Some strict higher homotopy groupoids: intuitions, examples, applications, prospects. CT2010, Genoa

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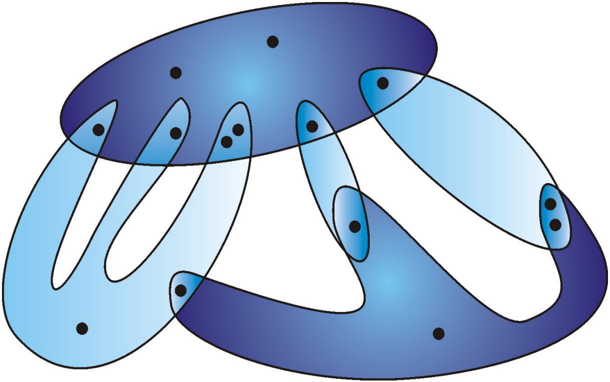
Origin of these ideas: van Kampen theorem for the fundamental groupoid on a set of base points:

$$\begin{array}{ccc} \pi_1(W, W_0) & \longrightarrow & \pi_1(U, W_0) \\ \downarrow & & \downarrow \\ \pi_1(V, W_0) & \longrightarrow & \pi_1(X, W_0) \end{array}$$

Pushout of groupoids if $X = IntU \cup IntV, W = U \cap V$ $W_0 \subseteq W$ meets each path component of W

This allows the complete computation of $\pi_1(X, x)$ as a small part of the larger structure $\pi_1(X, W_0)$.

Such computation involves choices and may not be algorithmic.



This success is contrary to the general philosophy of homological algebra.

Nonabelian cohomology yields only exact sequences.

It seems the success is because

groupoids have structure in dimensions 0 and 1

and so can model the geometry of the interactions of W_0, W, U, V allowing integration of homotopy 1-types.

Can one do analogous things in higher dimensions using homotopically defined objects with structure in dimensions $0, 1, \ldots, n$?

Can there be homotopy invariants with universal properties in dimensions > 1?

Clue: Whitehead's Theorem (1941-1948):

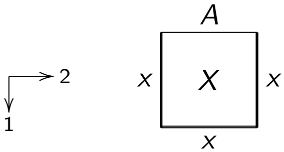
$$\pi_2(A \cup \{e^2_\lambda\}, A, x) \to \pi_1(A, x)$$

second relative homotopy group of A union 2-cells is a free crossed $\pi_1(A, x)$-module.

This freeness looks like a universal property in dimension 2!

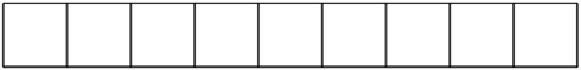
What are the 2nd relative homotopy groups

$$\pi_2(X, A, x) \to \pi_1(A, x)?$$



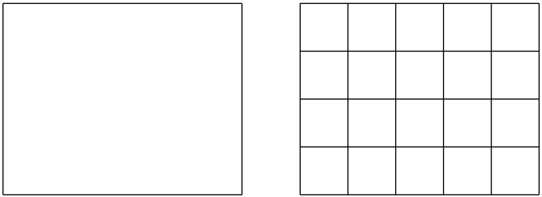
where thick lines show constant paths.

Compositions are as follows:



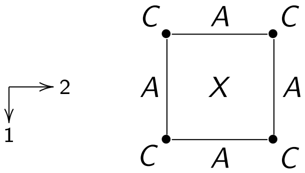
Whole construction involves choices, which is unaesthetic.

Consider the figures:



From left to right gives subdivision.
From right to left should give composition.
What we need for local-to-global problems is:
Algebraic inverses to subdivision.
We know how to cut things up, but how to control algebraically putting them together again?

Brown-Higgins 1974 $\rho_2(X, A, C)$:
homotopy classes rel vertices of maps $[0,1]^2 \to X$
with edges to A and vertices to C

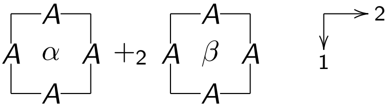


$$\rho_2(X, A, C) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \pi_1(A, C) \;\substack{\longrightarrow\\\longrightarrow}\; C$$

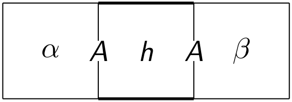
Childish idea: glue two squares if and only if the right side of one is the same as the left side of the other. These algebraic compositions are defined under geometric conditions.
That is my definition of higher dimensional algebra.

We would like to make a horizontal composition of classes:

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle$$



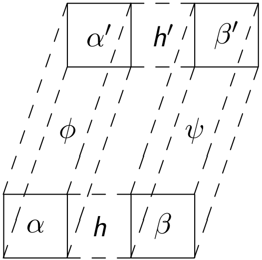
But the condition for the composition $+_2$ to be defined on classes in ρ_2 gives at least one homotopy h in A.
So we can form



where thick lines show constant paths, and define

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha +_2 h +_2 \beta \rangle\!\rangle$$

To show $+_2$ well defined, let $\phi : \alpha \equiv \alpha'$ and $\psi : \beta \equiv \beta'$, and let $\alpha' +_2 h' +_2 \beta'$ be defined. We get a picture in which dash-lines denote constant paths.



Can you see why the middle 'hole' can be filled appropriately? Thus $\rho(X, A, C)$ has in dimension 2 compositions in directions 1,2 satisfying the interchange law and is a double groupoid, containing as a substructure $\pi_2(X, A, x), x \in C$ and $\pi_1(A, C)$.

One needs extra structure of connections, or thin structure:

double groupoids (with connection)	$\simeq$	crossed modules over groupoids
$\rho(X, A, C)$ as double groupoid	$\simeq$	$\pi_2(X, A, C) \to \pi_1(A, C)$
van Kampen theorem for the double groupoid $\rho(X, A, C)$	$\simeq$	van Kampen theorem for the crossed module over groupoid $\pi_2(X, A, C)$

So you can calculate some nonabelian crossed modules, i.e. some homotopy 2-types!

Calculation of the corresponding $\pi_2(X, x)$ may be tricky!

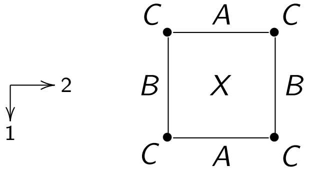
Higher dimensions?
Category FTop of filtered spaces:

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X$$

Example: n-cube I^n_*
$(RX_*)_n = \mathrm{FTop}(I^n_*, X_*)$
RX_* = cubical set with connections and compositions
$p : RX_* \to \rho X_* = (RX_*)/\equiv$
where $\equiv$ is thin homotopy, i.e. homotopy through filtered maps rel vertices of I^n
Amazing facts:
1) The natural structure on RX_* of cubical set with compositions and connections is inherited by ρX_*, the chief problem being the compositions, making ρX_* a strict ω-groupoid.
2)$p : RX_* \to \rho X_* = (RX_*)$ is a Kan fibration of cubical sets.
The last fact gives the strong link between
the lax structures on RX_* and the strict structures on ρX_*.

3) Strict cubical ω-groupoids with connections are equivalent to crossed complexes and ρX_* is thus equivalent to ΠX_* the fundamental crossed complex of the filtered space, defined using relative homotopy groups.
4) This gives a different foundation for algebraic topology whose full consequences have yet to be worked out.
Philosophy: spaces often come with structure, or are replaced by spaces with structure, so it is reasonable to base algebraic topology on spaces with structure rather than just bare spaces.

Tri-ads : $A, B \subseteq X$; set of base points $C \subseteq A \cap B$.
Consider the set $\Phi_2(X; A, B; C)$ of maps $I^2 \to X$



This forms a lax double category with the obvious compositions.
Not generally inherited by homotopy classes rel vertices.
Amazing fact: these compositions are inherited by the fundamental group

$$\pi_1(\Phi_2(X; A, B; x), \bar{x})$$

making it a
strict double groupoid internal to groups, i.e. a cat^2-group.

This generalises to $(n+1)$-ads, or even n-cubes of spaces, and so to cat^n-groups.
Strict n-fold groupoids model weak homotopy n-types, so there is still a lot to be said for studying the relations between strict and non strict structures.

Pushouts and Cubical Tricks

Suppose we have a homotopical functor Π of pairs which preserves certain pushouts of pairs of spaces- HHvKT.

If $X = A \cup B, C = A \cap B$, we get a pushout square

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

which can be turned into a pushout square of pairs

$$\begin{array}{ccc} (C, C) & \longrightarrow & (A, A) \\ \downarrow & & \downarrow \\ (B, C) & \xrightarrow[\varepsilon]{} & (X, A) \end{array}$$

where ε is the excision map. Applying Π gives an excision theorem for Π.

This is how we got a strong generalisation of Whitehead's theorem involving induced crossed modules and so bifibrations of categories. Three papers by Brown-Wensley include some group computation to do the sums: we obtain specific groups and numbers.

Suppose now we have a homotopical functor Π of squares of spaces which preserves certain pushouts of squares of spaces- HHvKT.

Consider again the first pushout square:

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

this gives rise to a new square

$$\begin{array}{ccc} \begin{matrix} C\,C \\ C\,C \end{matrix} & \longrightarrow & \begin{matrix} C\,A \\ C\,A \end{matrix} \\ \downarrow & & \downarrow \\ \begin{matrix} C\,C \\ B\,B \end{matrix} & \longrightarrow & \begin{matrix} C\,A \\ B\,X \end{matrix} \end{array}$$

which is a pushout of squares of spaces.

By applying Π to this pushout, we got the nonabelian tensor product of groups which act on each other.

Computes certain nonabelian triad homotopy groups $\pi_3(X; A, B; x)$ as built up by generalised Whitehead products from lower relative homotopy groups.

If $X = X_1 \cup X_2 \cup X_3$ we get a pushout 3-cube X_{***} of spaces.
Like to know what is excision in this situation.
But X_{***} can be regarded as a map $x : X_{-**} \to X_{+**}$ of squares, and so
as a map of squares of squares, and so
as a 3-cube of squares of spaces
which is a 3-pushout of squares of spaces!
This is how we got a totally new triadic Hurewicz Theorem, essentially conjectured by Loday, and proved as a consequence of our van Kampen theorem for n-cubes of spaces.
All these tricks extend easily to n-cubes of spaces, and the consequences have been largely unexplored, or merely scratched the surface.
Conclusion: There are some advantages in using strict higher homotopical groupoids and we know they can be defined for certain structured spaces.

Higher dimensional category theory contrasted with
higher dimensional group theory.
Prospects: Colimit theorems in applications of higher groupoids to algebraic topology, algebraic geometry, algebraic number theory.!!!???
Grothendieck: Extract from Letter 02.05.1983
Don't be amazed at my supposed efficiency in digging out the right kind of notions- I have just been following, rather let myself be pulled along, by that very strong thread (roughly: understand noncommutative cohomology of topoi!) which I kept trying to sell for about ten or twenty years now, without anyone ready to "buy" it, namely, to do the work. So finally I got mad and decided to work out at least an outline by myself.
Yours very cordially,
Alexander