

COVERING GROUPS OF NON-CONNECTED TOPOLOGICAL GROUPS REVISITED

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Introduction

All spaces are assumed to be locally path connected and semi-locally 1-connected. Let X be a connected topological group with identity e , and let $p : \tilde{X} \rightarrow X$ be the universal cover of the underlying space of X . It follows easily from classical properties of lifting maps to covering spaces that for any point \tilde{e} in \tilde{X} with $p\tilde{e} = e$ there is a unique structure of topological group on \tilde{X} such that \tilde{e} is the identity and $p : \tilde{X} \rightarrow X$ is a morphism of groups. We say that the structure of topological group on X *lifts* to \tilde{X} .

It is less generally appreciated that this result fails for the non-connected case. The set $\pi_0 X$ of path components of X forms a non-trivial group which acts on the abelian group $\pi_1(X, e)$ via conjugation in X . R.L. Taylor [20] showed that the topological group X determines an *obstruction class* k_X in $H^3(\pi_0 X, \pi_1(X, e))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the topological group structure on X to a universal covering so that the projection is a morphism. Further, recent work, for example Huebschmann [15], shows there are geometric applications of the non-connected case.

The purpose of this paper is to prove generalisations of this result on coverings of topological groups using modern work on coverings of groupoids (see for example, Higgins [12], Brown [3]), *via* the following scheme. We first use the fact that covering spaces of space X are equivalent to covering morphisms of the fundamental groupoid $\pi_1 X$ (section 1). This extends easily to the group case: if X is a topological group, then the fundamental groupoid inherits a group structure making it what is called a *group-groupoid*, i.e. a group object in the category of groupoids; then topological group coverings of X are equivalent to group-groupoid coverings of $\pi_1 X$ (Proposition 2.3).

The next input is the equivalence between group-groupoids, and crossed modules (Brown and Spencer [10]). Here a crossed module is a morphism $\mu : M \rightarrow P$ of groups together with an action of the group P on the group M , with two axioms satisfied. It is easy to translate notions from covering morphisms of group-groupoids to corresponding notions for crossed modules (Proposition 4.2).

The existence of simply connected covering groups of a topological group now translates to the existence of extensions of groups of “the type of a given crossed module” (Definition 5.1). This generalisation of the classical extension theory is due to Taylor [19] and Dedecker [11]. We formulate a corresponding notion of abstract kernels (Theorem 5.2), analogous to that due to Eilenberg-Mac Lane [17]. This leads to our main result, Theorem 5.4, which determines when a morphism $\theta : \Phi \rightarrow \pi_0 X$ of groups is realised by a covering morphism $p : \tilde{X} \rightarrow X$ of topological groups such that \tilde{X} is simply connected with $\pi_0 \tilde{X}$ isomorphic to Φ . We deduce that any topological group X admits a simply connected covering group covering all the components of X (Corollary 5.6). According to comments in [20], results of this type were known to Taylor.

Our proof of Theorem 5.2 uses methods of crossed complexes, as in Brown and Higgins [4]. This seems the natural setting for these results, since crossed complexes contain information on resolutions and on crossed modules. The exposition is analogous to that of Berrick [1] for the ordinary theory of extensions, in that fibrations are used, but in the algebraic context of crossed complexes. A direct account of a special case of these results, in the context of Lie groupoids, is given by Mackenzie in [16], and this account could also be adapted to the general case.

Section 6 deals with coverings other than simply connected ones.

The results of this paper formed part of Part I of Mucuk [18].

1 Groupoids and coverings

The main tool is the equivalence between covering maps of a topological space X and covering morphisms of the fundamental groupoid $\pi_1 X$ of X . Our main reference for groupoids and this result is Brown [3] but we adopt the following notations and conventions.

A topological space X is called *simply connected* if each loop in X is contractible in X , and X is called *1-connected* if it is connected and simply connected. A map $f : X \rightarrow Y$ is called *π_0 -proper* if $\pi_0(f)$ is a bijection.

If X is a topological space, the category $TCov/X$ of covering spaces of X is the full subcategory of the slice category Top/X of spaces over X in which the objects are the covering maps. It is standard that if $h : Y \rightarrow Z$ is a map in $TCov/X$, i.e. is a map over X , then h is a covering map. Further, if $f : Y \rightarrow X$ is a covering map such that Y is simply connected, then for any other cover $g : Z \rightarrow X$, there is a covering map $h : Y \rightarrow Z$ over X . This is summarised by saying that Y covers any other cover of X , and a covering map with this property is called *universal*. A necessary and sufficient condition for this is that Y be simply-connected.

For a groupoid G , we write O_G for the set of objects of G , and G for the set of arrows, or elements. We write $s, t : G \rightarrow O_G$ for the source and target maps. The product $g \circ h$ is defined if and only if $tg = sh$. The identity at $x \in O_G$ is written 1_x . The inverse of an element g is written g^{-1} .

The category of groupoids and morphisms of groupoids is written Gd .

For $x \in O_G$ we denote the *star* $\{g \in G \mid sg = x\}$ of x by G^x , and the *costar* $\{g \in G \mid tg = x\}$ of x by G_x , and write G_y^x for $G^x \cap G_y$. The object group at x is $G(x) = G_x^x$. An element of some G_x^x is called a *loop* of G .

We say G is *transitive* (resp. *1-transitive*, *simply transitive*) if for all $x, y \in O_G$, $G(x, y)$ is non-empty (resp. is a singleton, has not more than one element). The transitive component of an object x of G is the largest transitive subgroupoid of G with x as an object, and is written $C(G, x)$. The set

of transitive components of G is written $\pi_0 G$. A morphism p of groupoids is called π_0 -proper if $\pi_0(p)$ is a bijection.

Covering morphisms and universal covering groupoids of a groupoid are defined in Brown [2] (see also Higgins [12], Brown [3]) as follows:

Let $p: H \rightarrow G$ be a morphism of groupoids. Then p is called a *covering morphism* if for each $x \in O_H$, the restriction $G^x \rightarrow G^{px}$ of p is bijective. The covering morphism p is called *regular* if for all objects x of G and all $g \in G(x)$ the elements of $p^{-1}(g)$ are all or none of them loops. This is equivalent to the condition that for all objects y of H , the subgroup $pH(y)$ of $G(py)$ is a normal subgroup [3].

If G is a groupoid, the category $GdCov/G$ of coverings of G is the full subcategory of the slice category Gd/G of groupoids over G in which the objects are the covering morphisms.

A covering morphism $p: H \rightarrow G$ is called *universal* if H covers every covering of G , i.e. if for every covering morphism $a: A \rightarrow G$ there is a morphism of groupoids $a': H \rightarrow A$ such that $aa' = p$ (and hence a' is also a covering morphism). It is common to consider universal covering morphisms which are π_0 -proper.

We recall the following standard result (Brown [3], Chapter 9), which summarises the theory of covering spaces.

Proposition 1.1 *For any space X , the fundamental groupoid functor defines an equivalence of categories*

$$\pi_1 : TCov/X \rightarrow GdCov/(\pi_1 X).$$

One crucial step in the proof of this equivalence is the result (Brown [3], 9.5.5) that if $q: H \rightarrow \pi_1 X$ is a covering morphism of groupoids, then there is a topology on O_H such that $O_q: O_H \rightarrow X$ is a covering map, and there is an isomorphism $\alpha: \pi_1 O_H \rightarrow H$ such that $q\alpha = \pi_1(O_q)$. This result, which translates the usual covering space theory into a more base-point free context, yields the inverse equivalence.

We also remark that the universal cover of X at $x \in X$ is given by the target map $(\pi_1 X)^x \rightarrow X$ with the subspace topology from a topology on $\pi_1 X$.

Recall that an *action* of a groupoid G on sets *via* w consists of a function $w: A \rightarrow O_G$, where A is a set, and an assignment to each $g \in G(x, y)$ of a function $g_\#: w^{-1}(x) \rightarrow w^{-1}(y)$, written $a \mapsto a \circ g$, satisfying the usual rules for an action, namely $a \circ 1 = a$, $a \circ (g \circ h) = (a \circ g) \circ h$ when defined. A *morphism* $f: (A, w) \rightarrow (A', w')$ of such actions is a function $f: A \rightarrow A'$ such that $w'f = w$ and $f(a \circ g) = (fa) \circ g$ whenever $a \circ g$ is defined. This gives a category $Act(G)$ of actions of G on sets. For such an action, the *action groupoid* $A \rtimes G$ is defined to have object set A , arrows the pairs (a, g) such that $w(a) = sg$, source and target maps $s(a, g) = a$, $t(a, g) = a \circ g$, and composition

$$(a, g) \circ (b, h) = (a, g \circ h)$$

whenever $b = a \circ g$. The projection $q: A \rtimes G \rightarrow G$, $(a, g) \mapsto g$, is a covering morphism of groupoids, and the functor sending an action to this covering morphism gives an equivalence of categories $Act(G) \rightarrow GdCov/G$. (See for example Brown [3].)

Let x be an object of the transitive groupoid G , and let $N(x)$ be a subgroup of the object group $G(x)$. Then G acts on the set A of cosets $N(x) \circ g$ for $g \in G^x$, *via* the map $N(x) \circ g \mapsto tg$. So we can

form the corresponding covering morphism $p: H \rightarrow G$, where $H = A \times G$, and the object $\tilde{x} = N(x)$ of H satisfies $p(H(\tilde{x})) = N(x)$. This construction yields an equivalence of categories between the lattice $\mathcal{L}G(x)$ of subgroups of $G(x)$ and the category of pointed transitive coverings of G, x .

Suppose further that $a \in G_y^x$, $N(y) = a^{-1} \circ N(x) \circ a$, and $q: K \rightarrow G$ is the covering of G determined as above by $N(y)$, with $\tilde{y} \in O_K$ satisfying $q[K(\tilde{y})] = N(y)$. Then there is a unique isomorphism $h: H \rightarrow K$ such that $qh = p$ and $h\tilde{x} = \tilde{y}$. That is, conjugate subgroups of a transitive groupoid G determine isomorphic coverings, and we obtain an equivalence of categories between the lattice of conjugacy classes of subgroups of G and the isomorphism classes of transitive coverings of G .

If G is not transitive then π_0 -proper coverings may be constructed by working on each transitive component. We choose a transversal for the set $I = \pi_0 G$ of components of G , i.e. an object τ_i for each component G_i of G , choose a subgroup $N(\tau_i) \subseteq G(\tau_i)$, and get a covering $\tilde{G}_i \rightarrow G_i$ for each component G_i of G . The disjoint union of these coverings is a covering $p: \tilde{G} \rightarrow G$, which is universal if and only if all the $N(\tau_i)$ are trivial groups.

2 Group-groupoids and covering morphisms

The notion of group-groupoid, and the first parts of Propositions 2.1 and 2.3 below are taken from Brown-Spencer [10], although the term used there is \mathcal{G} -groupoid.

By a *group-groupoid* we mean a groupoid G with a morphism of groupoids $G \times G \rightarrow G$, $(g, h) \mapsto gh$, yielding a group structure internal to the category of groupoids. Since the multiplication is a morphism of groupoids, we obtain the *interchange law*, that $(a \circ g)(b \circ h) = (ab) \circ (gh)$, for all $g, h, a, b \in G$ such that $a \circ g$ and $b \circ h$ are defined. If the identity for the group structure on O_G is written e , then 1_e is the identity for the group structure on the arrows. The group inverse of an arrow g is written \bar{g} . Then $g \mapsto \bar{g}$ is a morphism $G \rightarrow G$ of groupoids.

It is a standard consequence of the interchange law that the groupoid composition in a group-groupoid can be recovered from the group law, as shown in the first part of the following proposition.

Proposition 2.1 *Let G be a group-groupoid, and suppose $a \circ b$ is defined in G , where $a \in G(x, y)$. Then $a \circ b = a\bar{1}_y b$. If further $g \in G(e)$, then*

$$a \circ (1_y g) \circ a^{-1} = 1_x g,$$

and

$$ag\bar{a} = 1_x g\bar{1}_x.$$

Further, $G(e)$ is abelian.

Proof Suppose $ta = y$. Then

$$\begin{aligned} a \circ b &= ((a\bar{1}_y)1_y) \circ (1_e b) \\ &= ((a\bar{1}_y) \circ 1_e)(1_y \circ b) \\ &= a\bar{1}_y b. \end{aligned}$$

Further

$$\begin{aligned}
a \circ (1_y g) \circ a^{-1} &= a \circ ((1_y g) \circ (a^{-1} 1_e)) = a \circ ((1_y \circ a^{-1})(g \circ 1_e)) \\
&= a \circ (a^{-1} g) &&= (a 1_e) \circ (a^{-1} g) \\
&= (a \circ a^{-1})(1_e \circ g) &&= 1_x g.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \circ (1_y g) \circ a^{-1} &= (a \bar{1}_y 1_y g) \circ a^{-1} &&= (ag) \circ (1_y \bar{a} 1_x) \\
&= ag \bar{1}_y 1_y \bar{a} 1_x &&= ag \bar{a} 1_x.
\end{aligned}$$

Hence $ag\bar{a} = 1_x g \bar{1}_x$. That $a \circ g = g = g \circ a$ for $a, g \in G(e)$ is immediate. \square

Corollary 2.2 *Let $N(e)$ be a subgroup of $G(e)$, and let N be the family of subsets $N(x) = 1_x N(e)$ for all $x \in O_G$. Then $N(x)$ is a normal subgroupoid of G . In particular, all the object groups of G are isomorphic, and are abelian.*

Proof That $N(x)$ is a subgroup follows from

$$1_x(b \circ a) = (1_x \circ 1_x)(b \circ a) = (1_x b) \circ (1_x a),$$

for $b, a \in N(x)$. The normality follows from the second formula of the Proposition, on taking $g \in N(e)$. It is immediate that all the object groups are isomorphic. \square

This result implies that all coverings of a group-groupoid are regular. It also shows that a choice τ of transversal for the components of a group-groupoid G induces an equivalence between the category $\mathcal{LG}(e)$ of subgroups of $G(e)$ under inclusion and the category of isomorphism classes of π_0 -proper coverings of G .

We now consider coverings in the category of group-groupoids.

A *morphism* of group-groupoids is a morphism of the underlying groupoids which preserves the group structure. Then group-groupoids and morphisms of them form a category which we will denote by $GpGd$. Let G be a group-groupoid. Then $GpGdCov/G$ denotes the full subcategory of the slice category $GpGd/G$ whose objects are group-groupoids $p : H \rightarrow G$ over G such that p is a covering morphism of the underlying groupoids.

We can now translate Proposition 1.1 to this situation.

Proposition 2.3 *Let X be a topological group. Then the fundamental groupoid $\pi_1 X$ is a group-groupoid with group structure induced by that of X . Further, the fundamental groupoid functor π_1 gives an equivalence from the category $GpTCov/X$ to the category $GpGd/\pi_1 X$.*

Proof We show that the inverse equivalence of Proposition 1.1 determines an inverse equivalence in this case also.

Suppose then that $q : H \rightarrow \pi_1 X$ is a morphism of group-groupoids such that the underlying groupoid morphism is a covering morphism. Then there is a topology on $\tilde{X} = O_H$ and an isomorphism

$\alpha: \pi_1 \tilde{X} \rightarrow H$ such that $p = O_q: \tilde{X} \rightarrow X$ is a covering map and $q\alpha = \pi_1(p)$. The group structure on H transports via α to a morphism of groupoids

$$\tilde{m}: \pi_1 \tilde{X} \times \pi_1 \tilde{X} \rightarrow \pi_1 \tilde{X}$$

such that $\pi_1(p) \circ \tilde{m} = m \circ (\pi_1(p) \times \pi_1(p))$, where m is the group multiplication on X , and clearly \tilde{m} is a group structure on $\pi_1 \tilde{X}$. By 9.5.5 of Brown [3], \tilde{m} induces a continuous map on \tilde{X} . This gives the multiplication on \tilde{X} . The fact that this is a group structure follows from the fact that \tilde{m} is a group structure. \square

3 Actions of group-groupoids on groups

In this section we relate group-groupoid covering morphisms to a notion of action of a group-groupoid on a group. The results are a special case of results of Section 1 of Brown and Mackenzie [8], and are included here for completeness.

Let G be a group-groupoid. An *action* of the group-groupoid G on a group A *via* w consists of a morphism $w: A \rightarrow G$ from the group A to the underlying group of O_G and an action of the groupoid G on the underlying set A via w such that the following interchange law holds:

$$(a \circ g)(b \circ h) = (ab) \circ (gh)$$

whenever both sides are defined. A *morphism* $f: (A, w) \rightarrow (A', w')$ of such operations is a morphism $f: A \rightarrow A'$ of groups and of the underlying operations of G . This gives a category $GpGdAct(G)$. For an action of G on the group A via w , the action groupoid $A \rtimes G$ is defined. It inherits a group structure by

$$(a, g)(c, k) = (ac, gk).$$

It is easily checked that $A \rtimes G$ is then a group-groupoid, and the projection $p: A \rtimes G \rightarrow G$ is an object of the category $GpGd/G$. By means of this construction, one obtains the following, which is a special case of Theorem 1.7 of Brown and Mackenzie [8] which considers the case of actions of Lie double groupoids.

Proposition 3.1 *The categories $GpGdCov/G$ and $GpGdAct(G)$ are equivalent.*

4 Group-groupoids and crossed modules

A *crossed module* (M, P, μ) is defined in Whitehead [21] to consist of two groups M and P together with a homomorphism $\mu: M \rightarrow P$, and an action of P on M on the right, written $(m, p) \mapsto m^p$, such that the following conditions are satisfied:

$$\text{CM1) } \mu(mp) = p^{-1}(\mu m)p$$

$$\text{CM2) } n^{\mu m} = m^{-1}nm$$

for all $m, n \in M$ and $p \in P$.

Standard examples of crossed modules are:

- (i) the inclusion $M \rightarrow P$ of a normal subgroup,
- (ii) the zero morphism $M \rightarrow P$ when M is a P -module,
- (iii) the inner automorphism map $\chi_M : M \rightarrow \text{Aut } rM$ for any group M ,
- (iv) a morphism $M \rightarrow P$ of groups which is surjective and has central kernel,
- (v) the free crossed P -module $C(w) \rightarrow P$ arising from a function $w: R \rightarrow P$ (see Brown and Huebschmann [7]),
- (vi) the induced morphism $\pi_1(F, x) \rightarrow \pi_1(E, x)$ of fundamental groups for any fibration of spaces $F \rightarrow E \rightarrow B$.

Standard consequences of the axioms (see for example [7]) are that μM is a normal subgroup of P , that $\text{Ker } \mu$ is central in M , and that μM acts trivially on $\text{Ker } \mu$ which thereby becomes a module over $\text{Coker } \mu$.

A morphism $(f, g): (M, P, \mu) \rightarrow (N, Q, \nu)$ of crossed modules consists of group morphisms $f: M \rightarrow N$ and $g: P \rightarrow Q$ such that $g\mu = \nu f$ and f is an operator homomorphism, that is, $f(m^p) = fm^{(gp)}$ for $m \in M$ and $p \in P$. So crossed modules and morphisms of them, with the obvious composition of morphisms $(f', g')(f, g) = (f'f, g'g)$, form a category, which we write $\text{Crs}M$.

The following theorem was found by Verdier in 1965, but not published, and found independently by Brown and Spencer [10]. We give a sketch of the proof, since we need some of its detail.

Theorem 4.1 *The category GpGd of group-groupoids is equivalent to the category $\text{Crs}M$ of crossed modules. If a group-groupoid G has associated crossed module (M, P, μ) then the underlying groupoid of G is transitive (resp. simply transitive, 1-transitive) if and only if μ is an epimorphism (resp. a monomorphism, isomorphism). Further, the group $\pi_0 G$ is $\text{Coker } \mu$.*

Sketch Proof: A functor $\delta: \text{GpGd} \rightarrow \text{Crs}M$ is defined as follows. For a group-groupoid G we let $\delta(G)$ be the crossed module (M, P, μ) where P is the group O_G of objects of G ; M is the costar G_e of G at the identity e of the group O_G ; $\mu: M \rightarrow P$ is the restriction of the source map s ; the group structures on M and P are induced by that on G ; and P acts on M by $m^p = \bar{1}_p m 1_p$ for $p \in P$ and $m \in M$. The results on transitivity follow immediately.

Conversely define a functor $\beta: \text{Crs}M \rightarrow \text{GpGd}$ in the following way. For a crossed module (M, P, μ) , $\beta(M, P, \mu)$ is the group-groupoid whose object set (group) is P and whose group of arrows is the semi-direct product $P \ltimes M$ with the standard group structure

$$(p, m)(q, n) = (pq, m^q n).$$

The source and target maps s, t are defined to be $s(p, m) = p$ and $t(p, m) = p(\mu m)$, while the composition of arrows is given by

$$(p, m) \circ (q, n) = (p, mn)$$

whenever $p(\mu m) = q$. □

If X is a topological group with identity e , then the fundamental groupoid $\pi_1 X$ becomes a group-groupoid, the associated crossed module is $t: (\pi_1 X)^e \rightarrow X$ (Brown and Spencer [10]), and $(\pi_1 X)^e$ has a topology making it the universal cover based at e of the path component of e .

It is easy to obtain results for morphisms of group-groupoids corresponding to Theorem 4.1, as follows.

Proposition 4.2 *Let $f: H \rightarrow G$ be a morphism of group-groupoids and let $(f_1, f_2): (N, Q, \nu) \rightarrow (M, P, \mu)$ be the morphism of crossed modules corresponding to f as in Theorem 4.1. Then, on underlying groupoids, f is a covering morphism if and only if $f_1: N \rightarrow M$ is an isomorphism. Further, f is a universal covering morphism if and only if f_1 is an isomorphism, ν is a monomorphism, and the induced morphism $\text{Coker}\nu \rightarrow \text{Coker}\mu$ is an isomorphism.*

We therefore define a morphism (f_1, f_2) of crossed modules as in the proposition to be a *covering morphism* if f_1 is an isomorphism, and so obtain a category $\text{CrsMCov}/(M \rightarrow P)$ of coverings of $M \rightarrow P$ as a full category of the slice category $\text{CrsM}/(M \rightarrow P)$.

Corollary 4.3 *The category GTCov/X of topological group coverings of a topological group X is equivalent to the category $\text{CrsMCov}/((\pi_1 X)^e \rightarrow X)$ of crossed module coverings of $(\pi_1 X)^e \rightarrow X$.*

5 Extensions, crossed modules and cohomology

We now recall the notion of an extension of groups of the type of a crossed module, due to Taylor [19] and Dedecker [11]. See also [9].

Definition 5.1 Let \mathcal{M} denote the crossed module $\mu: M \rightarrow P$. An *extension* (i, p, σ) of type \mathcal{M} of the group M by the group Φ is first an exact sequence of groups

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} \Phi \longrightarrow 1$$

so that E operates on M by conjugation, and $i: M \rightarrow E$ is hence a crossed module. Second, there is given a morphism of crossed modules

$$\begin{array}{ccccc} 1 & \longrightarrow & M & \xrightarrow{i} & E \\ & & \parallel & & \downarrow \sigma \\ & & M & \xrightarrow{\mu} & P \end{array}$$

i.e. $\sigma i = \mu$ and $m^e = m^{\sigma e}$, for all $m \in M$, $e \in E$.

Two such extensions of type \mathcal{M}

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} \Phi \longrightarrow 1,$$

$$1 \longrightarrow M \xrightarrow{i'} E' \xrightarrow{p'} \Phi \longrightarrow 1,$$

are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{p} & \Phi & \longrightarrow & 1, \\
 & & \parallel & & \downarrow \phi & & \parallel & & \\
 1 & \longrightarrow & M & \xrightarrow{i'} & E' & \xrightarrow{p'} & \Phi & \longrightarrow & 1,
 \end{array}$$

such that $\sigma'\phi = \sigma$. Of course in this case ϕ is an isomorphism, by the 5-lemma, and hence equivalence of extensions is an equivalence relation. Denote by $\text{Ext}_{\mathcal{M}}(\Phi, M)$ the set of equivalence classes of all extensions of type \mathcal{M} of M by Φ .

An extension of M by Φ of type \mathcal{M} determines a morphism $\theta: \Phi \rightarrow Q$, where $Q = \text{Coker}\mu$, which is dependent only on the equivalence class of the extension, and θ is here called the *abstract \mathcal{M} -kernel* of the extension. The set of extension classes with a given abstract \mathcal{M} -kernel θ is written $\text{Ext}_{(\mathcal{M},\theta)}(G, M)$.

The usual theory of extensions of a group M by a group Φ considers extensions of the type of the crossed module $\chi_M: M \rightarrow \text{Aut } M$. The advantages of replacing this by a general crossed module are first that the group $\text{Aut } M$ is not a functor of M , so that the relevant cohomology theory in terms of χ_M appears to have no coefficient morphisms, and second, that the more general case occurs geometrically, as in [20] and in this paper.

We now show there is an obstruction to realizability, analogous to the classical result of Eilenberg-Mac Lane ([17, Ch.V, Prop.8.3]). The cohomology groups $H_{\theta}^*(\Phi, A)$ referred to here are defined later.

Theorem 5.2 *Let \mathcal{M} be the crossed module $\mu: M \rightarrow P$ with $A = \text{Ker}\mu$, $Q = \text{Coker}\mu$. Let $\theta: \Phi \rightarrow Q$ be an abstract \mathcal{M} -kernel. Then there is an obstruction class $k(\mathcal{M}, \theta) \in H_{\theta}^3(\Phi, A)$ whose vanishing is necessary and sufficient for there to exist an extension of M by Φ of type \mathcal{M} with abstract \mathcal{M} -kernel θ . Further, if the obstruction class is zero, then the equivalence classes of such extensions are bijective with $H_{\theta}^2(\Phi, A)$.*

We give an exposition of a proof of this theorem using the methods of crossed complexes as given for example in Brown and Higgins [4] or [6]. The point is that crossed complexes allow for methods analogous to those of chain complexes as in standard homological algebra, but including non-abelian information of the type given by crossed modules. The obstruction result arises from an exact sequence of a fibration of crossed complexes. This allows us to give a proof analogous to that given for the classical case using topological methods by Berrick in [1]. A direct proof may also be given by extending the methods of Mackenzie [16] to more general crossed modules than $M \rightarrow \text{Aut}(M)$.

We assume the definition of crossed complex as given for example in Brown and Higgins [4] or [6], and in particular the notion of pointed morphism. Recall that a *reduced* crossed complex has a single vertex. A *homotopy* $h: f \simeq g$ of pointed morphisms $f, g: C \rightarrow D$ of crossed complexes is a family of functions $h_i: C_i \rightarrow D_{i+1}$ such that

i) $h_1: C_1 \rightarrow D_2$ is a derivation over g_1 , that is,

$$h_1(x + y) = h_1(x)^{g_1 y} + h_1(y),$$

where $g_1(y) = g_1(y)$, for $x, y \in C_1$.

ii) For $n \geq 2$, $h_n: C_n \rightarrow D_{n+1}$ is an operator morphism over g_1 , that is,

$$h_n(x^a + y) = (h_n x)^{g_a} + h_n(y),$$

where $g_a = g_1 a$.

iii) If $x \in C_1$, then

$$g x = f x \delta h_1 x.$$

iv) If $n \geq 2$ and $c \in C_n$, then

$$g x = f x h_{n-1} \delta x - \delta h_n x.$$

We will also use the morphism crossed complex $CRS_*(C, D)$ defined in Brown and Higgins [5] whose elements in dimension 0 are the pointed morphisms $C \rightarrow D$, in dimension 1 are the homotopies, and in higher dimensions are the “higher homotopies”.

A crossed module $\mu: M \rightarrow P$ can also be extended by trivial groups to give a crossed complex

$$\cdots \rightarrow 1 \rightarrow \cdots 1 \rightarrow 1 \rightarrow M \rightarrow P.$$

Denote this crossed complex again by \mathcal{M} .

Let Φ be a group. We write $C\Phi$ for the standard crossed resolution of Φ . This is defined in Huebschmann [14] and shown in Brown and Higgins [4] to be the fundamental crossed complex of the (Kan) simplicial set, $Nerv(\Phi)$, the nerve of the group Φ .

Write $[C\Phi, \mathcal{M}]$ for the set of pointed homotopy classes of morphisms $C\Phi \rightarrow \mathcal{M}$.

Theorem 5.3 *There is a bijection*

$$[C\Phi, \mathcal{M}] \cong \text{Ext}_{\mathcal{M}}(\Phi, M).$$

The proof is given in Brown and Higgins [4]. The key point is that $C_1\Phi$ is the free group on elements $[g]$, $g \in \Phi$, $C_2\Phi$ is the free crossed $C_1\Phi$ -module on $\delta: \Phi \times \Phi \rightarrow C_1\Phi$, where $\delta(g, h) = [g][h][gh]^{-1}$, and, for $i \geq 3$, $C_i\Phi$ is the free Φ -module on $[g_1, \dots, g_i]$, for $g_1, \dots, g_i \in \Phi$. Further because of the form of the boundary morphism $\delta: C_3\Phi \rightarrow C_2\Phi$, a morphism $C\Phi \rightarrow \mathcal{M}$ is equivalent to a factor set (with values in \mathcal{M}), and a homotopy of morphisms is essentially an equivalence of factor sets.

Recall that \mathcal{M} is the crossed module $\mu: M \rightarrow P$, and $A = \text{Ker}\mu$, $Q = \text{Coker}\mu$. Let $\xi\mathcal{M}$, $\zeta\mathcal{M}$ denote the crossed complexes in the following diagram of morphisms of crossed complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 1 & \longrightarrow & M & \longrightarrow & P \\ & & \downarrow & & \text{id} \downarrow & & \text{id} \downarrow \\ \cdots & \longrightarrow & A & \longrightarrow & M & \longrightarrow & P \\ & & \text{id} \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & A & \longrightarrow & 1 & \longrightarrow & Q \end{array} \qquad \begin{array}{c} \mathcal{M} \\ \downarrow \\ \xi\mathcal{M} \\ q \downarrow \\ \zeta\mathcal{M} \end{array}$$

where q is determined by the quotient morphism $P \rightarrow Q$. Since q is an epimorphism in each dimension, it is also a fibration of crossed complexes and therefore, since $C\Phi$ is free, the induced morphism of morphism complexes

$$q_*: CRS_*(C\Phi, \xi\mathcal{M}) \rightarrow CRS_*(C\Phi, \zeta\mathcal{M})$$

is also a fibration of crossed complexes (Brown and Higgins [6], Prop.6.2). Since $C\Phi$ is free and $\xi\mathcal{M}$ is acyclic, there is an identification

$$\pi_0CRS_*(C\Phi, \xi\mathcal{M}) \cong Hom(\Phi, Q).$$

Further, each morphism $\theta: \Phi \rightarrow Q$ determines an action of Φ on A and so a cohomology group $H_\theta^3(\Phi, A)$. Then $\pi_0CRS_*(C\Phi, \zeta\mathcal{M})$ is the union of all these cohomology groups for all such θ . The function $\pi_0(q_*)$ takes a morphism θ to a cohomology class

$$k(\mathcal{M}, \theta) \in H_\theta^3(\Phi, A),$$

called the *obstruction class* of (\mathcal{M}, θ) . If $k: C\Phi \rightarrow \xi\mathcal{M}$ is a realisation of θ , then qk represents $k(\mathcal{M}, \theta)$. If this class is 0, then there is a homotopy $h: qk \simeq l$, say, where $l_1 = qk_1$, $l_3 = 0$. Hence $k_3 = h_2\delta$. So there is a homotopy $k \simeq k'$ where $k'_1 = k_1$, $k'_2 = k_2 - \delta h_2$, $k'_3 = 0$.

Let F be the fibre of q_* over l . Then π_0F may be identified with the set $[C\Phi, \mathcal{M}]$ of homotopy classes of morphisms $C\Phi \rightarrow \mathcal{M}$, and so with the classes of extensions of A by Φ of type \mathcal{M} . The exact sequence of the fibration q_* with fibre F yields, given the above identifications, the exact sequence

$$0 \rightarrow H_\theta^2(\Phi, A) \rightarrow \text{Ext}_{\mathcal{M}}(\Phi, M) \rightarrow Hom(\Phi, Q) \rightarrow H_\theta^3(\Phi, A) \quad (\star)$$

where the three right hand terms have base points the class of the split extension, the morphism θ , and zero respectively. The obstruction part of Theorem 5.2 follows immediately. The standard theory of the exact sequence of a fibration of crossed complexes [13] also yields that the group $H_\theta^2(\Phi, A)$ operates on $\text{Ext}_{\mathcal{M}}(\Phi, M)$ so that the classes of extensions of type \mathcal{M} with abstract kernel θ are given by this group. This completes the proof of Theorem 5.2. \square

We can translate Theorem 5.2 to the following.

Theorem 5.4 *Let X be a topological group. Let Φ be a group, and let $\theta: \Phi \rightarrow \pi_0X$ be a morphism of groups. Then there is a covering morphism $p: \tilde{X} \rightarrow X$ of topological groups and an isomorphism $\alpha: \pi_0\tilde{X} \rightarrow \Phi$ such that $\theta\alpha = \pi_0(p)$ and \tilde{X} is simply connected if and only if the obstruction class*

$$k(\mathcal{M}, \theta) \in H_\theta^3(\Phi, \pi_1(X, e))$$

is zero, where \mathcal{M} is the associated crossed module $(\pi_1X)^e \rightarrow X$. Further, the isomorphism classes of such coverings are bijective with $H_\theta^2(\Phi, A)$.

Proof We write $\mu: M \rightarrow X$ for \mathcal{M} . If the obstruction class is zero then there is an extension $1 \rightarrow M \rightarrow E \rightarrow \Phi \rightarrow 1$ of type \mathcal{M} , and the crossed module $M \rightarrow E$ corresponds to a simply transitive group-groupoid \tilde{G} . The morphism from $M \rightarrow E$ to \mathcal{M} yields a covering morphism of group-groupoids $\tilde{G} \rightarrow \pi_1X$. Hence we obtain the required covering space $\tilde{X} = \text{Ob}(\tilde{G})$. The converse follows from Theorem 5.2, as does the classification of these coverings. \square

If \mathcal{M} is an arbitrary crossed module with cokernel Q , and one takes $\Phi = Q$ and $\theta = \text{id}$ in 5.2, then the class $k(\mathcal{M}, \text{id}) \in H^3(Q, A)$, where the action of Q on A is the given one, is called the *obstruction class* $k(\mathcal{M})$ of the crossed module \mathcal{M} . As a consequence of Theorem 5.4 we recover the result of Taylor [20].

Corollary 5.5 *Let X be a (possibly disconnected) topological group and let $p: \tilde{X} \rightarrow X$ be a π_0 -proper universal covering. Then the group structure of X lifts to \tilde{X} such that \tilde{X} is a topological group and p is a morphism of topological groups if and only if the obstruction class $k(\mathcal{M}) \in H^3(\pi_0 X, \pi_1(X, e))$ is zero.*

We remark that this obstruction class is shown in Brown and Spencer [10], to be the first k -invariant of the classifying space of the topological group X .

The following result is referred to in [20].

Corollary 5.6 *Let X be a (possibly disconnected) topological group. Then there exists a simply connected covering group $p: \tilde{X} \rightarrow X$ of X such that $\pi_0 p$ is surjective.*

Proof It is enough to choose an epimorphism $\theta: \Phi \rightarrow \pi_0 X$ such that the induced morphism on cohomology

$$\theta^*: H^3(\pi_0 X, \pi_1(X, e)) \rightarrow H_\theta^3(\Phi, \pi_1(X, e))$$

is trivial. This can be done with Φ a free group. □

Of course, there is no uniqueness result for this simply connected cover.

In the next section, we generalise Theorem 5.4 to a wider class of coverings.

6 General coverings of topological groups

We now deal with other coverings than simply connected ones, as does Taylor in [20] for the proper case.

We first recall two basic constructions which will be used later. The first essentially gives the usual forward coefficient morphism in cohomology.

Proposition 6.1 *Let $\mu: M \rightarrow P$ be a crossed module with $A = \text{Ker } \mu$ and $Q = \text{Coker } \mu$. Let $\phi: A \rightarrow B$ be a morphism of Q -modules. Then there is a crossed module $\mu': M' \rightarrow P$ and a morphism of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \\ & & \phi \downarrow & & \phi' \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow{j} & M' & \xrightarrow{\mu'} & P & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

such that (ϕ', id) is a morphism of crossed modules.

Proof The proof is easy on taking $M' = (B \times M)/C$, where $C = (\phi, i)(A)$, and defining μ' by $[b, m] \mapsto \mu m$, ϕ' by $m \mapsto [m, 1]$, where $[b, m]$ denotes the class of (b, m) in M' . □

Proposition 6.2 *Let \mathcal{M} be the crossed module $\mu: M \rightarrow P$, let $Q = \text{Coker}\mu$, and let $\theta: \Phi \rightarrow Q$ be an abstract kernel. Then*

$$k(\mathcal{M}, \theta) = k(\mathcal{N}, \text{id})$$

where \mathcal{N} is the crossed module $\nu: M \rightarrow P \times_Q \Phi$, $m \mapsto (m, 1)$. Further, there is a bijection

$$\text{Ext}_{(\mathcal{M}, \theta)}(\Phi, M) \cong \text{Ext}_{(\mathcal{N}, \text{id})}(\Phi, M).$$

Proof This follows from the morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & M & \xrightarrow{\nu} & P \times_Q \Phi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \text{id} \downarrow & & \text{id} \downarrow & & \downarrow & & \theta \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

□

Now we can give the following theorem.

Theorem 6.3 *Let X be a topological group, let $\theta: \Phi \rightarrow \pi_0 X$ be a morphism of groups, and let N be a $\pi_0 X$ -invariant subgroup of $\pi_1(X, e)$. Then there is a covering morphism $p: \tilde{X} \rightarrow X$ of topological groups and an isomorphism $\alpha: \pi_0 \tilde{X} \rightarrow \Phi$ such that $\theta\alpha = \pi_0(p)$ and $p(\pi_1(\tilde{X}, \tilde{e})) = N$ if and only if the obstruction class*

$$k(\mathcal{M}, \theta) \in H_\theta^3(\Phi, \pi_1(X, e)),$$

where \mathcal{M} is the associated crossed module $(\pi_1 X)^e \rightarrow X$, is mapped to zero by the morphism induced by the coefficient morphism

$$\pi_1(X, e) \rightarrow (\pi_1(X, e))/N.$$

Proof Write the crossed module \mathcal{M} as $\mu: M \rightarrow P$, and let $Q = \text{Coker}\mu$, $A = \text{Ker}\mu$. Suppose that there is such a covering morphism of topological groups and isomorphism α as given in the theorem. Let the crossed module \mathcal{N} associated to \tilde{X} be written as $\nu: \tilde{M} \rightarrow E$, so that $\text{Ker}\nu = N$. Then \mathcal{N} maps to \mathcal{M} as part of the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & \tilde{M} & \xrightarrow{\nu} & E & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & i \downarrow & & \cong \downarrow & & \sigma \downarrow & & \theta \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

where N is now a Φ -module via θ . Let \mathcal{M}' and $\mathcal{M} \rightarrow \mathcal{M}'$ be the crossed module and morphism of crossed modules constructed from the quotient mapping $A \rightarrow A/N$ as in Proposition 6.1. Let $k: C\Phi \rightarrow \xi\mathcal{N}$ be a realisation of the identity morphism on Φ . Then the composite $C\Phi \rightarrow \xi\mathcal{N} \rightarrow \zeta\mathcal{M} \rightarrow \zeta\mathcal{M}'$ realises $k(\mathcal{N}, \text{id})$. Clearly the composition

$$C\Phi \rightarrow \xi\mathcal{N} \rightarrow \zeta\mathcal{M} \rightarrow \zeta\mathcal{M}'$$

realises the zero class in $H_\theta^3(\Phi, A/N)$, as required.

Suppose conversely that $k(\mathcal{M}, \theta)$ maps to zero in $H_\theta^3(\Phi, A/N)$. Again, let \mathcal{M}' be the crossed module constructed in Proposition 6.1, with morphism $\phi' : M \rightarrow M'$. Then, by assumption, the obstruction class $k(\mathcal{M}', \theta)$ is zero, and so there is an extension of type \mathcal{M}' and with abstract kernel θ

$$1 \longrightarrow M' \xrightarrow{i'} E \longrightarrow \Phi \longrightarrow 1$$

It is easy to check that $\nu = i'\phi' : M \rightarrow E$ becomes a crossed module when E acts on M via σ , and that $\text{Ker}\nu = \text{Ker}\phi' = N$. Hence we have the following morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{\nu} & E & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \sigma & & \downarrow \theta & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

The morphism of crossed modules this includes can be realised by a covering morphism of group-groupoids and so of topological groups as required. \square

Example 6.4 We mention some nice examples of Taylor [20]. He shows there are exactly three non-isomorphic topological group extensions of $SO(2)$ by \mathbf{Z}_2 , namely the direct sum of the two groups, the orthogonal group $O(2)$, and finally the multiplicative group of all quaternions $a + bi + cj + dk$, of norm 1, such that $(a^2 + b^2)(c^2 + d^2) = 0$. Other examples of non-connected coverings of topological groups are given in section 8 of [20].

This completes our account of the theory of covering groups of topological groups.

Of course these theorems on spaces have analogues for group-groupoids which we leave the reader to state.

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