Homotopy classification the J.H.C. Whitehead way*

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Introduction

Almost 40 years ago J.H.C. Whitehead showed in [11] that, for connected CW-complexes X, Y with dim $X \leqslant n$ and $\pi Y = 0$ for $2 \leqslant i \leqslant n-1$, the homotopy classification of maps $X \to Y$ can be reduced to a purely algebraic problem of classifying, up to an appropriate notion of homotopy, the π_1-equivariant chain homomorphisms $C_*\widetilde{X} \to C_*\widetilde{Y}$ between the cellular chain complexes of the universal covers. The classification of homotopy equivalences $Y \simeq Y$ can similarly be reduced to a purely algebraic problem. Moreover, the algebra of the cellular chains of the universal covers closely reflects the topology, and provides pleasant and interesting exercises.

These results ought to be a standard piece of elementary algebraic topology. Yet, perhaps because of the somewhat esoteric exposition given in [11], and perhaps because of a lack of worked examples, they have remained largely ignored. The purpose of the present paper is to rectify this situation. We shall show the utility of Whitehead's results by using them to give new and clearer treatments of various known classifications.

In Section 1 we recall the relevant definitions and theorems from [11], and then in Section 2 we use the notions of equivariant cohomology and twisted degree to obtain three fairly general classification results as corollaries. In the subsequent sections these three corollaries are applied to specific examples: in Section 3, 4 and 6 we give a succinct algebraic account of work of P. Olum [8] on maps between n-manifolds, including maps of lens spaces and maps from a surface to the projective plane; in Section 5 we recover a classification of generalized lens spaces which was first obtained by S. Jajodia in [6].

In view of the ease with which Whitehead's methods handle the classifications of Olum and Jajodia, it is surprising that the papers [8] and [6] (both of which were written after the publication of [11]) make respectively no use, and so little use, of [11].

We note here that B. Schellenberg, who was a student of Olum, has rediscovered in [9] the main classification theorems of [11]. The paper [9] relies heavily on earlier work of Olum.

The present paper is a revised version of part of my M.Sc. dissertation [5]. I would like to thank Professor R. Brown for suggesting that this work be carried out, and for his permission to reproduce in Section 6 the calculations of [3].

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1 Whitehead's results

All spaces are assumed to be connected CW-complexes with a 0-cell chosen as base-point, and all maps are assumed to preserve base-points.

The universal cover $\widetilde{X}$ of a space X has a canonical CW-structure for which the projection of $p : \widetilde{X} \to X$ is cellular. For each k-cell e^k of X we can choose a "preferred" k-cell of $\widetilde{X}$ which covers e^k, the base-point of $\widetilde{X}$ being the preferred 0-cell covering the base-point of X. We shall abuse notation slightly and denote the preferred k-cell also by e^k.

There is a left action of $\pi_1 X$ on $\widetilde{X}$, and this induces a left $\pi_1 X$-action on the chain group $C_k\widetilde{X} = H_k(\widetilde{X}^k, \widetilde{X}^{k-1})$. This group can be regarded as the free $\mathbb{Z}[\pi_1 X]$-module with basis the preferred k-cell e^k of $\widetilde{X}$; and the boundary homomorphism ∂_k: $C_k\widetilde{X} \to C_{k-1}\widetilde{X}$ can be regarded as a $\mathbb{Z}[\pi_1 X]$-module homomorphism.

Any cellular map of spaces $f : X \to Y$ determines a unique base-point preserving lift $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$; and a homomorphism $\pi_1 f : \pi_1 X \to \pi_1 Y$ which in turn yields a homomorphism $\pi_1 f : \mathbb{Z}[\pi_1 X] \to \mathbb{Z}[\pi_1 Y]$; and a $\pi_1 f$-equivariant chain homomorphism $f_* : C_*\widetilde{X} \to C_*\widetilde{Y}$ which sends the base-point in $C_0\widetilde{X}$ to the base-point in $C_0\widetilde{Y}$. For any homomorphism $\theta : \pi_1 X \to \pi_1 Y$ we denote by $(X, Y)^\theta$ the set of pointed cellular maps from X and Y over θ, and we denote by $(C_*\widetilde{X}, C_*\widetilde{Y})^\theta$ the set of θ-equivariant base-point preserving chain homomorphisms from $C_*\widetilde{X}$ to $C_*\widetilde{Y}$. Thus there is a function

$$(X, Y)^\theta \to (C_*\widetilde{X}, C_*\widetilde{Y})^\theta.$$

Now two base-point preserving chain homomorphisms $f_*, g_* : C_*\widetilde{X} \to C_*\widetilde{Y}$, where f_* is θ-equivariant and g_* is θ'-equivariant say, are *free homotopic* (written $f_* \simeq g_*$) if and only if there is an element $w \in \pi_1 Y$ and a family of θ-equivariant homomorphisms $\mu_k : C_k\widetilde{X} \to C_{k+1}\widetilde{Y}$ for $k \geqslant 0$, such that

$$wg_k - f_k = \partial_{k+1}\mu_k + \mu_{k-1}\partial_k \quad (k \geqslant 0 \text{ and } \mu_{-1}\partial_0 = 0).$$

If w can be chosen to be the identify element of $\pi_1 Y$ then f_* and g_* are said to be *based homotopic* (and we write $f_* \underset{\cdot}{\simeq} g_*$). It is easily checked that both $\simeq$ and $\underset{\cdot}{\simeq}$ are equivalence relations.

Recall that two homomorphisms $\theta, \phi : \pi_1 X \to \pi_1 Y$ are *conjugate* if there is an element $w \in \pi_1 Y$ such that $\theta x = w(\phi x)w^{-1}$ for all $x \in \pi_1 X$. We shall denote the conjugacy class of θ by $[\theta]$, and the set of conjugacy classes of homomorphisms $\pi_1 X \to \pi_1 Y$ by $[\pi_1 X, \pi_1 Y]$.

We denote by $[X, Y]$ the set of free homotopy (i.e.base-point allowed to traverse a loop) classes of maps $X \to Y$, and by $[X, Y]_{\cdot}$ the set of based homotopy (i.e. base-point fixed) classes of maps $X \to Y$. Similarly we denote by $[C_*\widetilde{X}, C_*\widetilde{Y}]$ and $[C_*\widetilde{X}, C_*\widetilde{Y}]_{\cdot}$ the set of free and based homotopy classes of equivariant base-point preserving chain homomorphisms $C_*\widetilde{X} \to C_*\widetilde{Y}$. There are obvious quotient maps

$$q_0 : [X, Y] \to [\pi_1 X, \pi_1 Y], \qquad q'_0 : [X, Y]_{\cdot} \to [\pi_1 X, \pi_1 Y]$$
$$q_1 : [C_*\widetilde{X}, C_*\widetilde{Y}] \to [\pi_1 X, \pi_1 Y], \quad q'_1 : [C_*\widetilde{X}, C_*\widetilde{Y}]_{\cdot} \to [\pi_1 X, \pi_1 Y].$$

For any homomorphism $\theta : \pi_1 X \to \pi_1 Y$ we write

$$[X, Y]^\theta = q_0^{-1}[\theta], \qquad [X, Y]^\theta_{\cdot} = {q'}_0^{-1}[\theta],$$
$$[C_*\widetilde{X}, C_*\widetilde{Y}]^\theta = q_1^{-1}[\theta], \quad [C_*\widetilde{X}, C_*\widetilde{Y}]^\theta_{\cdot} = {q'}_1^{-1}[\theta].$$

The above function $(X,Y)^\theta \to (C_*\widetilde{X}, C_*\widetilde{Y})^\theta$ induces functions

$$[X,Y]^\theta \to [C_*\widetilde{X}, C_*\widetilde{Y}]^\theta \quad (*)$$

$$[X,Y]^\theta_{\cdot} \to [C_*\widetilde{X}, C_*\widetilde{Y}]^\theta_{\cdot} \quad (**)$$

Theorem 1.1 *(J.H.C. Whitehead [11]). Let X,Y be reduced CW-complexes such that dim $X \leqslant n$ and $\pi_i Y = 0$ for $2 \leqslant i \leqslant n-1$. Let $\theta : \pi_1 X \to \pi_1 Y$ be any homomorphism. Then the functions (*) and (**) are bijections. Moreover, these bijections respect free and based homotopy equivalences.*

This theorem follows immediately from Theorems 5, 6, 7, 9, 10 and 12 in [11]. An outline of an alternative proof can be found in [4].

2 Equivariant cohomology and twisted degree

Throughout this section X and Y are n-dimensional $(n \geqslant 2)$ reduced CW-complexes with $\pi_i Y = 0$ for $2 \leqslant i \leqslant n-1$.

Let $\theta : \pi_1 X \to \pi_1 Y$ be a homomorphism. Then the group $\pi_1 X$ acts on $\pi_n Y$ via θ and the standard action of $\pi_1 Y$ on $\pi_n Y$. We denote the θ-equivariant cohomology of $\widetilde{X}$ with coefficients in $\pi_n Y$ by $H^*(\widetilde{X}, \pi_n Y, \theta)$. This cohomology can equivalently be thought of as the cohomology of X with local coefficients in $\pi_n Y$. (See for example [10] for a general discussion on equivariant cohomology.)

Corollary 2.1 *There is a (non-canonical) bijection*

$$[X,Y]^\theta_{\cdot} \cong H^n(\widetilde{X}, \pi_n Y, \theta).$$

There is also a (non-canonical) surjection

$$H^n(\widetilde{X}, \pi_n Y, \theta) \to [X,Y]^\theta. \qquad \square$$

This corollary is obtained from the above theorem of Whitehead using the following lemma.

Lemma 2.2 *If $f_*, g_* : C_*\widetilde{X} \to C_*\widetilde{Y}$ are respectively θ-, θ'-equivariant chain homomorphisms such that $[\theta] = [\theta']$, then there exists a θ-equivariant chain homomorphism $g_* : C_*\widetilde{X} \to C_*\widetilde{Y}$ such $g'_k = f_k$ for $0 \leqslant k \leqslant n-1$ and $g'_* \simeq g_*$.* $\square$

The proof of the corollary and lemma involves only standard arguments of homological algebra, and is left to the reader.

By the *twisted degree* $\deg(f)$ of a map $f : X \to Y$ we shall mean the induced homomorphism of equivariant cohomology groups

$$\deg(f) : H^n(\widetilde{Y}, \pi_n Y, \mathrm{id}) \to H^n(\widetilde{X}, \pi_n Y, \pi_1 f).$$

When X and Y are n-manifolds (see section 3) this is exactly the notion of twisted degree which was introduced in [8].

Now let $\theta = \pi_1 f$, and for a given $w \in \pi_1 Y$ let $\phi = w(\pi_1 f)w^{-1} = \pi_1 X \to \pi_1 Y$. Then there is an isomorphism

$$\bar{w} : H^n(\widetilde{X}, \pi_n Y, \theta) \cong H^n(\widetilde{X}, \pi_n Y, \phi)$$

which maps the element represented by $\alpha \in \mathrm{Hom}_\theta(C_n\widetilde{X}, \pi_n Y)$ to the element represented by $w\alpha \in \mathrm{Hom}_\phi$ $(C_n\widetilde{X}, \pi_n Y)$.

If two maps $f, g : X \to Y$ are homotopic, with $w \in \pi_1 Y$ the element involved in the corresponding chain homotopy $f_* \simeq g_*$ (so $\pi_1 f = w(\pi_1 g)w^{-1}$), then it is readily seen that

$$\deg(f) = \bar{w}\deg(g).$$

Thus we have a necessary condition for two maps to be homotopic. The following corollary to the above theorem of Whitehead gives necessary and sufficient conditions. Again, the proof contains no surprises and is left to the reader.

Corollary 2.3 *Let $f, g : X \to Y$ be maps such that:*
i) the homomorphism

$$H^n(\widetilde{Y}, \pi_n Y, \mathrm{id}) \to \mathrm{Hom}_{\mathrm{id}}(H_n\widetilde{Y}, \pi_n Y)$$

which is induced by the inclusion $H_n\widetilde{Y} \to C_n\widetilde{Y}$ has an injective homomorphism in the image;
ii) the surjection

$$\mathrm{Hom}_{\pi_1 f}(C_n\widetilde{X}, \pi_n Y) \to H^n(\widetilde{X}, \pi_n Y, \pi_1 f)$$

is also injective.

Then $f \simeq g$ if and only if there is an element $w \in \pi_1 Y$ such that $\pi_1 g = w^{-1}(\pi_1 f)w$ and $\deg(f) = \bar{w}\deg(g)$.

Also, $f \mathrel{\dot{\simeq}} g$ if and only if $\pi_1 f = \pi_1 g$ and $\deg(f) = \deg(g)$. □

Finally in this section we give a result for calculating all the possible degrees of maps $X \to Y$.

Note that any θ-equivalent homomorphism $\alpha : C_n\widetilde{X} \to H_n\widetilde{Y}$ induces a homomorphism

$$\bar{\alpha} : H^n(\widetilde{Y}, \pi_n Y, \mathrm{id}) \to H^n(\widetilde{X}, \pi_n Y, \theta);$$

$\bar{\alpha}$ maps an n-cocycle $\phi : C_n\widetilde{Y} \to \pi_n Y$ to the n-cocycle $\phi x : C_n\widetilde{X} \to \pi_n Y$.

Corollary 2.4 *Let $f : X \to Y$ be a map with $\pi_1 f = \theta$ say. Then the degrees of the maps $X \to Y$ including the homomorphism θ on fundamental groups are precisely the homomorphisms*

$$\deg(f) + \bar{\alpha} : H^n(\widetilde{Y}, \pi_n Y, \mathrm{id}) \to H^n(\widetilde{X}, \pi_n Y, \theta)$$

with $\alpha \in \mathrm{Hom}_\theta(C_n\widetilde{X}, H_n\widetilde{Y})$. □

Again the proof contains no surprises and is left to the reader.

3 Maps of n-manifolds

Throughout this section X and Y are n-manifolds $(n \geqslant 2)$ with a reduced CW-structure, and $\pi_i Y = 0$ for $2 \leqslant i \leqslant n-1$.

We shall use the corollaries obtained in Section 2 to recover some classification results in [8].

Now the elements of $\pi_1 X$ can be divided into two classes, those whose representative paths are *orientation-preserving*, and those represented by *orientation reversing* paths, with the obvious meanings. The product of two elements is orientation-reversing if and only if exactly one of the elements is orientation-preserving. A homomorphism $\theta : \pi_1 X \to \pi_1 Y$ is *orientation-true* if it respects both orientation-preserving and orientation-reversing elements. If θ is not orientation-true then it is *orientation-false*.

Let us recall the following proposition of Olum [8, 2.1] on the equivariant cohomology of $\widetilde{X}$. The proof is analogous to its counterpart in ordinary cohomology.

Proposition 3.1 *Suppose Y is compact, and $\theta : \pi_1 X \to \pi_1 Y$ is any homomorphism. Then*

$$H^n(\widetilde{X}, \pi_n Y, \theta) = \begin{cases} 0 & \text{if } X \text{ is not compact,} \\ \mathbb{Z} & \text{if } X \text{ is compact and } \theta \text{ is orientation-true,} \\ \mathbb{Z}_2 & \text{if } X \text{ is compact and } \theta \text{ is orientation-false.} \end{cases}$$

□

Theorem 3.2 (Olum [8]). *Suppose either X or $\widetilde{Y}$ is not compact. Then two maps $f, g : X \to Y$ are free homotopic if and only if $[\pi_1 f] = [\pi_1 g]$. Furthermore, every homomorphism $\pi_1 X \to \pi_1 Y$ is induced by some map $X \to Y$.*

Proof If f and g are homotopic then the induced chain homomorphisms f_* and g_* are also homotopic. It is a simple algebraic exercise to deduce from the above definition of a chain homotopy that if $f_* \simeq g_*$ then $[\pi_1 f] = [\pi_1 g]$.

If $\widetilde{Y}$ is not compact then $\pi_n Y = H_n \widetilde{Y} = 0$ and so the result is very well known. Alternatively it follows from the preceding paragraphs and the surjection of Corollary 2.1.

If $\widetilde{X}$ is not compact then the result follows from Proposition 3.1 and the surjection of Corollary 2.1. □

Theorem 3.3 (Olum [8]). *Suppose both X and $\widetilde{Y}$ are compact. If $\theta : \pi_1 X \to \pi_1 Y$ is an orientation-true homomorphism then there is a bijection.*

$$[X, Y]^{\theta}_{\cdot} \cong \mathbb{Z}.$$

If θ is orientation-false then there is a bijection

$$[X, Y]^{\theta}_{\cdot} \cong \mathbb{Z}_2.$$

Proof The bijections follow from Corollary 2.1 and Proposition 3.1. □

In this section, the twisted degree of a map $X \to Y$ is simply (multiplication by) an integer. Also, the fundamental group $\pi_1 Y$ is finite. We shall denote its order by $|\pi_1 Y|$.

In the following two theorems we shall assume that X and Y have just a single n-cell (and are thus compact). This assumption is made to simplify the proofs. □

Theorem 3.4 (Olum [8]). *Suppose $\widetilde{Y}$ is compact and that X and Y each have just a single n-cell. Let $f,\ g : X \to Y$ be maps with $\pi_1 f$ orientation-true. Then f and g are free homotopic if and only if $\pi_1 f = w(\pi_1 g)w^{-1}$ for some $w \in \pi_1 Y$ and $\deg(f) = +\deg(g)$ or $-\deg(g)$ according as w is orientation-preserving or orientation-reversing.*

Furthermore, the degrees of the maps $X \to Y$ inducing $\pi_1 f$ on fundamental groups are precisely the integers $\deg(f) + k|\pi_1 Y|$ with $k = 0, \pm 1, \pm 2, \ldots$

Proof Let PY be the subgroup of $\pi_1 Y$ of index 1 or 2, consisting of the orientation-preserving elements, and let $RY = \pi_1 Y \setminus PY$. Let $NY \in \mathbb{Z}[\pi_1 Y]$ be group-ring element

$$NY = \sum_{x \in PY} x - \sum_{y \in RY} y.$$

Thus for $x \in PX$, $y \in RY$ we have

$$xNY = NY \quad \text{and} \quad yNY = -NY.$$

Let e'^n be the single n-cell of Y. Then the homology group $H_n\widetilde{Y}$ is the subgroup of $C_n\widetilde{Y}$ consisting of all integer multiples of NYe'^n.

The last assertion of the theorem follows easily from Corollary 2.4. The first assertion will follow from Corollary 2.3 once we have proved that the hypotheses of Corollary 2.3 are satisfied.

Since every non-trivial homomorphism $H_n\widetilde{Y} = \mathbb{Z} \to \pi_n Y = \mathbb{Z}$ is injective, hypothesis (i) is certainly satisfied.

To see that (ii) is satisfied, let $\bar{X}$ be the compact orientable cover of X with $\pi_1\bar{X} = PX$, the group of orientation preserving elements of $\pi_1 X$. Let e^n be the n-cell of X. Then the homology group $H_n\bar{X}$ is the subgroup of $C_n\bar{X}$ generated by $(1-\bar{y})e^n$ where $\bar{y} \in \pi_1 X : PX$ is the element represented by a $y \in RX$ (if $RX = \emptyset$ then $\bar{y} = 0$). Let $q_* : C_*\widetilde{X} \to C_*\bar{X}$ be the canonical quotient map. Now any $\pi_1 f$-equivariant homomorphism $\phi : C_{n-1}\widetilde{X} \to \pi_n Y$ induces an equivariant homomorphism $\bar{\phi} : C_{n-1}\bar{X} \to \pi_n Y$ such that $\phi = \bar{\phi} q_{n-1}$. Considering e^n now to be an element of $C_n\widetilde{X}$ we have

$$\begin{aligned}
\phi\partial_n((1-y)e^n) &= \bar{\phi} q_{n-1}\partial_n((1-y)e^n) \\
&= \bar{\phi}\partial_n q_n((1-y)e^n) \\
&= \bar{\phi}\partial_n((1-\bar{y})e^n) \\
&= \bar{\phi}(0) \\
&= 0.
\end{aligned}$$

But also, since $\pi_1 f$ is orientation-true.

$$\begin{aligned}\phi\partial_n((1-y)e^n) &= (1-\pi_1 fy)\phi\partial_n e^n\\ &= 2\phi\partial_n e^n.\end{aligned}$$

It follows that $\phi\partial_n e^n = 0$, and hence hypothesis (ii) is satisfied. □

The modifications needed to obtain based homotopy versions of Theorems 3.2 and 3.3 are easy and are left to the reader. □

Theorem 3.5 (Olum [8]). *Suppose $\widetilde{Y}$ is compact, that X, Y have just a single n-cell, and that $\pi_i X = \pi_i Y = 0$ for $2 \leqslant i \leqslant n-1$. If $f : X \to Y$ is a map with $\pi_1 f = \theta$ say, then there exists a based homotopy equivalence $X \to Y$ inducing θ if only and only if θ is an orientation-true isomorphism and* $\deg(f) \equiv \pm 1 \text{ modulo } |\pi_1 Y|$.

Proof Suppose $g : X \to Y$ is a based homotopy equivalence with $\pi_1 g = \theta$, and with based homotopy inverse $g' : X \to Y$. Then θ is an isomorphism. Now if $\deg(g) = m$ and $\deg(g') = m'$, then $\deg$(identity)$= 1$ and hence $\deg(g) = \pm 1$. Thus by Theorem 3.4 $\deg(f) \equiv \pm 1 \text{ modulo } |\pi_1 Y|$. Since the identity homomorphism $\deg(g'g)$ factors through $H^n(\widetilde{X}, \pi_n, \theta)$, it follows from Proposition 3.1that θ must be orientation-true.

Conversely suppose θ is an orientation-true isomorphism and $\deg(f) \equiv \pm 1 \text{ modulo } |\pi_1 Y|$. Then, by Theorem 3.4 there is a map $g : X \to Y$ with $\pi_1 g = \theta$ and $\deg(g) = \pm 1$. Let $g' : Y \to X$ be any map with $\pi_1 g' = \theta^{-1}$. Then $\pi_1(g'g) =$ identity and hence, by Theorem 3.4, $\deg(g'g) \equiv 1 \text{ modulo } |\pi_1 Y|$. If $\deg(g') = m'$ then $1 \equiv \deg(g'g) = m'(\pm 1) \text{ modulo } |\pi_1 Y|$. Therefore $m' \equiv \pm 1 \text{ modulo } |\pi_1 Y|$ and, by Theorem 3.4 we can choose g' such that $m' = \pm 1$. Then $\deg(g'g) = \deg(gg') = 1$ and the result follows from Theorem 3.4. □

4 Lens spaces

In this section we recover a result of Olum [8] on the homotopy type of lens spaces.

A $(2n-1)$-dimensional $(n \geqslant 1)$ lens space L is determined by an integer $m \geqslant 3$ and integers $q_1, \ldots, q_n$ each coprime to m (see for example [8]); we write $L = L(m, q_1, \ldots, q_n)$. The fundamental group $\pi_1 L$ is the cyclic group $\mathbb{Z}_m$ of order m, and we denote one of its generators by x. The space L can be given a CW-structure with one cell e^k in each dimension k. The chain group $C_k\widetilde{L}$ is then the free $\mathbb{Z}[\mathbb{Z}_m]$-module on e^k, and the boundary homomorphisms $\partial_{k+1} : C_{k+1}\widetilde{\widetilde{L}} \to C_k\widetilde{L}$ are given by

$$\begin{aligned}2\partial_{2k}(e^{2k}) &= (1 + x + x^2 + \cdots + x^{m-1})e^{2k-1} \qquad & 1 \leqslant k \leqslant n,\\ \partial_{2k+1}(e^{2k+1}) &= (x^{(\bar{q}k)} - 1)e^{2k} & 0 \leqslant k \leqslant n\end{aligned}$$

where $1 \leqslant \bar{q}_k \leqslant m$ such that $q_k\, \bar{q}_k \equiv 1 \text{ modulo } m, \bar{q}_0 = 1$.

Let $L' = L(m', q'_1, \ldots q'_n)$ be another lens space with cells denoted by e'^k and with fundamental group $\mathbb{Z}_{m'}$ generated by an element y.

For each integer $p, 1 \leqslant p \leqslant m'$, such that $pm \equiv 0$ modulo m', there is a homomorphism $\mathbb{Z}_m \to \mathbb{Z}_{m'}$, $x \to y^p$, and this exhausts the possible homomorphisms.

Theorem 4.1 Olum [8]. *Let p be an integer such that there is a homomorphism $\theta_p : \mathbb{Z}_m \to \mathbb{Z}_{m'}$. The (twisted) degrees of the maps $L \to L'$ inducing θ_p are precisely those integers congruent to*

$$p(pm/m')^n q_1' q_2' \dots q_n' \bar{q}_1 \dots \bar{q}_n$$

modulo m'. *There is a homotopy equivalence $L \to L'$ inducing θ_p if and only if $m = m'$ and*

$$q_1 q_2 \dots q_n \equiv \pm p^{n+1} q_1' \dots q_n' \quad \text{modulo } m'$$

for some integers p comprime to m.

Proof Let us simplify notation by setting

$$\sigma_j(w) = 1 + w + w^2 + \dots + w^{j-1}$$

for $w \in \pi_1 Y$. For a fixed integer p such that $1 \leqslant p \leqslant m'$ and $pm \equiv 0$ modulo m', let $f_* : C_*\widetilde{L} \to C_*\widetilde{L}'$ be a base-point preserving θ_p-equivariant chain homomorphism given by

$$f_k(e^k) = F_k e'^k$$

say, where $F_k \in \mathbb{Z}[\mathbb{Z}_{m'}]$. Since f_* is base-point preserving we have

$$F_0 = 1 \qquad \text{(i)}$$

and the equations $f_k \partial_{k+1} = \partial_{k+1} f_{k+1}$ are equivalent to

$$F_{2k+1}(y^{(\bar{q}'k)} - 1) = F_{2k}(y^{p\bar{q}k} - 1). \qquad \text{(ii)}$$

$$F_{2k+2}\sigma_{m'}(Y) = F_{2k+1}\sigma_m(y^p) \qquad \text{(iii)}$$

Multiplying both sides of (ii) by $\sigma_{p(\bar{q}k)(qk)}(y^{(\bar{q}k)})$ we obtain the equivalent equations

$$F_{2k+1} \equiv F_{2k}\sigma_{p(\bar{q}k)(qk)}(y^{(\bar{q}k)}) \quad \text{modulo } \sigma_{m'}(y) \qquad \text{(ii)}'$$

Let $\in : \mathbb{Z}[\mathbb{Z}_{m'}] \to \mathbb{Z}$ be the augmentation map. Then (iii) is equivalent to

$$m' \in (F_{2k+2}) = m \in (F_{2k+1}) \qquad \text{(iii)}'$$

Now $\deg(f_*) = \in (F_{2n+1})$ and so, from (i), (ii)$'$ and (iii)$'$ we have

$$\begin{aligned} \deg(f_*) &= \in (F_{2n+1}) \\ &= pq_n'\bar{q}_n \in (F_{2n}) \quad \text{modulo } m' \\ &= (pm/m')q_n'\bar{q}_n \in (F_{2n-1}) \\ &\vdots \\ &\equiv p(pm/m')^n q_1' \dots q_n' \bar{q}_1 \dots \bar{q}_n \quad \text{modulo } m'. \end{aligned}$$

The theorem now follows from Theorems 3.4 and 3.5. □

5 Generalized lens spaces

In this section we recover a result of S. Jajodia [6] on the homotopy classification of generalized lens spaces.

Let Γ be a group with presentation $\langle x_1, \ldots, x_n : R^m \rangle$ where R is a word in the x_i and m is an integer. For a fixed integer q coprime to m, a *generalized lens space* $L = L(\Gamma, q)$ is obtained by attaching a 3-cell to the standard cellular model of the presentation via an attaching map representing $R^q - 1$. This space has a CW-structure with one cell e^k in dimensions $k = 0,\ 2,\ 3$, and n cells $e_1, e_2, \ldots, e_n$ in dimension 1. The chain group $C_k\widetilde{L}$ is the free $\mathbb{Z}[\Gamma]$-module on the k-cell. The boundary homomorphisms $\partial_{k+1} : C_{k+1}\widetilde{L} \to C_k\widetilde{L}$ are given by

$$\partial_1(e_1) = (x_1 - 1)e^0,$$
$$\partial_2(e^2) = (1 + R + R^2 + \cdots + R^{m-1}) \sum_{1 \leqslant i \leqslant n} (\partial R/\partial x_1)e_i$$

when $\partial R/\partial x_i$ is the Reidemeister-Fox derivative (see for example [2]),

$$\partial_3(e^3) = (R^q - 1)e^2.$$

Let $L' = L(\Gamma, q')$ be another generalized lens space, where q' is coprime to m. We identify the 2-skeleton of L' with that of L, and we denote the 3-cell of L' by e'^3.

Let us introduce a homotopy invariant $D(f)$ of a map $f : L \to$ omit $\pm L'$ which is more easily handled than the twisted degree $\deg(f)$. Suppose that the corresponding $\pi_1 f$-equivariant chain homomorphism $f_* : C_*\widetilde{L} \to C_*\widetilde{L}$ is given in dimension 3 by $f_3(e^3) = F_3 e'^3$ where $F_3 \in \mathbb{Z}[\Gamma]$. Then, letting $\in : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ denote the augmentation map, we define

$$D(f) = \in (F_3).$$

Proposition 5.1 *Two maps $f, g : L \to L'$ are free homotopic if and only $[\pi_1 f] = [\pi_1 g]$ and $D(f) = D(g)$. There is a free homotopy equivalence $L \to L'$ inducing $\pi_1 f : \Gamma \to \Gamma$ if and only if $\pi_1 f$ is an automorphism and $D(f) \equiv \pm 1$ modulo m.* □

Proof It is straightforward to check that if $f \simeq g$ then $[\pi_1 f] = [\pi_1 g]$ and $D(f) = D(g)$. The converse of this follows from the lemma in Section 2 and the fact that $H_3(\widetilde{L}')$ is generated as $\mathbb{Z}[\Gamma]$-module by $(1 + R + \cdots + R^{m-1})e'^3$.

This second claim of the proposition is proved in a similar fashion to the proof in Theorem 3.5. □

Theorem 5.2 (Jajodia [6]). *Suppose the word R is a primitive element of the free group on the x_i (i.e. R is contained in some basis of the group). Then the spaces L, L' have the same free homotopy type if and only if $\pm qq'^{\pm 1}$ is a quadratic residue* modulo m *(i.e. if and only if there is an integer t coprime to m such that $qq' \equiv \pm(tq)^2$* modulo m*).*

Proof Suppose there is a free homotopy equivalence $L \to L'$ inducing $\theta : \Gamma \to \Gamma$. Then θ is an automorphism. By Proposition 4.13 in [7], there exists a $w \in \Gamma$ and an integer t coprime to m such that $\theta R = wR^t w^{-1}$. Set $\theta' = w^{-1}\theta w$. It is easily checked that there is a θ'-equivariant homomorphism $f_* : C_*\widetilde{L} \to C_*\widetilde{L}'$ with

$$f_0(e^0) = e^0,$$
$$f_1(e_i) = \sum_{1\leqslant i\leqslant n} (\partial\theta' x_i/\partial x_j)e_j,$$
$$f_2(e^2) = \sigma_t(R)e^2,$$

where $\sigma_j(R) = 1 + R + R^2 + \cdots R^{j-1}$. It follows from the lemma in Section 2 that any θ-equivariant chain homomorphism $C_*\widetilde{L} \to C_*\widetilde{L}'$ is free homotopic to such an f_*. Now for such an f_* let us suppose that $f_3(e^3) = F_3 e'^3$ where $F_3 \in \mathbb{Z}[\Gamma]$. Then the equality $f_2\partial_3 = \partial_3 f_3$ implies

$$(R^{q'} - 1)F_3 = (R^{tq} - 1)\sigma_1(R)$$

or equivalently

$$\sigma_{q'}(R)F_3 \equiv \sigma_{tq}(R)\sigma_t(R) \quad \text{modulo } \sigma_m(R).$$

If f_* is induced by $f : L \to \widetilde{L}'$, then this last equation yields

$$q'D(f) = q'\varepsilon(F_3) \equiv t^2 q \qquad \text{modulo } m$$

or equivalently

$$qq'D(f) \equiv (tq)^2 \quad \text{modulo } m.$$

Since we have supposed the existence of an homotopy equivalence $L \to L'$, it follows from Proposition 5.1 that $qq' \equiv \pm(tq)^2$modulo m.

Conversely, suppose that there is a t coprime to m such that $qq' \equiv \pm(tq)^2$ modulo m. Since R is primitive, it is shown in Section 2 of [6] that there is an automorphism $\theta : \Gamma \to \Gamma$ such that $\theta R = wR^t w^{-1}$ for some $w\varepsilon\Gamma$. Let $f : L \to L'$ be any map over θ. From the preceding paragraph we see that $D(f) \equiv \pm 1$ modulo m. It follows from Proposition 5.1 that L and L' are free homotopy equivalent. □

6 Maps from a surface to the projective plane

In this section we recover some results of [8]. The calculations of this section are essentially those of the preprint [3] mentioned in [1].

We consider the homotopy classification of maps from a compact, connected, closed surface to the projective plane. The case of maps inducing an orientation-true homomorphism of fundamental groups is covered by

Theorem 3.5, so here we just consider the case of maps over orientation-false homomorphisms. Since the based-homotopy classification of such maps is essentially covered by Theorem 3.3, we restrict ourselves to the classification up to free homotopy.

Let P be the projective plane. It has a CW-structure with one cell e'^k in each dimension $k = 0, 1, 2$. The fundamental group $\mathbb{Z}_2$ with generator y. The chain group $C_k\widetilde{P}$ is the free $\mathbb{Z}[\mathbb{Z}_2]$-module on e'^k, and the boundary homomorphisms are given by

$$\partial_1(e'^1) = (y-1)e'^0,$$
$$\partial_2(e'^2) = (y+1)e'^1.$$

Let M be a compact, connected, closed surface. Then M has a CW-structure with a 0-cell e^0, 1-cells $e_1, e_2, \ldots, e_m$, and a 2-cell e^2. The fundamental group $\pi_1 M$ has a presentation $\langle x_1, \ldots, x_m : R\rangle$ where the x_1 correspond to the e_i. The boundary homomorphism ∂_1 is given by

$$\partial_1(e_i) = (x_1 - 1)e^0 \text{ for } 1 \leqslant i \leqslant m.$$

The boundary homomorphism ∂_2 is given by the Reidemeister-Fox derivative and depends on R. To write down ∂_2 we consider separately the orientable and non-orientable cases.

M **non-orientable**

In this case $R = x_1^2 x_2^2 \ldots x_m^2$ and ∂_2 is given by

$$\partial_2(e^2) = \sum_{1\leqslant i\leqslant m} x_1^2 x_2^2 \ldots x_{i-1}^2 (1 + x_i) e_i.$$

Theorem 6.1 (Olum [8]). *Suppose M is non-orientable and let $\theta : \pi_1 M \to \pi_1 P$ be an orientation-false homomorphism. If there are precisely an odd number k of x_i such that $\theta x_i = y$, then there is just one free homotopy class of maps $M \to P$ inducing θ. If k is even then there are two free homotopy classes of maps (corresponding to the twisted degree being even or odd).*

Proof Since θ is orientation-false we have $k < m$. Now for any two integers p and q whose sum is k, it is easily checked that there is a θ-equivariant chain homomorphism $f_* : C_*\widetilde{M} \to C_*\widetilde{P}$ given by

$$f_0(e^0) = e'^0,$$
$$f_1(e_i) = \begin{cases} e'^1 & \text{if } \theta x_i = y, \\ 0 & \text{if } \theta x_i = 1, \end{cases}$$
$$f_2(e^2) = (p+q)e'^2.$$

By the lemma in Section 2 this exhausts, up to homotopy, all the possible θ-equivariant chain homomorphisms. Let $g_* : C_*\widetilde{M} \to C_*\widetilde{P}$ be the corresponding chain homomorphism for integers $p' + q' = k$. Now if $\mu_j : C_j\widetilde{M} \to C_{j+1}\widetilde{P}$ for $j = 0, 1$ are θ-equivariant homomorphisms and w is an element of $\pi_1 P$ satisfying

$$wg_1 - f_1 = \partial_2\mu_1 + \mu_0\partial_1,$$

then clearly there exist integers s_i such that

$$\mu_0\partial_1(e_i) = s_i(y-1)e'^1 \quad (1 \leqslant i \leqslant m)$$

and hence $\partial_2\mu_1 = 0$. Thus there are integers r_i such that

$$\mu_i(e_i) = r_i(1-y)e'^2 \quad (1 \leqslant i \leqslant m)$$

It follows that

$$\begin{aligned}\mu_1\partial_2(e^2) &= \sum_{1\leqslant i\leqslant m} r_i(1+\theta x_i)(1-y)e'^2 \\ &= 2r(1-y)\end{aligned}$$

for some r.

Thus if $f_* \simeq g_*$ then there is a $w \in \mathbb{Z}_2$ and an integer r such that

$$w(p'+q'y) - (p+qy) = 2r(1-y),$$

Conversely, if there is a w and r such that this equality holds, then it is readily seen that $f_* \simeq g_*$.

If k is odd then w and r can always be found and there is just one free homotopy class of chain homomorphism. If however k is even then w and r can only be found if either p and p' are both odd or both even; there are thus two free homotopy classes of chain homomorphisms. The theorem follows from the theorem of Whitehead in Section 1. □

M orientable

In this case m is even and R is the product

$$R = [x_1, x_2][x_3, x_4]\cdots[x_{m-1}, x_m]$$

of the commutators $[x_i, x_{i+1}] = x_i x_{i+1} x_i^{-1} x_{i+1}^{-1}$. The boundary homomorphism ∂_2 is given by

$$\partial_2(e^2) = \sum_{1\leqslant i\leqslant m/2} \gamma_i(e_{2i-1} + x_{2i-1}e_{2i} - x_{2i-1}x_{2i}x_{2i-1}^{-1}e_{2i-1} - [x_{2i-1}, x_{2i}]e_{2i})$$

where γ_i is a product of commutators.

Theorem 6.2 (Olum [8]). *Suppose M is orientable and let $\theta : \pi_1 M \to \pi_1 P$ be an orientation-false homomorphism. Then there are two homotopy classes of maps $M \to P$ inducing θ (corresponding to the twisted degree being even or odd).*

Proof It is easily checked that for each integers p there is a θ-equivariant chain homomorphsm $f_* : C_*\widetilde{M} \to C_*\widetilde{P}$ given by

$$f_0(e^0) = e'^0,$$
$$f_1(e_i) = \begin{cases} e'^1 & \text{if } \theta x_i = y, \\ 0 & \text{if } \theta x_i = 1, \end{cases}$$
$$f_2(e^2) = p(1-y)e'^2.$$

By the lemma in Section 2 this exhausts up to homotopy all the possible chain homomorphisms. Let $g_* : C_*\widetilde{M} \to C_*\widetilde{P}$ be the chain homomorphism corresponding to the integers p'. Arguing as in the preceding proof one can show that $f_* \simeq g_*$ if and only if there is a $w \in \mathbb{Z}_2$ and integers r_i such that

$$(p'w - p)(1-y) = \sum_{1 \leqslant i \leqslant m/2} \{r_{2i-1}(1 - \theta x_{2i}) + r_{2i}(\theta x_{2i-1} - 1)\}(1-y).$$

Now since θ is orientation-false at least one x_i maps onto y. It follows that there are precisely two free homotopy classes of chain homomorphisms, and the theorem follows from the theorem of Whitehead in Section 1. □

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