B \to U$. The homotopy exact sequence of $(CA \times B, A \times B)$ gives

$$\pi_i(CA \times B, A \times B) \cong \pi_{i-1} A, i \geqslant 2, \text{ and } \pi_1(CA \times B, A \times B) = 0.$$

Suppose now that $n > 2$ and A is $(n-2)$-connected. Then $\pi_1 A = 0$. We conclude from Theorem E that (X, U) is $(n-1)$-connected and $\pi_n(X, U)$ is the $\pi_1 U$-module induced from $\pi_{n-1} A$, considered as trivial $\pi_1 B$-module, by $\lambda = f_* : \pi_1 B \to \pi_i U$. Hence $\pi_n(X, U)$ is the $\pi_1 U$-module

$$\pi_{n-1} A \otimes_{\mathbb{Z}(\pi_1 B)} \mathbb{Z}(\pi_1 U).$$

Example 2. In Example 1, let B be a point. Then $X = U \cup_f CA$ and we deduce that if A is $(n-2)$-connected then (X, U) is $(n-1)$-connected and

$$\pi_n(X, U) \cong \pi_{n-1} A \otimes \mathbb{Z}(\pi_1 U).$$

Example 3. In Example 2, let U also be a point. Then $X = SA$ the (unreduced) suspension of A, and we deduce that if A is $(n-2)$-connected then SA is $(n-1)$ - connected and

$$\pi_n SA \cong \pi_{n-1} A.$$

All this is for $n > 2$. However, as shown in [7], the case $n = 2$ of Theorem E implies that if A is path-connected then

$$\pi_2 SA \cong (\pi_1 A)^{ab},$$

while of course Van Kampen's theorem (which is itself a special case of Theorem D) implies that

$$\pi_1 S^1 = \mathbb{Z}, \qquad \pi_1 S^2 = 0$$

(the first of these equations requires the use of groupoids in these theorems). Thus Theorem D implies that S^n is $(n-1)$-connected and

$$\pi_n S^n = \mathbb{Z}, \qquad n \geqslant 1.$$

Example 4. Any space $\bar{U} = U \cup \{e^n_\alpha\}$ obtained from the path-connected space U by attaching n-cells is homotopy equivalent, rel U , to a space $X = U \cup_f CA$ where A is a wedge of $(n-1)$-spheres. Suppose $n > 2$. Then A is $(n-2)$-connected and $\pi_{n-1}A$ is a free abelian group (by Corollary 7.2). Thus Example 2 specialises to the well-known fact that $(\bar{U}, U)$ is $(n-1)$-connected and $\pi_n(\bar{U}, U)$ is the free $\pi_1 U$-module with one generator for each n-cell attached. In the case $n = 2$, the same argument shows that $\pi_2(\bar{U}, U)$ is a free crossed module (see [7, p. 211]).

Example 5. Let (V, W) be a based pair, and let $X = V \cup CW$. Suppose that (V, W) is $(n-1)$-connected $(n \geqslant 2)$, and that V, W are path connected. We determine $\pi_i X, i \leqslant n$.

To this end, let $U = CW$, let $W' = W \times [0, \frac{1}{2}] \subseteq U$ be the bottom half of the cone, and let $V' = V \cup W'$. The inclusions $(V, W) \to (V', W'), (X, *) \to (X, U)$ induce isomorphisms of all relative homotopy groups. By Theorem E, with V, W replaced by V', W' and $f : W' \to U$ equal to the inclusion, so that Hypothesis $\mathcal{A}$ applies, we deduce that (X, CW) is $(n-1)$-connected and $\pi_n(X, CW) = \lambda_* \pi_n(V', W')$. Hence X is $(n-1)$-connected and $\pi_n X = \lambda_* \pi_n(V, W)$. Since $\lambda = f_* : \pi_1 W \to \pi_1 U$, and $\pi_1 U = 0$, we deduce that $\pi_n X$ is obtained from $\pi_n(V, W)$ by killing the action of $\pi_1 W$. In the case $n = 2$, this means simply that $\pi_2 X$ is the group $\pi_2(V, W)$ made abelian.

Example 6. Continuing the previous example, the absolute Hurewicz theorem (proved here in Section 8) gives $H_1 X = 0, 0 < i < n$, and $H_n X = \pi_n X$. However, for $i > 0$

$$H_i X \cong H_i(X, CW) \cong H_i(V', W') \cong H_i(V, W).$$

So we deduce that $H_i(V, W) = 0, 0 < i < n$, and that $H_n(V, W)$ is obtained from $\pi_n(V, W)$ by killing the action of $\pi_1 W$ - this is the relative Hurewicz Theorem.

Example 7. As a final example, we note that by Proposition 5.3 and the above we have a method of constructing connected filtered spaces X_* , namely by taking X_0 to be a point and $X_{n+1} = X_n \cup_{f_n} CA_n$ where A_n is an $(n-1)$–connected space and f_n is a map $A_n \to X_n$.

Remark. C.T.C. Wall has shown us that Theorem E for $n > 2$ and when U, V, W, X are CW-complexes may be proved using covering spaces and the relative Hurewicz theorem. Curiously enough, no other proof of the case $n = 2$ of Theorem E is known, although a proof of Whitehead's theorem, that $\pi_2(U \cup \{e^2_\alpha\}, U)$ is a free crossed module, has been given by J. Ratcliffe in his Ph.D. thesis [20] using methods of covering spaces, the relative Hurewicz theorem, and a homological characterisation of free crossed modules. Whitehead's proof [21, 23] is still interesting because of its use of the fundamental group of the complement of a link obtained by using methods essentially of transversality; an exposition of this proof is given in [6].

8 Homotopy and homology

There are standard definitions of homology groups for any cubical complex, and of homotopy groups for Kan complexes (cubical complexes satisfying Kan's extension condition, that any box has a filler).

The homology groups of KX, the cubical singular complex of X, are simply the (cubical) singular homology groups of X. Also KX is a Kan complex and its homotopy groups can be easily seen to be identical with those of X.

Let X_* be a filtered space. Then RX_* is a Kan complex and ϱX_* is an ω-groupoid, and hence a Kan complex (by [10, (7.2)]). (A direct proof that ϱX_* is a Kan complex can be given using Theorem 3.2.)

The following proposition is one step towards the Hurewicz theorem.

Proposition 8.1 *Let X_* be a filtered space that the following conditions $\psi(X_*, m)$ hold for all $m \geqslant 0$:*

$\psi(X_*, 0)$: *The map $\pi_0 X_0 \to \pi_0 X$ induced by inclusion is surjective.*

$\psi(X_*, m)(m \geqslant 1)$: *For all $\nu \in X_0$, the map*

$$\pi_m(X_m, X_{m-1}, \nu) \to \pi_m(X, X_{m-1}, \nu)$$

induced by inclusion is surjective.

Then the inclusion $i : RX_* \to KX$ is a homotopy equivalence of cubical sets.

Proof There exist maps $h_m : K_m X \to K_{m+1} X,\ r_m : K_m X \to K_m X$ for $m \geqslant 0$ such that

(i) $\partial^0_{m+1} h_m = 1, \partial^1_{m+1} h_m = r_m$,

(ii) $r_m(KX) \subseteq R_m X_*$ and $h_m \mid R_m X_* = \varepsilon_{m+1}$,

(iii) $\partial^\tau_i h_m = h_{m-1} \partial^\tau_i$ for $1 \leqslant i \leqslant m$ and $\tau = 0, 1$,

(iv) $h_m \varepsilon_j = \varepsilon_j h_{m-1}$ for $1 \leqslant j \leqslant m$.

Such r_m, h_m are easily constructed by induction, starting with $h_{-1} = \emptyset$, and using $\psi(X_*, m)$ to define $h_m \alpha$ for elements α of $K_m X$ which are not degenerate and do not lie in $R_m X_*$.

These maps define a retraction $r : KX \to RX_*$ and a homotopy $h \simeq ir$ rel RX_*. □

Corollary 8.2 *If the conditions $\psi(X_*, m)$ of the proposition hold for all $m \geqslant 0$, then the inclusion $i : RX_* \to KX$ induces an isomorphism of all homology and homotopy groups.* □

Remark. That a similar inclusion (in the simplicial case) induces an equivalence of the associated chain complexes is proved by Blakers in [3]. It is used by him to prove results related to the Hurewicz theorem. For completeness, we outline a proof of the Hurewicz theorem using Corollary 8.2 and the homotopy addition lemma in the following form. Let $n \geqslant 2$, and let $\beta : (I^{n+1}, I^{n+1}_{n-1}) \to (X, \nu)$ be a map. Then each $\partial^\tau_i \beta$ represents an element β^τ_i of $\pi_n(X, \nu)$, and we have

$$\sum_{i=1}^{n+1} (-1)^i (\beta^0_i - \beta^1_i) = 0. \tag{8.3}$$

This follows from the form of the homotopy addition lemma given in [10, (7.1)], applied to the ω-groupoid ϱX_* where X_* is the filtered space with $X_i = \{\nu\}, i < n, X_i = X, i \geqslant n$.

Theorem 8.4 (The Hurewicz Theorem). *If $n \geqslant 2$ and X is an $(n-1)$-connected space, then $H_i X = 0$ for $0 < i < n$ and the Hurewicz map $\omega_n : \pi_n X \to H_n X$ is an isomorphism.*

Proof Let X_* be the filtered space defined immediately above. Then X_* satisfies $\psi(X_*, m)$ for all $m \geqslant 0$ and so $i : RX_* \to KX$ is a homotopy equivalence. But $H_i RX_* = 0$ for $0 < i < n$; hence $H_i X = H_i KX = 0$ for $0 < i < n$.

Let $C_m X_*$ denote the group of (normalised) m-chains of RX_*. Then every element of $C_n X_*$ is a cycle, and the basis elements $\alpha \in R_n X_*$ of $C_n X_*$ are maps $I^n \to X$ with $\alpha(\boldsymbol{\partial} I^n) = \{\nu\}$. So they determine elements $\tilde\alpha$ of $\pi_n(X, \nu)$, and $\alpha \mapsto \tilde\alpha$ determines a morphism $C_n X_* \to \pi_n(X, \nu)$. But by (8.3), this morphism annihilates the group of boundaries. So it induces a map $H_n X \to \pi_n(X, \nu)$ which is easily seen to be inverse to the Hurewicz map. □

We know that if X_* is a filtered space, then $p : RX_* \to \varrho X_*$ is a Kan fibration.

Theorem 8.5 *Let X_* be a filtered space, and let $\nu \in X_0$. Let F be the fibre of $p : RX_* \to \varrho X_*$ over ν. Then $\pi_n(F, \nu)$ is isomorphic to the image of the morphism*

$$i_n : \pi_n(X_{n-1}, \nu) \to \pi_n(X_n, \nu)$$

induced by inclusion.

Proof We define a map $\theta : \pi_n(F, \nu) \to \pi_n(X_n, \nu)$.

Let $\alpha \in F_n$ have all its faces at the base point ν. Then α determined $\alpha' : (I^n, I^n) \to (X_n, \nu)$ with the same values as α, and $\alpha \mapsto \alpha'$ induces θ.

If $\alpha \in F_n$, then $p\alpha = \varepsilon^n_1 \bar\nu$ in $\varrho_n X$, and so α is thin homotopic to $\bar\nu$, the constant map at ν. Suppose further that α has all its faces at the base point. Let B be the box in I^n with base $\partial^0_n I^n$. By Corollary 1.4, the constant thin homotopy $\bar\nu \mid B \equiv \alpha \mid B$ extends to a thin homotopy $h : \bar\nu \equiv \alpha$. Let $\beta = \partial^1_n h, k = \Gamma_n \beta$. Then $h +_n k$ is a thin homotopy $\bar\nu +_n \beta \simeq \alpha +_n \bar\nu$, rel $\boldsymbol{\partial} I^n$. Let $\beta' : (I^n, \boldsymbol{\partial} I^n) \to (X_{n-1}, \nu)$ be the map with the same values as β. Then $\alpha' \simeq i\beta'$. This proves Im $\theta \subseteq$ Im i_n.

Let $\alpha' : (I^n, \boldsymbol{\partial} I^n) \to (X_{n-1}, \nu)$ represent an element of $\pi_n(X_{n-1}, \nu)$. Let $\alpha : I^n_* \to X_*$ have the same values as α'. Then $\Gamma_n \alpha$ is a thin homotopy $\alpha \equiv \bar\nu$, so that $\alpha \in F_n$. Clearly $\theta\bar\alpha = i_n \alpha'$, and this proves Im $i_n \subseteq$ Im θ.

Finally, we prove θ injective. Suppose $\theta\bar{\alpha} = 0$. Then there is a homotopy $h : \alpha' \simeq \bar{\nu}$ of maps $(I^n, \boldsymbol{\partial} I^n) \to (X_n, \nu)$. Clearly $h \in R_{n+1}X_*$. However, $\Gamma_{n+1}h$ is a thin homotopy $h \equiv \bar{\nu}$. Therefore $h \in F_{n+1}$, and so $\bar{\alpha} = 0$. □

We say X_* is a J_n*-filtered space* if for $0 \leqslant i < n$ and $\nu \in X_0$, the map

$$\pi_{i+1}(X_i, \nu) \to \pi_{i+1}(X_{i+1}, \nu)$$

induced by inclusion is trivial.

Corollary 8.6 *If X_* is a J_n-filtered space, then each fibre of $p : RX_* \to \varrho X_*$ is n-connected, and the induced maps $\pi_i RX_* \to \pi_i \varrho X_*, H_i RX_* \to H_i \varrho X_*$, of homotopy and homology, are isomorphisms for $i \leqslant n$ and epimorphisms for $i = n + 1$.*

The conclusion of Corollary 8.6 as regards homology may be regarded as a version of Theorem I of [3].

Remark. Let X_* be the skeletal filtration of a CW-complex X with one vertex ν. It is proved in [23] that the group $H_n \Pi X_*$ is for $n \geqslant 2$ isomorphic to $H_n \widetilde{X}$, where $\widetilde{X}$ is the universal cover of X based at ν. Also, Theorem 8.5 shows that if F is the fibre of $p : RX_* \to \varrho X_*$ over ν then (F, ν) is isomorphic to the group $\Gamma_n X$ considered in [24]; the homotopy exact sequence of the fibration $p : RX_* \to \varrho X_*$ is in fact equivalent to Whitehead's exact sequence

$$\to \pi_{n+1}X \to H_{n+1}\widetilde{X} \to \Gamma_n X \to \pi_n X \xrightarrow{\omega_n} H_n \widetilde{X} \to \cdots$$

(all based at ν) where ω_n is the Hurewicz map. Further, the condition that X_* be a J_n-filtered space is in this case precisely the condition that X is a J_n-complex in the sense of [22], and is also by Theorem 8.5 equivalent to $p : RX_* \to \varrho X_*$ being an n-equivalence. Thus these results are related to the results of [1] which give necessary and sufficient conditions for X to be a J_n-complex.

9 The free ω-groupoid on one generator

Let G be any ω-groupoid and define G^m to be the ω-subgroupoid of G generated by all elements of dimension $\leqslant m$. Then G^m has only thin elements in dimension greater than m and is the largest such ω-groupoid. In fact,

$$G^m \cong Sk^m G = sk^m(tr^m G)$$

as described in [10, Section 5], and by abuse of language we call it the *m-skeleton of G* (not to be confused with the m-skeleton of G considered as a cubical complex). We define the *skeletal filtration* of G to be

$$G^* : G^0 \subseteq G^1 \subseteq \cdots .$$

The elements of G_n^m are the same as those of G_n for $n \leqslant m$; and for $n > m, G_n^m$ can be described inductively as the set of thin elements of G_n whose faces are in G_{n-1}^m.

Since G^m is an ω-groupoid, it is a Kan complex. Therefore if $p \in G_0$, and $0 < l < m$, the rth relative homotopy group $\pi_r(G^m, G^1, p)$ is defined for $r \geqslant 2$. So there is a crossed complex ΠG^* which in dimension $n \geqslant 2$ is the family of groups $\pi_n(G^n, G^{n-1}, p), p \in G_0$, and in dimension 1 is the groupoid $\pi_1 G^1$.

Proposition 9.1 *If G^* is the skeletal filtration of an ω-groupoid G then the crossed complex ΠG^* is naturally isomorphic to γG. Further, G^* is connected.*

Proof The elements of $\pi_n(G^n, G^{n-1}, p), p \in G_0, n \geqslant 2$, are classes of elements x of G_n such that $\partial_i^\tau x = \varepsilon_1^{n-1} p$ for $(\tau, i) \neq (0,1)$, two such elements x, y being equivalent if there is an $h \in G_{n+1}^n$ such that $\partial_{n+1}^0 h = x, \partial_{n+1}^1 h = y, \partial_i^\tau h = \varepsilon_1^n p$ for $(\tau, i) \neq (0,1)$ and $i \neq n+1$, and $\partial_1^0 h \in G_n^{n-1}$. Then h is thin, as is dh for any face operator d not involving ∂_{n+1}^0 or ∂_{n+1}^1. It follows from Proposition 4.4 that $x = y$. Thus $\pi_n(G^n, G^{n-1}, p)$ can be identified with $C_n(p) = (\gamma_n G)(p)$.

The identification of the groupoid $\pi_1 G^1$ with G_1 is simple, as is the identification of the boundary maps. The identification of the operations may be carried out in a similar manner to the proof of Theorem 5.1.

Finally, that G^* is connected follows from the fact that $G_n^r = G_n$ for $r \geqslant n$. □

The *geometric realisation* $|A|$ of a cubical complex A is defined in a manner similar to that of the simplicial case [15], using identifications involving only the face operators ∂_i^τ and degeneracy operators ε_j. Details are given in [14], where it is also proved that if X is a space and KX is the singular cubical complex of X, then the natural map $j_x : |KX| \to X$ induces an isomorphism of homotopy groups.

It is proved in [18] that if A is a Kan cubical complex, then the natural map $i_A : A \to K|A|$ induces isomorphisms of homotopy groups.[3] So if (A, B) is a pair of Kan cubical complexes, then the natural map $i : (A, B) \to (K|A|, K|B|)$ induces isomorphisms of relative homotopy groups. Since $\pi_n(KX, KY)$ may be identified with $\pi_n(X, Y)$ for any pair of spaces X, Y, it follows that we have a natural isomorphism $\pi_n(A, B, \nu) = \pi_n(|A|, |B|, \nu)$ for any Kan pair (A, B).

If G is an ω-groupoid, then $|G|$ denotes the geometric realisation of the underlying cubical complex of G.

Proposition 9.2 *Let G be an ω-groupoid, G^* its skeletal filtration, and let $X_* = |G^*|$ be the filtration of $X = |G|$ given by $X_n = |G^n|$. Then there is a natural isomorphism of ω-groupoids*

$$G \cong \varrho |G^*|.$$

Proof By the previous remarks and Proposition 9.1 we have natural isomorphisms

$$\gamma G \cong \Pi G \cong \Pi |G|.$$

The result follows since $\Pi |G| \cong \gamma \varrho |G|$ and γ is an equivalence. □

[3] More recent, and published, work on cubical theory is by Jardine, J. F., Categorical homotopy theory, Homology, Homotopy Appl., 8 (2006), 71–144.

Corollary 9.3 *If C is a crossed complex, there is a filtered space X_* such that C is isomorphic to ΠX_*.*

Proof Let G be the ω-groupoid λC (cf. [10, Section 6]) and let $X = |G|$. By Proposition 9.2, $C \cong \Pi X_*$. $\Box$

Remark 1 This result contrasts with Whitehead's example of a crossed complex C which is of dimension 5, has $\pi_1 C = Z_2$, is free in each dimension but is not isomorphic to ΠX_* for the skeletal filtration X_* of any CW-complex X see [23]).

Remark 2. Note also that when $X = |\lambda C|$, the absolute homotopy groups $\pi_n(X, \nu)$ are isomorphic to $\pi_1(C, \nu)$ for $n = 1, H_n(C, \nu)$ for $n \geqslant 2$ by Remark 2 of [10, Section 7]. Thus Corollary 9.3 generalises the construction of Eilenberg-Mac Lane spaces.

Recall from Section 5 that if X_* is a filtered space then there is a natural isomorphism of crossed complexes $\theta : \Pi X_* = \gamma\varrho X_*$; and from [10, Section 4], that there is a 'folding map' $\Phi : \varrho_n X_* \to \gamma_n \varrho X_*$.

Proposition 9.4 *Let $n \geqslant 2$ and let $c^n \in \varrho_n I^n_*$ be the class of the identity map $I^n_* \to I^n_*$. Then $\pi_n(I^n, I^n, 1)$ is isomorphic to $\mathbb{Z}$ and is generated by $\theta^{-1}\Phi c^n$.*

Proof There is an alternative definition of relative homotopy groups, namely $\pi'_n(X, Y, \nu)$ is the set of homotopy classes of maps $(I^n, I^n, 1) \to (X, Y, \nu)$, with addition induced by a map $I^n \to I^n \bigvee I^n$. An isomorphism $\xi : \pi_n(X, Y, \nu) \to \pi'_n(X, Y, \nu)$ is induced by $\alpha \mapsto \alpha'$ where (in the notation of the proof of Theorem 5.1) $\alpha : (I^n, \partial^0_1 I^n, B) \to (X, Y, \nu)$, and $\alpha' : (I^n, I^n, 1) \to (X, Y, \nu)$ has the same values as α. (Here $1 = (1, \cdots, 1)$ is the base point of I^n.)

Let $\varrho_n(I^n, 1)$ be the set of x in $\varrho_n I^n$ such that $(\partial^1_1)^n x = 1$. Then a map

$$\eta : \varrho_n(I^n, 1) \to \pi'_n(I^n, I^n, 1)$$

is induced by $\beta \mapsto \beta'$ where $\beta : I^n \to I^n$ satisfies $\beta(1) = 1$, and β' has the same values as β. Clearly $\eta\theta = \xi$.

A standard deduction from the results of Section 7 is that $\pi'_n(I^n, I^n, 1)$ is isomorphic to $\mathbb{Z}$ and is generated by α^n , the class of the identity map. Now clearly $\eta c^n = \alpha^n$. Also, it is easily checked that for any $x \in \varrho_n(I^n, 1)$ and $j = 1, \cdots, n-1$, we have $\eta\Phi_j x = \eta x$. Hence $\eta\Phi c^n = \eta c^n = \alpha^n$. The result now follows. $\Box$

From now on, we identify ΠX_* with $\theta\Pi X_* = \gamma\varrho X_*$ for any filtered space X_*.

We now describe the crossed complex ΠI^n_*. The cell complex I^n has one cell for each cubical face operator d from dimension n to $r, 0 \leqslant r \leqslant n$, and d determines a characteristic map $\tilde{d} : I^r_* \to I^n_*$ for this cell. Then $\tilde{d}$ induces $\varrho(\tilde{d}) : \varrho I^r_* \to \varrho I^n_*$ and $\varrho(\tilde{d})(c^r) = dc^n$. Since $\varrho(\tilde{d})$ is a morphism of ω-groupoids, it follows that $\varrho(\tilde{d})(\Phi c^r) = \Phi dc^n$. Hence ΠI^n_* has generators Φdc^n for each face operator d from dimension n to $r, 0 \leqslant r \leqslant n$. The boundary $\delta\Phi dc^n$ is given by the homotopy addition lemma [10, (7.1)].

Proposition 9.5 *The homotopy ω-groupoid ϱI^n_* is the free ω-groupoid on the class $c^n \in \varrho_n I^n_*$ of the identity map.*

Proof Let G be an ω-groupoid and let $x \in G_n$. We have to prove there is a unique morphism $f : \varrho I^n_* \to G$ of ω-groupoids such that $f(c^n) = x$.

By Proposition 9.2, we may assume $G = \varrho X_*$ for a suitable filtered space X_*. Then x is the class of a map $\alpha : I^n_* \to X_*$ and it is clear that $f = \varrho(\alpha) : \varrho I^n_* \to \varrho X_*$ satisfies $f(c^n) = x$. This proves the existence of f.

Suppose $g : \varrho I^n_* \to G$ is another morphism such that $g(c^n) = x$. Then $\gamma f, \gamma g : \Pi I^n \to \Pi X_*$ agree on the element $\Phi c^n \in \pi_n(I^n, I^n, 1)$ of Proposition 9.4.

However, ΠI^n_* is generated as crossed complex by the elements $\Phi d c^n \in \pi_r(I^n_r, I^n_{r-1}, 1)$ for all face operators d from dimension n to $r, 0 \leqslant r \leqslant n$. Since f, g are morphisms of ω-groupoids, $f(\Phi d c^n) = \Phi d(f c^n) = \Phi d(g c^n) = g(\Phi d c^n)$. Therefore f and g agree on ΠI^n_*. But the latter generates ϱI^n_* as ω-groupoid. So $f = g$. $\square$

Corollary 9.6 *If G is an ω-groupoid, then G_n is naturally isomorphic to $\mathcal{C}(\Pi I^n_*, \gamma G)$.*

Proof $G_n \cong \mathcal{G}(\varrho I^n_*, G) \cong \mathcal{C}(\Pi I^n_*, \gamma G)$. $\square$

Remark. This corollary gives another description of the functor $\lambda : \mathcal{C} \to \mathcal{G}$, the inverse equivalence of γ, namely that λ is naturally equivalent to $C \mapsto \mathcal{C}(\Pi I^n_*, C)$. In view of the explicit description of ΠI^n_* given above, a morphism $f : \Pi I^n_* \to C$ of crossed complexes is describable as a family $\{f(d)\}$ where d runs through all the cubical face operators from dimension n to dimension $r (0 \leqslant r \leqslant n), f(d) \in C_r$, and the elements $f(d)$ are required to satisfy the relations (cf. [10, Theorem 7.1])

$$\delta f(d) = \begin{cases} \sum_{i=1}^{r} (-1)^i \{ f(\partial^1_i d) - f(\partial^0_i d)^{f(u_i d)} \} & (r \geqslant 4), \\ -f(\partial^1_3 d) - f(\partial^0_2 d)^{f(u_2 d)} - f(\partial^1_1 d) + f(\partial^0_3 d)^{f(u_3 d)} + f(\partial^1_2 d) + f(\partial^0_1 d)^{f(u_1 d)} & (r = 3), \\ -f(\partial^1_1 d) - f(\partial^0_2 d) + f(\partial^0_1 d) + d(\partial^1_2 d) & (r = 2), \end{cases}$$

and $\delta^\tau f(d) = f(\partial^\tau_1 d)(r = 1)$. (These relations imply that $f(d) \in C_r(p)$ where $p = f(\beta d)$.)

Similar functors have been used by Blakers [3] (from crossed complexes to simplicial complexes) and Ashley [2] (from crossed complexes to simplicial T-complexes); in particular, Ashley shows that such a functor generalises a functor of Dold-Kan [19, Theorem 22.4] from chain complexes to simplicial abelian groups.

Acknowledgement

We must again thank Keith Dakin for the insights afforded by his work on T-complexes [12] and for discussions on the problems involved in analysing the structures possessed by ϱX_*. We must thank Nicholas Ashley for discussions which led to considerable improvements in some proofs (in Sections 1, 3 and 5). We are grateful to the Science Research Council for support.

References

[1] J. F. Adams, Four applications of the self-obstruction invariant, J. London Math. Soc. 31 (1956) 148-159.

[2] N. M. Ashley, T-complexes and crossed complexes, Ph.D. thesis, University of Wales (1978), Dissertationes Math. 265 (1988) 1-61. http://pages.bangor.ac.uk/~mas010/doctorates.html

[3] A. L. Blakers, Some relations between homology and homotopy groups, Ann. of math. 49 (1948) 428-461.

[4] R. Brown, Elements of modern topology (McGraw-Hill, Maidenhead, 1968). (Revised and republished as 'Topology and Groupoids', Booksurge PLC, S. Carolina, 2006.) http://pages.bangor.ac.uk/~mas010/topgpds.html

[5] R. Brown, Fibrations of groupoids, J. Algebra 15 (1970) 103-132.

[6] R. Brown, On the second relative homotopy group of an adjunction space: an exposition of a theorem of J. H. C. Whitehead, J. London Math. Soc. (2) 22 (1980) 146-152.

[7] R. Brown and P. J. Higgins, On the connection between the second relative homotopy groups of some related spaces, Proc. London Math. Soc. (3) 36 (1978) 193-212.

[8] R. Brown and P. J. Higgins, Sur les complexes croisés, ω-groupoïdes, et T-complexes, C. R. Acad. Sci. Paris, Sér. A 285 (1977) 997-999.

[9] R. Brown and P. J. Higgins, Sur les complexes croisés d'homotopie associés à quelques espaces filtrés, C. R. Acad. Sci. Paris, Sér. A 286 (1978) 91-93.

[10] R. Brown and P. J. Higgins, On the algebra of cubes, J. Pure Appl. Algebra 21 (1981) 233-260.

[11] R. H. Crowell, On the Van Kampen theorem, Pacific J. Math. 9 (1959) 43-50.

[12] M. K. Dakin, Kan complexes and multiple groupoid structures, Ph.D. thesis, University of Wales (1977). http://pages.bangor.ac.uk/~mas010/doctorates.html

[13] T. tom Dieck, K. H. Kamps and D. Puppe, Homotopie-theorie, Lecture Notes in math. 157 (Springer, Berlin, 1970).

[14] H. Federer, Lectures in algebraic topology, Brown University, Providence, R. I. (1962).

[15] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 35 (Springer, Berlin, 1967).

[16] P. J. Higgins, Presentations of groupoids, with application to groups, Proc. Cambridge Phil. Soc. 60 (1964) 7-20.

[17] P. J. Higgins, Categories and Groupoids (Van Nostrand, New York, 1971). (Available as Theory and Applications of Categories Reprint, 7 (2005).)

[18] S. Hintze, Polysets, $\square$-sets and semi-cubical sets, M.Phil. Thesis, Warwick (1973).

[19] P. J. May, Simplicial Methods in Algebraic Topology (Van Nostrand, New York, 1967).

[20] J. Ratcliffe, Free and projective crossed modules, J. London Math. Soc. (2) 22 (1980) 66-74.

[21] J. H. C. Whitehead, On adding relations to homotopy groups, Annals of Math. 42 (1941) 409-28.

[22] J. H. C. Whitehead, Combinatorial homotopy I, Bull. American Math. Soc. 55 (1949) 213-245.

[23] J. H. C. Whitehead, Combinatorial homotopy II, Bull. American Math. Soc. 55 (1949) 453-96.

[24] J. H. C. Whitehead, A certain exact sequence, Annals of Math. 52 (1950) 51-110.

Additional bibliography

[25] R. Brown, P.J. Higgins, R. Sivera, *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids*, EMS Tracts in Mathematics Vol. 15, 703 pages. (August 2011). http://pages.bangor.ac.uk/~mas010/nonab-a-t.html

[26] R. Brown and P.J. Higgins, Tensor products and homotopies for ω-groupoids and crossed complexes, J. Pure Appl. Algebra 47 (1) (1987) 1–33.

R. Brown and P.J. Higgins, The classifying space of a crossed complex. Math. Proc. Cambridge Philos. Soc. 110 (1) (1991) 95–120.