COVERING GROUPS OF NON-CONNECTED TOPOLOGICAL GROUPS REVISITED

Ronald Brown and Osman Mucuk
School of Mathematics
University of Wales
Bangor, Gwynedd LL57 1UT, U.K.

Received 11 March 1993 , Revision 31 May 1993

Published in Math. Proc. Camb. Phil. Soc, 115 (1994) 97-110.
This version: 18 Dec. 2006, with some new references.

Introduction

All spaces are assumed to be locally path connected and semi-locally 1-connected. Let X be a connected topological group with identity e, and let $p : \tilde{X} \to X$ be the universal cover of the underlying space of X. It follows easily from classical properties of lifting maps to covering spaces that for any point $\tilde{e}$ in $\tilde{X}$ with $p\tilde{e} = e$ there is a unique structure of topological group on $\tilde{X}$ such that $\tilde{e}$ is the identity and $p\colon \tilde{X} \to X$ is a morphism of groups. We say that the structure of topological group on X *lifts* to $\tilde{X}$.

It is less generally appreciated that this result fails for the non-connected case. The set $\pi_0 X$ of path components of X forms a non-trivial group which acts on the abelian group $\pi_1(X, e)$ via conjugation in X. R.L. Taylor [20] showed that the topological group X determines an *obstruction class* k_X in $H^3(\pi_0 X, \pi_1(X, e))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the topological group structure on X to a universal covering so that the projection is a morphism. Further, recent work, for example Huebschmann [15], shows there are geometric applications of the non-connected case.

The purpose of this paper is to prove generalisations of this result on coverings of topological groups using modern work on coverings of groupoids (see for example, Higgins [12], Brown [3]), *via* the following scheme. We first use the fact that covering spaces of space X are equivalent to covering morphisms of the fundamental groupoid $\pi_1 X$ (section 1). This extends easily to the group case: if X is a topological group, then the fundamental groupoid inherits a group structure making it what is called a *group-groupoid*, i.e. a group object in the category of groupoids; then topological group coverings of X are equivalent to group-groupoid coverings of $\pi_1 X$ (Proposition 2.3).

The next input is the equivalence between group-groupoids, and crossed modules (Brown and Spencer [10]). Here a crossed module is a morphism $\mu : M \to P$ of groups together with an action of the group P on the group M, with two axioms satisfied. It is easy to translate notions from covering morphisms of group-groupoids to corresponding notions for crossed modules (Proposition 4.2).

The existence of simply connected covering groups of a topological group now translates to the existence of extensions of groups of "the type of a given crossed module" (Definition 5.1). This generalisation of the classical extension theory is due to Taylor [19] and Dedecker [11]. We formulate a corresponding notion of abstract kernels (Theorem 5.2), analogous to that due to Eilenberg-Mac Lane [17]. This leads to our main result, Theorem 5.4, which determines when a morphism $\theta : \Phi \to \pi_0 X$ of groups is realised by a covering morphism $p : \tilde{X} \to X$ of topological groups such that $\tilde{X}$ is simply connected with $\pi_0 X$ isomorphic to Φ. We deduce that any topological group X admits a simply connected covering group covering all the components of X (Corollary 5.6). According to comments in [20], results of this type were known to Taylor.

Our proof of Theorem 5.2 uses methods of crossed complexes, as in Brown and Higgins [4]. This seems the natural setting for these results, since crossed complexes contain information on resolutions and on crossed modules. The exposition is analogous to that of Berrick [1] for the ordinary theory of extensions, in that fibrations are used, but in the algebraic context of crossed complexes. A direct account of a special case of these results, in the context of Lie groupoids, is given by Mackenzie in [16], and this account could also be adapted to the general case.

Section 6 deals with coverings other than simply connected ones.

The results of this paper formed part of Part I of Mucuk [18].

1 Groupoids and coverings

The main tool is the equivalence between covering maps of a topological space X and covering morphisms of the fundamental groupoid $\pi_1 X$ of X. Our main reference for groupoids and this result is Brown [3] but we adopt the following notations and conventions.

A topological space X is called *simply connected* if each loop in X is contractible in X, and X is called 1-*connected* if it is connected and simply connected. A map $f : X \to Y$ is called π_0-*proper* if $\pi_0(f)$ is a bijection.

If X is a topological space, the category $TCov/X$ of covering spaces of X is the full subcategory of the slice category Top/X of spaces over X in which the objects are the covering maps. It is standard that if $h\colon Y \to Z$ is a map in $TCov/X$, i.e. is a map over X, then h is a covering map. Further, if $f\colon Y \to X$ is a covering map such that Y is simply connected, then for any other cover $g\colon Z \to X$, there is a covering map $h\colon Y \to Z$ over X. This is summarised by saying that Y covers any other cover of X, and a covering map with this property is called *universal.* A necessary and sufficient condition for this is that Y be simply-connected.

For a groupoid G, we write O_G for the set of objects of G, and G for the set of arrows, or elements. We write $s, t\colon G \to O_G$ for the source and target maps. The product $g \circ h$ is defined if and only if $tg = sh$. The identity at $x \in O_G$ is written 1_x. The inverse of an element g is written g^{-1}.

The category of groupoids and morphisms of groupoids is written Gd.

For $x \in O_G$ we denote the *star* $\{g \in G \mid sg = x\}$ of x by G^x, and the *costar* $\{g \in G \mid tg = x\}$ of x by G_x, and write G^x_y for $G^x \cap G_y$. The object group at x is $G(x) = G^x_x$. An element of some G^x_x is called a *loop* of G.

We say G is *transitive* (resp. 1-*transitive*, *simply transitive*) if for all $x, y \in O_G$, $G(x, y)$ is non-empty (resp. is a singleton, has not more than one element). The transitive component of an object x of G is the largest transitive subgroupoid of G with x as an object, and is written $C(G, x)$. The set of transitive components of G is written $\pi_0 G$. A morphism p of groupoids is called π_0-*proper* if $\pi_0(p)$ is a bijection.

Covering morphisms and universal covering groupoids of a groupoid are defined in Brown [2] (see also Higgins [12], Brown [3]) as follows:

Let $p\colon H \to G$ be a morphism of groupoids. Then p is called a *covering morphism* if for each $x \in O_H$, the restriction $G^x \to G^{px}$ of p is bijective. The covering morphism p is called *regular* if for all objects x of G and all $g \in G(x)$ the elements of $p^{-1}(g)$ are all or none of them loops. This is equivalent to the condition that for all objects y of H, the subgroup $pH(y)$ of $G(py)$ is a normal subgroup [3].

If G is a groupoid, the category $GdCov/G$ of coverings of G is the full subcategory of the slice category Gd/G of groupoids over G in which the objects are the covering morphisms.

A covering morphism $p\colon H \to G$ is called *universal* if H covers every covering of G, i.e. if for every covering morphism $a\colon A \to G$ there is a morphism of groupoids $a'\colon H \to A$ such that $aa' = p$ (and hence a' is also a covering morphism). It is common to consider universal covering morphisms which are π_0-proper.

We recall the following standard result (Brown [3], Chapter 9), which summarises the theory of covering spaces.

Proposition 1.1 *For any space X, the fundamental groupoid functor defines an equivalence of categories*

$$\pi_1 : TCov/X \to GdCov/(\pi_1 X).$$

One crucial step in the proof of this equivalence is the result (Brown [3], 9.5.5) that if $q\colon H \to \pi_1 X$ is a covering morphism of groupoids, then there is a topology on O_H such that $O_q\colon O_H \to X$ is a covering map, and there is an isomorphism $\alpha\colon \pi_1 O_H \to H$ such that $q\alpha = \pi_1(O_q)$. This result, which translates the usual covering space theory into a more base-point free context, yields the inverse equivalence.

We also remark that the universal cover of X at $x \in X$ is given by the target map $(\pi_1 X)^x \to X$with the subspace topology from a topology on $\pi_1 X$.

Recall that an *action* of a groupoid G on sets *via* w consists of a function $w\colon A \to O_G$, where A is a set, and an assignment to each $g \in G(x, y)$ of a function $g_\sharp\colon w^{-1}(x) \to w^{-1}(y)$, written $a \mapsto a \circ g$, satisfying the usual rules for an action, namely $a \circ 1 = a$, $a \circ (g \circ h) = (a \circ g) \circ h$ when defined. A *morphism* $f\colon (A, w) \to (A', w')$ of such actions is a function $f\colon A \to A'$ such that $w'f = w$ and $f(a \circ g) = (fa) \circ g$ whenever $a \circ g$ is defined. This gives a category *Act*(G) of actions of G on sets. For such an action, the *action groupoid* $A \rtimes G$ is defined to have object set A, arrows the pairs (a, g) such that $w(a) = sg$, source and target maps $s(a, g) = a$, $t(a, g) = a \circ g$, and composition

$$(a, g) \circ (b, h) = (a, g \circ h)$$

whenever $b = a\circ g$. The projection $q\colon A\rtimes G \to G$, $(a, g) \mapsto g$, is a covering morphism of groupoids, and the functor sending an action to this covering morphism gives an equivalence of categories *Act*$(G) \to GdCov/G$. (See for example Brown [3].)

Let x be an object of the transitive groupoid G, and let $N(x)$ be a subgroup of the object group $G(x)$. Then G acts on the set A of cosets $N(x) \circ g$ for $g \in G^x$, *via* the map $N(x) \circ g \mapsto tg$. So we can form the corresponding covering morphism $p\colon H \to G$, where $H = A \rtimes G$, and the object $\tilde{x} = N(x)$ of H satisfies $p(H(\tilde{x})) = N(x)$. This construction yields an equivalence of categories between the lattice $\mathcal{L}G(x)$ of subgroups of $G(x)$ and the category of pointed transitive coverings of G, x.

Suppose further that $a \in G^x_y$, $N(y) = a^{-1} \circ N(x) \circ a$, and $q\colon K \to G$ is the covering of G determined as above by $N(y)$, with $\tilde{y} \in O_K$ satisfying $q[K(\tilde{y})] = N(y)$. Then there is a unique isomorphism $h\colon H \to K$ such that $qh = p$ and $h\tilde{z} = \tilde{y}$. That is, conjugate subgroups of a transitive groupoid G determine isomorphic coverings, and we obtain an equivalence of categories between the lattice of conjugacy classes of subgroups of G and the isomorphism classes of transitive coverings of G.

If G is not transitive then π_0-proper coverings may be constructed by working on each transitive component. We choose a transversal for the set $I = \pi_0 G$ of components of G, i.e. an object τ_i for each component G_i of G, choose a subgroup $N(\tau_i) \subseteq G(\tau_i)$, and get a covering $\tilde{G_i} \to G_i$ for each component G_i of G. The disjoint union of these coverings is a covering $p\colon \tilde{G} \to G$, which is universal if and only if all the $N(\tau_i)$ are trivial groups.

2 Group-groupoids and covering morphisms

The notion of group-groupoid, and the first parts of Propositions 2.1 and 2.3 below are taken from Brown-Spencer [10], although the term used there is $\mathcal{G}$-groupoid.

By a *group-groupoid* we mean a groupoid G with a morphism of groupoids $G \times G \to G$, $(g, h) \mapsto gh$, yielding a group structure internal to the category of groupoids. Since the multiplication is a morphism of groupoids, we obtain the *interchange law*, that $(a \circ g)(b \circ h) = (ab) \circ (gh)$, for all $g, h, a, b \in G$ such that $a \circ g$ and $b \circ h$ are defined. If the identity for the group structure on O_G is written e, then 1_e is the identity for the group structure on the arrows. The group inverse of an arrow g is written $\bar{g}$. Then $g \mapsto \bar{g}$ is a morphism $G \to G$ of groupoids.

It is a standard consequence of the interchange law that the groupoid composition in a group-groupoid can be recovered from the group law, as shown in the first part of the following proposition.

Proposition 2.1 *Let G be a group-groupoid, and suppose $a \circ b$ is defined in G , where $a \in G(x, y)$. Then $a \circ b = a\bar{1}_y b$. If further $g \in G(e)$, then*

$$a \circ (1_y g) \circ a^{-1} = 1_x g,$$

and

$$a g \bar{a} = 1_x g \bar{1}_x.$$

Further, G(e) is abelian.

Proof Suppose $ta = y$. Then

$$\begin{aligned} a \circ b &= ((a\bar{1}_y)1_y) \circ (1_e b) \\ &= ((a\bar{1}_y) \circ 1_e)(1_y \circ b) \\ &= a\bar{1}_y b. \end{aligned}$$

Further

$$\begin{aligned} a \circ (1_y g) \circ a^{-1} &= a \circ ((1_y g) \circ (a^{-1} 1_e)) &&= a \circ ((1_y \circ a^{-1})(g \circ 1_e)) \\ &= a \circ (a^{-1} g) &&= (a 1_e) \circ (a^{-1} g) \\ &= (a \circ a^{-1})(1_e \circ g) &&= 1_x g. \end{aligned}$$

On the other hand

$$\begin{aligned} a \circ (1_y g) \circ a^{-1} &= (a \bar{1}_y 1_y g) \circ a^{-1} &&= (ag) \circ (1_y \bar{a} 1_x) \\ &= a g \bar{1}_y 1_y \bar{a} 1_x &&= a g \bar{a} 1_x. \end{aligned}$$

Hence $ag\bar{a} = 1_x g \bar{1}_x$. That $a \circ g = g = g \circ a$ for $a, g \in G(e)$ is immediate. □

Corollary 2.2 *Let $N(e)$ be a subgroup of $G(e)$, and let N be the family of subsets $N(x) = 1_x N(e)$ for all $x \in O_G$. Then $N(x)$ is a normal subgroupoid of G. In particular, all the object groups of G are isomorphic, and are abelian.*

Proof That $N(x)$ is a subgroup follows from

$$1_x(b \circ a) = (1_x \circ 1_x)(b \circ a) = (1_x b) \circ (1_x a),$$

for $b, a \in N(x)$. The normality follows from the second formula of the Proposition, on taking $g \in N(e)$. It is immediate that all the object groups are isomorphic. □

This result implies that all coverings of a group-groupoid are regular. It also shows that a choice τ of transversal for the components of a group-groupoid G induces an equivalence between the category $\mathcal{L}G(e)$ of subgroups of $G(e)$ under inclusion and the category of isomorphism classes of π_0-proper coverings of G.

We now consider coverings in the category of group-groupoids.

A *morphism* of group-groupoids is a morphism of the underlying groupoids which preserves the group structure. Then group-groupoids and morphisms of them form a category which we will denote by *GpGd*. Let G be a group-groupoid. Then $GpGdCov/G$ denotes the full subcategory of the slice category $GpGd/G$ whose objects are group-groupoids $p : H \to G$ over G such that p is a covering morphism of the underlying groupoids.

We can now translate Proposition 1.1 to this situation.

Proposition 2.3 *Let X be a topological group. Then the fundamental groupoid $\pi_1 X$ is a group-groupoid with group structure induced by that of X . Further, the fundamental groupoid functor π_1 gives an equivalence from the category $GpTCov/X$ to the category $GpGd/\pi_1 X$.*

Proof We show that the inverse equivalence of Proposition 1.1 determines an inverse equivalence in this case also.

Suppose then that $q \colon H \to \pi_1 X$ is a morphism of group-groupoids such that the underlying groupoid morphism is a covering morphism. Then there is a topology on $\tilde{X} = O_H$ and an isomorphism

$\alpha\colon \pi_1\tilde{X} \to H$ such that $p = O_q\colon \tilde{X} \to X$ is a covering map and $q\alpha = \pi_1(p)$. The group structure on H transports via α to a morphism of groupoids

$$\tilde{m}\colon \pi_1\tilde{X} \times \pi_1\tilde{X} \to \pi_1\tilde{X}$$

such that $\pi_1(p) \circ \tilde{m} = m \circ (\pi_1(p) \times \pi_1(p))$, where m is the group multiplication on X, and clearly $\tilde{m}$ is a group structure on $\pi_1\tilde{X}$. By 9.5.5 of Brown [3], $\tilde{m}$ induces a continuous map on $\tilde{X}$. This gives the multiplication on $\tilde{X}$. The fact that this is a group structure follows from the fact that $\tilde{m}$ is a group structure. □

3 Actions of group-groupoids on groups

In this section we relate group-groupoid covering morphisms to a notion of action of a group-groupoid on a group. The results are a special case of results of Section 1 of Brown and Mackenzie [8], and are included here for completeness.

Let G be a group-groupoid. An *action* of the group-groupoid G on a group A *via* w consists of a morphism $w\colon A \to G$ from the group A to the underlying group of O_G and an action of the groupoid G on the underlying set A via w such that the following interchange law holds:

$$(a \circ g)(b \circ h) = (ab) \circ (gh)$$

whenever both sides are defined. A *morphism* $f\colon (A, w) \to (A', w')$ of such operations is a morphism $f\colon A \to A'$ of groups and of the underlying operations of G. This gives a category $GpGdAct(G)$. For an action of G on the group A via w, the action groupoid $A \rtimes G$ is defined. It inherits a group structure by

$$(a, g)(c, k) = (ac, gk).$$

It is easily checked that $A \rtimes G$ is then a group-groupoid, and the projection $p\colon A \rtimes G \to G$ is an object of the category $GpGd/G$. By means of this construction, one obtains the following, which is a special case of Theorem 1.7 of Brown and Mackenzie [8] which considers the case of actions of Lie double groupoids.

Proposition 3.1 *The categories GpGdCov/G and GpGdAct(G) are equivalent.*

4 Group-groupoids and crossed modules

A *crossed module* (M, P, μ) is defined in Whitehead [21] to consist of two groups M and P together with a homomorphism $\mu\colon M \to P$, and an action of P on M on the right, written $(m, p) \mapsto m^p$, such that the following conditions are satisfied:

CM1) $\mu(mp) = p^{-1}(\mu m)p$

CM2) $n^{\mu m} = m^{-1}nm$

for all $m, n \in M$ and $p \in P$.

Standard examples of crossed modules are:

(i) the inclusion $M \to P$ of a normal subgroup,

(ii) the zero morphism $M \to P$ when M is a P-module,

(iii) the inner automorphism map $\chi_M : M \to \mathrm{Aut}\, rM$ for any group M,

(iv) a morphism $M \to P$ of groups which is surjective and has central kernel,

(v) the free crossed P-module $C(w) \to P$ arising from a function $w \colon R \to P$ (see Brown and Huebschmann [7]),

(vi) the induced morphism $\pi_1(F, x) \to \pi_1(E, x)$ of fundamental groups for any fibration of spaces $F \to E \to B$.

Standard consequences of the axioms (see for example [7]) are that μM is a normal subgroup of P, that $Ker\mu$ is central in M, and that μM acts trivially on $Ker\mu$ which thereby becomes a module over $Coker\mu$.

A *morphism* $(f, g) \colon (M, P, \mu) \to (N, Q, \nu)$ of crossed modules consists of group morphisms $f \colon M \to N$ and $g \colon P \to Q$ such that $g\mu = \nu f$ and f is an operator homomorphism, that is, $f(m^p) = fm^{(gp)}$ for $m \in M$ and $p \in P$. So crossed modules and morphisms of them, with the obvious composition of morphisms $(f', g')(f, g) = (f'f, g'g)$, form a category, which we write $CrsM$.

The following theorem was found by Verdier in 1965, but not published, and found independently by Brown and Spencer [10]. We give a sketch of the proof, since we need some of its detail.

Theorem 4.1 *The category GpGd of group-groupoids is equivalent to the category $CrsM$ of crossed modules. If a group-groupoid G has associated crossed module (M, P, μ) then the underlying groupoid of G is transitive (resp. simply transitive, 1-transitive) if and only if μ is an epimorphism (resp. a monomorphism, isomorphism). Further, the group $\pi_0 G$ is $Coker\mu$.*

Sketch Proof: A functor $\delta \colon GpGd \to CrsM$ is defined as follows. For a group-groupoid G we let $\delta(G)$ be the crossed module (M, P, μ) where P is the group O_G of objects of G; M is the costar G_e of G at the identity e of the group O_G; $\mu \colon M \to P$ is the restriction of the source map s ; the group structures on M and P are induced by that on G ; and P acts on M by $m^p = \bar{1}_p m 1_p$ for $p \in P$ and $m \in M$. The results on transitivity follow immediately.

Conversely define a functor $\beta \colon CrsM \to GpGd$ in the following way. For a crossed module (M, P, μ), $\beta(M, P, \mu)$ is the group-groupoid whose object set (group) is P and whose group of arrows is the semi-direct product $P \ltimes M$ with the standard group structure

$$(p, m)(q, n) = (pq, m^q n).$$

The source and target maps s, t are defined to be $s(p, m) = p$ and $t(p, m) = p(\mu m)$, while the composition of arrows is given by

$$(p, m) \circ (q, n) = (p, mn)$$

whenever $p(\mu m) = q$. □

If X is a topological group with identity e, then the fundamental groupoid $\pi_1 X$ becomes a group-groupoid, the associated crossed module is $t \colon (\pi_1 X)^e \to X$ (Brown and Spencer [10]), and $(\pi_1 X)^e$ has a topology making it the universal cover based at e of the path component of e.

It is easy to obtain results for morphisms of group-groupoids corresponding to Theorem 4.1, as follows.

Proposition 4.2 *Let $f\colon H \to G$ be a morphism of group-groupoids and let $(f_1, f_2)\colon (N, Q, \nu) \to (M, P, \mu)$ be the morphism of crossed modules corresponding to f as in Theorem 4.1. Then, on underlying groupoids, f is a covering morphism if and only if $f_1\colon N \to M$ is an isomorphism. Further, f is a universal covering morphism if and only if f_1 is an isomorphism, ν is a monomorphism, and the induced morphism $Coker\nu \to Coker\mu$ is an isomorphism.*

We therefore define a morphism (f_1, f_2) of crossed modules as in the proposition to be a *covering morphism* if f_1 is an isomorphism, and so obtain a category $CrsMCov/(M \to P)$ of coverings of $M \to P$ as a full category of the slice category $CrsM/(M \to P)$.

Corollary 4.3 *The category $GTCov/X$ of topological group coverings of a topological group X is equivalent to the category $CrsMCov/((\pi_1 X)^e \to X)$ of crossed module coverings of $(\pi_1 X)^e \to X$.*

5 Extensions, crossed modules and cohomology

We now recall the notion of an extension of groups of the type of a crossed module, due to Taylor [19] and Dedecker [11]. See also [9].

Definition 5.1 Let $\mathcal{M}$ denote the crossed module $\mu\colon M \to P$. An *extension* (i, p, σ) *of type* $\mathcal{M}$ of the group M by the group Φ is first an exact sequence of groups

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} \Phi \longrightarrow 1$$

so that E operates on M by conjugation, and $i\colon M \to E$ is hence a crossed module. Second, there is given a morphism of crossed modules

$$\begin{array}{ccccc} 1 & \longrightarrow & M & \xrightarrow{i} & E \\ & & \| & & \downarrow{\scriptstyle \sigma} \\ & & M & \xrightarrow[\mu]{} & P \end{array}$$

i.e. $\sigma i = \mu$ and $m^e = m^{\sigma e}$, for all $m \in M$, $e \in E$.

Two such extensions of type $\mathcal{M}$

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} \Phi \longrightarrow 1,$$

$$1 \longrightarrow M \xrightarrow{i'} E' \xrightarrow{p'} \Phi \longrightarrow 1,$$

are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{p} & \Phi & \longrightarrow & 1, \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 1 & \longrightarrow & M & \xrightarrow{i'} & E' & \xrightarrow{p'} & \Phi & \longrightarrow & 1, \end{array}$$

such that $\sigma'\phi = \sigma$. Of course in this case ϕ is an isomorphism, by the 5-lemma, and hence equivalence of extensions is an equivalence relation. Denote by $\mathrm{Ext}_{\mathcal{M}}(\Phi, M)$ the set of equivalence classes of all extensions of type $\mathcal{M}$ of M by Φ.

An extension of M by Φ of type $\mathcal{M}$ determines a morphism $\theta\colon \Phi \to Q$, where $Q = Coker\mu$, which is dependent only on the equivalence class of the extension, and θ is here called the *abstract* $\mathcal{M}$*-kernel* of the extension. The set of extension classes with a given abstract $\mathcal{M}$-kernel θ is written $\mathrm{Ext}_{(\mathcal{M},\theta)}(G, M)$.

The usual theory of extensions of a group M by a group Φ considers extensions of the type of the crossed module $\chi_M\colon M \to \mathrm{Aut}\, M$. The advantages of replacing this by a general crossed module are first that the group $\mathrm{Aut}\, M$ is not a functor of M, so that the relevant cohomology theory in terms of χ_M appears to have no coefficient morphisms, and second, that the more general case occurs geometrically, as in [20] and in this paper.

We now show there is an obstruction to realizability, analogous to the classical result of Eilenberg-Mac Lane ([17, Ch.V, Prop.8.3]). The cohomology groups $H^*_\theta(\Phi, A)$ referred to here are defined later.

Theorem 5.2 *Let* $\mathcal{M}$ *be the crossed module* $\mu : M \to P$ *with* $A = Ker\mu$, $Q = Coker\mu$. *Let* $\theta : \Phi \to Q$ *be an abstract* $\mathcal{M}$*-kernel . Then there is an* obstruction class $k(\mathcal{M}, \theta) \in H^3_\theta(\Phi, A)$ *whose vanishing is necessary and sufficient for there to exist an extension of* M *by* Φ *of type* $\mathcal{M}$ *with abstract* $\mathcal{M}$*-kernel* θ*. Further, if the obstruction class is zero, then the equivalence classes of such extensions are bijective with* $H^2_\theta(\Phi, A)$.

We give an exposition of a proof of this theorem using the methods of crossed complexes as given for example in Brown and Higgins [4] or [6]. The point is that crossed complexes allow for methods analogous to those of chain complexes as in standard homological algebra, but including non-abelian information of the type given by crossed modules. The obstruction result arises from an exact sequence of a fibration of crossed complexes. This allows us to give a proof analogous to that given for the classical case using topological methods by Berrick in [1]. A direct proof may also be given by extending the methods of Mackenzie [16] to more general crossed modules than $M \to \mathrm{Aut}(M)$

We assume the definition of crossed complex as given for example in Brown and Higgins [4] or [6], and in particular the notion of pointed morphism. Recall that a *reduced* crossed complex has a single vertex. A *homotopy* $h\colon f \simeq g$ of pointed morphisms $f, g\colon C \to D$ of crossed complexes is a family of functions $h_i\colon C_i \to D_{i+1}$ such that

i) $h_1\colon C_1 \to D_2$ is a derivation over g_1, that is,

$$h_1(x + y) = h_1(x)^{gy} + h_1(y),$$

where $g(y) = g_1(y)$, for $x, y \in C_1$.

ii) For $n \geqslant 2$, $h_n \colon C_n \to D_{n+1}$ is an operator morphism over g_1, that is,

$$h_n(x^a + y) = (h_n x)^{ga} + h_n(y),$$

where $ga = g_1 a$.

iii) If $x \in C_1$, then

$$gx = fx\delta h_1 x.$$

iv) If $n \geqslant 2$ and $c \in C_n$, then

$$gx = fxh_{n-1}\delta x - \delta h_n x.$$

We will also use the morphism crossed complex $CRS_*(C, D)$ defined in Brown and Higgins [5] whose elements in dimension 0 are the pointed morphisms $C \to D$, in dimension 1 are the homotopies, and in higher dimensions are the "higher homotopies".

A crossed module $\mu \colon M \to P$ can also be extended by trivial groups to give a crossed complex

$$\cdots \to 1 \to \cdots 1 \to 1 \to M \to P.$$

Denote this crossed complex again by $\mathcal{M}$.

Let Φ be a group. We write $C\Phi$ for the standard crossed resolution of Φ. This is defined in Huebschmann [14] and shown in Brown and Higgins [4] to be the fundamental crossed complex of the (Kan) simplicial set, $Nerv(\Phi)$, the nerve of the group Φ.

Write $[C\Phi, \mathcal{M}]$ for the set of pointed homotopy classes of morphisms $C\Phi \to M$.

Theorem 5.3 *There is a bijection*

$$[C\Phi, \mathcal{M}] \cong \mathrm{Ext}_{\mathcal{M}}(\Phi, M).$$

The proof is given in Brown and Higgins [4]. The key point is that $C_1\Phi$ is the free group on elements $[g]$, $g \in \Phi$, $C_2\Phi$ is the free crossed $C_1\Phi$-module on $\delta \colon \Phi \times \Phi \to C_1\Phi$, where $\delta(g, h) = [g][h][gh]^{-1}$, and, for $i \geqslant 3$, $C_i\Phi$ is the free Φ-module on $[g_1, \ldots, g_i]$, for $g_1, \ldots, g_i \in \Phi$. Further because of the form of the boundary morphism $\delta \colon C_3\Phi \to C_2\Phi$, a morphism $C\Phi \to \mathcal{M}$ is equivalent to a factor set (with values in $\mathcal{M}$), and a homotopy of morphisms is essentially an equivalence of factor sets.

Recall that $\mathcal{M}$ is the crossed module $\mu \colon M \to P$, and $A = Ker\mu$, $Q = Coker\mu$. Let $\xi\mathcal{M}$, $\zeta\mathcal{M}$ denote the crossed complexes in the following diagram of morphisms of crossed complexes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & 1 & \longrightarrow & M & \longrightarrow & P & \qquad & \mathcal{M} \\
 & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\cdots \longrightarrow & A & \longrightarrow & M & \longrightarrow & P & & \xi\mathcal{M} \\
 & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\
\cdots \longrightarrow & A & \longrightarrow & 1 & \longrightarrow & Q & & \zeta\mathcal{M}
\end{array}$$

where q is determined by the quotient morphism $P \to Q$. Since q is an epimorphism in each dimension, it is also a fibration of crossed complexes and therefore, since $C\Phi$ is free, the induced morphism of morphism complexes

$$q_* \colon CRS_*(C\Phi, \xi\mathcal{M}) \to CRS_*(C\Phi, \zeta\mathcal{M})$$

is also a fibration of crossed complexes (Brown and Higgins [6], Prop.6.2). Since $C\Phi$ is free and ξM is acyclic, there is an identification

$$\pi_0 CRS_*(C\Phi, \xi\mathcal{M}) \cong Hom(\Phi, Q).$$

Further, each morphism $\theta\colon \Phi \to Q$ determines an action of Φ on A and so a cohomology group $H^3_\theta(\Phi, A)$. Then $\pi_0 CRS_*(C\Phi, \zeta\mathcal{M})$ is the union of all these cohomology groups for all such θ. The function $\pi_0(q_*)$ takes a morphism θ to a cohomology class

$$k(\mathcal{M}, \theta) \in H^3_\theta(\Phi, A),$$

called the *obstruction class* of $(\mathcal{M}, \theta)$. If $k\colon C\Phi \to \xi\mathcal{M}$ is a realisation of θ, then qk represents $k(\mathcal{M}, \theta)$. If this class is 0, then there is a homotopy $h\colon qk \simeq l$, say, where $l_1 = qk_1$, $l_3 = 0$. Hence $k_3 = h_2\delta$. So there is a homotopy $k \simeq k'$ where $k'_1 = k_1$, $k'_2 = k_2 - \delta h_2$, $k'_3 = 0$.

Let F be the fibre of q_* over l. Then $\pi_0 F$ may be identified with the set $[C\Phi, \mathcal{M}]$ of homotopy classes of morphisms $C\Phi \to \mathcal{M}$, and so with the classes of extensions of A by Φ of type $\mathcal{M}$. The exact sequence of the fibration q_* with fibre F yields, given the above identifications, the exact sequence

$$0 \to H^2_\theta(\Phi, A) \to \mathrm{Ext}_{\mathcal{M}}(\Phi, M) \to Hom(\Phi, Q) \to H^3_\theta(\Phi, A) \quad (\star)$$

where the three right hand terms have base points the class of the split extension, the morphism θ, and zero respectively. The obstruction part of Theorem 5.2 follows immediately. The standard theory of the exact sequence of a fibration of crossed complexes [13] also yields that the group $H^2_\theta(\Phi, A)$ operates on $\mathrm{Ext}_{\mathcal{M}}(\Phi, M)$ so that the classes of extensions of type $\mathcal{M}$ with abstract kernel θ are given by this group. This completes the proof of Theorem 5.2. □

We can translate Theorem 5.2 to the following.

Theorem 5.4 *Let X be a topological group. Let Φ be a group, and let $\theta\colon \Phi \to \pi_0 X$ be a morphism of groups. Then there is a covering morphism $p\colon \tilde{X} \to X$ of topological groups and an isomorphism $\alpha\colon \pi_0\tilde{X} \to \Phi$ such that $\theta\alpha = \pi_0(p)$ and $\tilde{X}$ is simply connected if and only if the obstruction class*

$$k(\mathcal{M}, \theta) \in H^3_\theta(\Phi, \pi_1(X, e))$$

is zero, where $\mathcal{M}$ is the associated crossed module $(\pi_1 X)^e \to X$. Further, the isomorphism classes of such coverings are bijective with $H^2_\theta(\Phi, A)$.

Proof We write $\mu\colon M \to X$ for $\mathcal{M}$. If the obstruction class is zero then there is an extension $1 \to M \to E \to \Phi \to 1$ of type $\mathcal{M}$, and the crossed module $M \to E$ corresponds to a simply transitive group-groupoid $\tilde{G}$. The morphism from $M \to E$ to $\mathcal{M}$ yields a covering morphism of group-groupoids $\tilde{G} \to \pi_1 X$. Hence we obtain the required covering space $\tilde{X} = Ob(\tilde{G})$. The converse follows from Theorem 5.2, as does the classification of these coverings. □

If $\mathcal{M}$ is an arbitrary crossed module with cokernel Q, and one takes $\Phi = Q$ and $\theta = \mathrm{id}$ in 5.2, then the class $k(\mathcal{M}, \mathrm{id}) \in H^3(Q, A)$, where the action of Q on A is the given one, is called the *obstruction class* $k(\mathcal{M})$ of the crossed module $\mathcal{M}$. As a consequence of Theorem 5.4 we recover the result of Taylor [20].

Corollary 5.5 *Let X be a (possibly disconnected) topological group and let $p\colon \tilde{X} \to X$ be a π_0-proper universal covering. Then the group structure of X lifts to $\tilde{X}$ such that $\tilde{X}$ is a topological group and p is a morphism of topological groups if and only if the obstruction class $k(\mathcal{M}) \in H^3(\pi_0 X, \pi_1(X, e))$ is zero.*

We remark that this obstruction class is shown in Brown and Spencer [10], to be the first k-invariant of the classifying space of the topological group X.

The following result is referred to in [20].

Corollary 5.6 *Let X be a (possibly disconnected) topological group. Then there exists a simply connected covering group $p : \tilde{X} \to X$ of X such that $\pi_0 p$ is surjective.*

Proof It is enough to choose an epimorphism $\theta : \Phi \to \pi_0 X$ such that the induced morphism on cohomology

$$\theta^* : H^3(\pi_0 X, \pi_1(X, e)) \to H^3_\theta(\Phi, \pi_1(X, e))$$

is trivial. This can be done with Φ a free group.

□

Of course, there is no uniqueness result for this simply connected cover.

In the next section, we generalise Theorem 5.4 to a wider class of coverings.

6 General coverings of topological groups

We now deal with other coverings than simply connected ones, as does Taylor in [20] for the proper case.

We first recall two basic constructions which will be used later. The first essentially gives the usual forward coefficient morphism in cohomology.

Proposition 6.1 *Let $\mu\colon M \to P$ be a crossed module with $A = Ker\mu$ and $Q = Coker\mu$. Let $\phi\colon A \to B$ be a morphism of Q-modules. Then there is a crossed module $\mu'\colon M' \to P$ and a morphism of exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{j} & M' & \xrightarrow{\mu'} & P & \longrightarrow & Q & \longrightarrow & 1
\end{array}$$

such that (ϕ', id) is a morphism of crossed modules.

Proof The proof is easy on taking $M' = (B \times M)/C$, where $C = (\phi, i)(A)$, and defining μ' by $[b, m] \mapsto \mu m$, ϕ' by $m \mapsto [m, 1]$, where $[b, m]$ denotes the class of (b, m) in M'. □

Proposition 6.2 *Let $\mathcal{M}$ be the crossed module $\mu\colon M \to P$, let $Q = Coker\mu$, and let $\theta\colon \Phi \to Q$ be an abstract kernel. Then*

$$k(\mathcal{M}, \theta) = k(\mathcal{N}, \mathrm{id})$$

where $\mathcal{N}$ is the crossed module $\nu\colon M \to P \times_Q \Phi$, $m \mapsto (m, 1)$. Further, there is a bijection

$$\mathrm{Ext}_{(\mathcal{M},\theta)}(\Phi, M) \cong \mathrm{Ext}_{(\mathcal{N},\mathrm{id})}(\Phi, M).$$

Proof This follows from the morphism of exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & M & \xrightarrow{\nu} & P \times_Q \Phi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

□

Now we can give the following theorem.

Theorem 6.3 *Let X be a topological group, let $\theta\colon \Phi \to \pi_0 X$ be a morphism of groups, and let N be a $\pi_0 X$-invariant subgroup of $\pi_1(X, e)$. Then there is a covering morphism $p\colon \tilde{X} \to X$ of topological groups and an isomorphism $\alpha\colon \pi_0 \tilde{X} \to \Phi$ such that $\theta\alpha = \pi_0(p)$ and $p(\pi_1(\tilde{X}, \tilde{e})) = N$ if and only if the obstruction class*

$$k(\mathcal{M}, \theta) \in H^3_\theta(\Phi, \pi_1(X, e)),$$

where $\mathcal{M}$ is the associated crossed module $(\pi_1 X)^e \to X$, is mapped to zero by the morphism induced by the coefficient morphism

$$\pi_1(X, e) \to (\pi_1(X, e))/N.$$

Proof Write the crossed module $\mathcal{M}$ as $\mu\colon M \to P$, and let $Q = Coker\mu$, $A = Ker\mu$. Suppose that there is such a covering morphism of topological groups and isomorphism α as given in the theorem. Let the crossed module $\mathcal{N}$ associated to $\tilde{X}$ be written as $\nu : \tilde{M} \to E$, so that $Ker\nu = N$. Then $\mathcal{N}$ maps to $\mathcal{M}$ as part of the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & N & \longrightarrow & \tilde{M} & \xrightarrow{\nu} & E & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

where N is now a Φ-module via θ. Let $\mathcal{M}'$ and $\mathcal{M} \to \mathcal{M}'$ be the crossed module and morphism of crossed modules constructed from the quotient mapping $A \to A/N$ as in Proposition 6.1. Let $k\colon C\Phi \to \xi\mathcal{N}$ be a realisation of the identity morphism on Φ. Then the composite $C\Phi \to \xi\mathcal{N} \to \zeta\mathcal{N}$ realises $k(\mathcal{N}, \mathrm{id})$. Clearly the composition

$$C\Phi \to \xi\mathcal{N} \to \zeta\mathcal{N} \to \zeta\mathcal{M} \to \zeta\mathcal{M}'$$

realises the zero class in $H^3_\theta(\Phi, A/N)$, as required.

Suppose conversely that $k(\mathcal{M}, \theta)$ maps to zero in $H^3_\theta(\Phi, A/N)$. Again, let $\mathcal{M}'$ be the crossed module constructed in Proposition 6.1, with morphism $\phi' : M \to M'$. Then, by assumption, the obstruction class $k(\mathcal{M}', \theta)$ is zero, and so there is an extension of type $\mathcal{M}'$ and with abstract kernel θ

$$1 \longrightarrow M' \xrightarrow{i'} E \longrightarrow \Phi \longrightarrow 1$$

It is easy to check that $\nu = i'\phi' \colon M \to E$ becomes a crossed module when E acts on M via σ, and that $Ker\nu = Ker\phi' = N$. Hence we have the following morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{\nu} & E & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\mu} & P & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

The morphism of crossed modules this includes can be realised by a covering morphism of group-groupoids and so of topological groups as required. □

Example 6.4 We mention some nice examples of Taylor [20]. He shows there are exactly three non-isomorphic topological group extensions of $SO(2)$ by $\mathbf{Z_2}$, namely the direct sum of the two groups, the orthogonal group $O(2)$, and finally the multiplicative group of all quaternions $a + bi + cj + dk$, of norm 1, such that $(a^2 + b^2)(c^2 + b^2) = 0$. Other examples of non-connected coverings of topological groups are given in section 8 of [20].

This completes our account of the theory of covering groups of topological groups.

Of course these theorems on spaces have analogues for group-groupoids which we leave the reader to state.

Acknowledgements

We would like to thank: Kirill Mackenzie and Johannes Huebschmann for helpful comments and conversations, and a referee for many helpful comments. The second author would like to thank the Turkish Government for support during his studies at Bangor.

References

[1] A.J. Berrick, *Group extensions and their trivialisation,* L'Enséignement Mathématique, t.31, 151-172, 1985.

[2] R. Brown, *Fibrations of groupoids*, J. Algebra 15, 103-132, 1970.

[3] R. Brown, *Topology: a geometric account of general topology, homotopy types and the fundamental groupoid*, Ellis Horwood, Chichester, Prentice Hall, New York, 1988. new edition as *Topology and Groupoids*, Booksurge PLC, S. Carolina, 2006.

[4] R. Brown and P.J. Higgins, *Crossed complexes and non-abelian extensions*, in Category Theory Proceedings, Gummersbach, 1981, Lecture Notes in Math. 962, edited K.H.Kamps *et al*, Springer, Berlin, 39-50, 1982.

[5] R. Brown and P.J. Higgins, *Homotopies and tensor products for ω-groupoids and crossed complexes*, J. Pure Appl. Algebra, 47, 1-33, 1987.

[6] R. Brown and P.J. Higgins, *The classifying space of a crossed complex*, Math. Proc. Camb. Phil. Soc., 110, 95-120, 1991.

[7] R.Brown and J. Huebschmann, *Identities among relations*, in *Low-Dimensional Topology*, edited R. Brown and T. L. Thickstun, London Math. Soc. Lecture Notes 48, Cambridge University Press, 153-202, 1982.

[8] R. Brown and K. Mackenzie, *Determination of a double Lie groupoid by its core diagram*, J. Pure App. Algebra, 80, 237-272, 1992.

[9] R. Brown and T. Porter, "*On the Schreier theory of non-abelian extensions: generalisations and computations*". Proceedings Royal Irish Academy 96A (1996) 213-227.

[10] R. Brown and C.B Spencer, *$\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group*, Proc. Konn. Ned. Akad. v. Wet., 79, 296-302, 1976.

[11] P. Dedecker, *Les foncteurs* Ext_Π, H^2_Π *et* H^2_Π *non abéliens* , C.R. Acad. Sci. Paris, 258, 4891-4894, 1964.

[12] P. J. Higgins, *Categories and groupoids*, Van Nostrand, 1971. Reprints in Theory and Applications of Categories, No. 7 (2005) pp 1-195.

[13] J.Howie, *Pullback functors and fibrations of crossed complexes*, Cah. Top. Géom. Diff. Cat., 20, 284-296, 1979.

[14] J.Huebschmann, *Crossed n-fold extensions and cohomology*, Comm. Math. Helv., 55, 302-314, 1980.

[15] J. Huebschman, *Holonomies of Yang-Mills connections for bundles on a surface with disconnected structure group*,Math. Proc. Cambr. Phil. Soc, 116 (1994), 375-384. .

[16] K.Mackenzie, *Classification of principal bundles and Lie groupoids with prescribed gauge group bundle*, J. Pure Appl. Algebra, 58, 181-208, 1989.

[17] S. Mac Lane, *Homology*, Springer-Verlag, Berlin, 1963.

[18] O. Mucuk, *Covering groups of non-connected topological groups, and the monodromy groupoid of a topological group*, University of Wales PhD Thesis, Bangor, 1993.

[19] R.L. Taylor, *Compound group extensions* I, Trans. Amer. Math. Soc. 75, 106-135, 1953.

[20] R.L. Taylor, *Covering groups of non connected topological groups*, Proc. Amer. Math. Soc. 5, 753-768, 1954.

[21] J.H.C. Whitehead, *Combinatorial homotopy* II, Bull. Amer. Math. Soc. 55, 453-496, 1949.