Memory Evolutive Systems; Hierarchy, Emergence, Cognition

By A Ehresmann and J.P. Vanbremeersch

"The theory of Memory Evolutive Systems represents a mathematical model for natural open self-organizing systems, such as biological, sociological or neural systems. In these systems, the dynamics are modulated by the cooperative and/or competitive interactions between the global system and a net of internal Centers of Regulation (CR) with a differential access to a central hierarchical Memory. The MES proposes a mathematical model for autonomous evolutionary systems and is based on the Category Theory of mathematics." (from the advertisement for the book).

This is a remarkable, bold, well worked out and very welcome book, which represents a start of the modelling of natural self-organising systems by significant and relevant mathematical structures from category theory. It is also well written, with good examples from a variety of areas related to self-organising systems. The authors are a mathematician (Ehresmann) who is a long standing expert in category theory, and a physician (Vanbremeersch) who specialises in gerontology and has long been interested in explaining the complex responses of organisms to illness or senescence.

This review is written from the viewpoint of a mathematician who has an amateur's interest in the area modelled, and has worked on developing new mathematical structures for the understanding of the behaviour of certain geometric objects glued together.

The key idea is that of a mathematical structure called a category, and one which evolves with time. So something should be said about the role of category theory in mathematics.

The subject was invented by Eilenberg and Mac Lane in 1945 as a method of describing the idea of 'naturality' in mathematics. It is not useful to go into that idea now, but it led to the notions of natural transformation, for which was needed the notion of functor, for which was needed the notion of category.

The notion of category allows for a home for the notion of a 'mathematical structure'. It is expected that examples of a mathematical structure such as a group will form the 'objects' of a category. Comparisons of such examples will be by what are called morphisms of such structures. It is of interest that this is an extension of Klein's famous Erlangern Programme: study a geometry through its group of automorphisms. The simple axioms for a category specify what we expect from such morphisms: that they compose and that the compositions satisfy the simple rules of associativity and the existence of left and right identities. However, and this is a key aspect, not all such morphisms compose. The basic reason is that if you compare X and Y, and also Y and Z, then you expect to compare X and Z; further information about comparing Y' and Z will not allow you to compare X and Z, unless for example Y is the same as Y'. So we have a *partial composition*, defined under a geometric condition exactly analogous to the condition for the composition of journeys, or, as above, of comparisons. This allows for the combination of the notion of comparison with a kind of geography of the objects being compared.

Many major properties of these objects in a category are then expressed not in terms of the internal properties of the objects, but in terms of the way they interact through the morphisms (or comparisons) with all other objects. Thus an individual structure, a particular abstract group, for example, is studied in terms of its behaviour with respect to all other groups, through the morphisms between all groups. Thus the category of all abstract groups may be thought of as a 'society' of abstract groups.

The notion of functor between categories allows for comparison of properties of such mathematical structures, and analogies between the ways different mathematical structures behave.

The surprise is that such concepts developed by mathematicians to understand the construction and behaviour of particular mathematical structures and of examples of these structures should turn out so widely useful. This reflects on the nature of mathematics. Some physicists (Deutsch, Penrose,...) have used in relation to mathematics the term 'absolute truth'. This term is certainly unclear and neglects the evolution of mathematics and of the notion of validity. Indeed 'validity' seems to depend on a notion of logic; this leads to the idea popularised by Russell of reducing mathematics to logic, thus hopefully ensuring some kind of universal validity. Yet category theory has easily revealed areas where traditional notions of logic such as the law of the excluded middle do not hold, and has suggested new ways of viewing set theory, in which the notion of function is taken as basic instead of the notion of membership.

Although the notion of category was developed to give a description of the methodology of certain mathematical processes, a kind of metamathematical role, it has also turned out to be a useful mathematical structure in itself. This seems to be because it combines the geometry of graphs with the algebra of composition. Indeed a special case of a category, that of a groupoid, gives a vast generalisation of the notion of group yet encompassing many of the aspects that we desire from group theory, for example an encompassing of the notion of, indeed a generalisation of, the notion of symmetry, replacing it with the wider idea of a reversible transition.

To return to neuroscience, an assumption must be that since the brain has evolved over hundreds of millions of years our attempts to model some of its manner of functioning must be very partial. It is also possible that some success in such modelling may need entirely new mathematical ideas. On the other hand, the success of fractal ideas in modelling some aspects of growth suggests that quite simple processes when iterated can lead to complex structures.

What does category theory have to offer? The interest is in "the" structure of the brain. So we need mathematical structures rich enough to model some of these structures, their relationships and their evolution in a lifetime and over generations. The mathematical area which deals with structure, analogy and comparison is surely category theory.

Why do we need analogy and comparison[1]? The brain seems to be a great device for modelling and governing action, with the aim of survival. The organism in which the brain resides needs some kind of model of its environment, an analogy of the environment, in order to take appropriate action to survive and reproduce. Such reactions to the environment are seen in even unicellular organisms. Indeed research has shown how a single cell of an organism is itself tightly packed with activity.

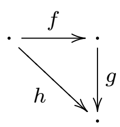
[1]See R. Brown and T. Porter, 'Category Theory: an abstract setting for analogy and comparison', In: What is Category Theory? Advanced Studies in Mathematics and Logic, Polimetrica Publisher, Italy, (2006) 257-274.

Then such models of the environment need to be related, and combined, and allowed to communicate. Some of these ideas can be encapsulated in category theory.

One intuitive source of category theory is very simple. It developed from the notation $f : X \to Y$ for a function f from the set X to the set Y, and the important notion of the geometry ore geography of composition. That is one can compose a function f as above with a function $g : Y' \to Z$ if and only if $Y = Y'$. The analogy with composing journeys is clear. The algebraic setup for this is a binary operation 'composition' which is defined only under a geometric condition: the target of one function has to be the source of the other. This apparently 'trivial' loosening of the algebraic notion of composition has surprisingly profound consequences, and allows for a large conglomeration of new concepts able to be expressed by category theory, and conveniently only by category theory.

One example, and a key to this book, is that of *colimit*. This corresponds to the intuitive idea of gluing, of combining, structures of the same type.

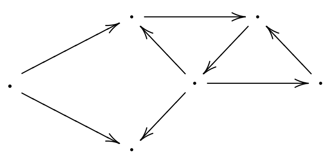
First we need that the notion of composition allows the definition of a *commutative* triangle of morphisms



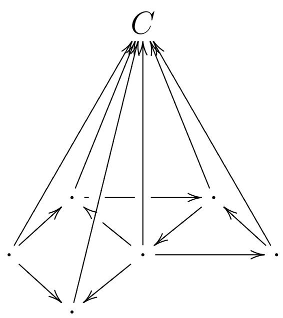
to mean that $fg = h$. Note here that the objects of a category allow the combination of a notion of multiplication fg (which usually defines a monoid) with the notion of position; the notion of commutative triangle applies only to the case where composition is defined.

A colimit has 'input data', a 'cocone', and output from the 'best' cocone (when it exists).

The 'input data' for a colimit is a *pattern* P, also called in the literature a 'diagram', that is a collection of some objects in a category C and some arrows between them, such as:

$P =$ 

The 'functional controls' for a colimit consist of a *cocone with base P and vertex an object C*.



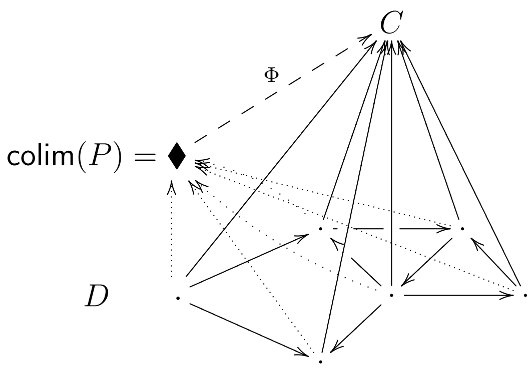
such that each of the 'sloping' triangular faces of this cocone is commutative.

The output will be an object $\mathrm{colim}(P)$ in our category C defined by a special *colimit cocone* such that any cocone on D factors uniquely through the colimit cocone. The commutativity condition on the cocone in essence forces interaction in the colimit of different parts of the diagram D.

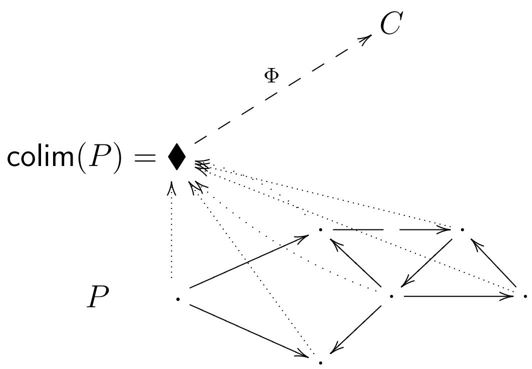
In the next picture the colimit is written

$$\blacklozenge = \mathsf{colim}(D),$$

the dotted arrows represent new morphisms which combine to make the colimit cocone:



and the broken arrow Φ is constructed from the other information. Again, all triangular faces of the combined picture are commutative. Now stripping away the 'old' cocone gives the factorisation of the cocone via the colimit:



Intuitions:

The object $\mathrm{colim}(P)$ is 'put together' from the constituent pattern P by means of the colimit cocone. From beyond (or above our pattern) P, an object C 'sees' the pattern P 'mediated' through its colimit, i.e. if C tries to interact with the whole of P, it has to do so via $\mathrm{colim}(P)$. The colimit cocone is a kind of program: given any cocone on P with vertex C, the output will be a morphism

$$\Phi : \mathrm{colim}(P) \to C$$

constructed from the other data. How is this done?

Email Analogy for the behaviour of a colimit

You want to send an email Φ of a document P to a receiver C. The document P is made up of lots of parts. What the email programme does is split P up in some way into pieces, label each piece at its beginning and end, and send these labelled pieces separately to C which combines them. Also the theory has to ensure that the final received email is independent of all the choices that have been made. The important relevance for neuroscience is that communication from a colimit is always done via the associated cocone, which describes how the colimit is put together: this is a kind of distributed communication.

In a sense, this idea pervades this book, but perhaps not so explicitly. They do refer to research which shows that some activities involve many areas of the brain; this reflects also common sense, when we describe for example an orchestra conductor as totally involved with the music. Also the appeal of music surely reflects its ability to conjure wide associations of rhythm and to stir up as we say 'deep recesses of the mind', arousing emotions which we are unable to express in words.

In any case, this email analogy of colimit, and the idea of a colimit as a model of a structure in the brain, focuses attention on the idea of communication between structures, and the distributed way it is obtained.

Evolution

The major concept which is explored in this book adds another aspect to the above, namely that of a category evolving with time, so that in this evolution the colimits of various patterns also change.

The potential influence of higher dimensional algebra and higher categories

Andrée Ehresmann and her late husband Charles Ehresmann were among the first to develop ideas of higher dimensional categories and of categories with additional structure. This area is now developing strongly, and could well become a major feature of mathematics of the 21st century and of its applications[2].

It is worth giving some background to these ideas[3].

It is somehow intuitively clear that the brain does not work only 1-dimensionally. We can take in a picture at a glance. Even in mathematics some things are clearer from pictures than from formulae: for example we can immediately read

$$\begin{matrix} ||| & ||||| \\ ||| & ||||| \end{matrix}$$

[2]See for example the 'n-category cafe' http://golem.ph.utexas.edu/category/
and 'the ncat lab' http://ncatlab.org/nlab/show/HomePage

[3]See also R. Brown and T. Porter, 'Category theory and higher dimensional algebra: potential descriptive tools in neuroscience', Proceedings of the International Conference on Theoretical Neurobiology, Delhi, February 2003, edited by Nandini Singh, National Brain Research Centre, Conference Proceedings 1 (2003) 80-92. arXiv:math/0306223

as

$$2 \times (3 + 5) = 2 \times 3 + 2 \times 5.$$

However in comparison with the picture the formula is somewhat barbarous; it requires the knowledge of all sorts of conventions as to the meaning of the specific symbols. Even the well known term $=$ can be confusing. It does not mean 'is the same as' but rather represent a transform. Thus $1 + 1 = 2$ does not mean that the left hand side is the 'same' as the right hand side, but rather that the left hand operation gives the right hand answer.

However we are forced in mathematics into formulae written on a line partly because of the exigencies of writing and printing, and partly because of modes of calculation, which are often best done in a linear fashion, one after the other. The idea that the brain works in such a linear fashion is clearly absurd, yet the mathematics to describe some different mode of operation is only beginning to be developed. This is partly why I described this book as a 'start on the modelling of natural self-organising systems by significant and relevant mathematical structures'. It would be surprising if some 'final answers' were to appear in this year, 2009.

The structure of mathematics

We have mentioned communication between structures. It is also clear that this communication is in a certain sense *symbolic*. By this I mean that the communication is an abstract representation of what is required from the communication: it clearly has to be this. This suggests that a study of symbolic communication should be an important part of further studies in theoretical neuroscience and in systems theory.

Mathematics is possibly the most strongly organised, complex and hierarchically structured systems involving symbolic communication in our culture. The authors of this book describe on p. 251 how the development of the theory of sketches by a group of workers might be described in terms of the kind of social organisation modelled in this book. However the structure of the mathematics itself is a different concern, and it is worth considering whether this is relevant. This would follow the nice mode of self reference, which is embedded in some famous results such as Gödel incompleteness.

First, mathematics has *concepts*. As examples of these, we have semigroups, groups, rings, topological spaces, categories, integers, divided polynomial rings, and so on. Indeed the theory of sketched mentioned above is one way of describing the organisation of these concepts.

Characteristics of these concepts that they have among others: inputs of other concepts; representations of objects, such as the elements of a ring, or the elements of an array, in terms of internal records; outputs in terms of printing; programming and applications of algorithms. Even the concept of a list is quite complex, since there are all kinds of operations needed on lists, such as: delete an element; add an element; join two lists of the same type; count the number of elements.

Second, these concepts are arranged hierarchically, from the simplest to the most complex, where a complex concept involves simpler ones and combines them. Thus a "divided power polynomial ring" involves not only polynomial rings but the extra notion of divided powers. The work of mathematicians over the centuries has developed these concepts, tested their interrelations, and found efficient notations.

Thirdly, the representation of these objects internally should reflect nicely the operations required on them. The problems with arithmetic in Latin numerals show why they were abandoned in favour of Arabic numerals. Another example is the record of a polynomial such as $3+4t-t^2$. In some systems this is recorded simply as an expression. In more advanced systems, such as AXIOM[4], polynomials are special cases of monoidal rings $R[M]$ where R is a ring, not necessarily commutative, and M is a monoid. This implies immediately that $R[M]$ is itself a ring, that is it has addition and multiplication satisfying all the axioms of a ring, and that the operations and internal records of rings are all available, and may be iterated, to form for example the ring $R[M][N]$ where N is another monoid. Thus while the expressions of polynomials in two variables seems complicated, from this abstract viewpoint the step is simple and economical. This reflects standard practices in mathematics, which have evolved over the last three centuries for economy, power, and suggestibility, i.e. for analogy.

Fourthly, these concepts, often called types, satisfy for the mathematician three important properties, which are not so often implemented in computational software.

1. *Inheritance* This means that a concept such as associative binary operation has to be set up only once, and then is used, with its notion of representation, and its algorithms, in all other cases where it applies, as for example at least twice for addition and multiplication in the concept, or type, of ring.

2. *Types as first class variables* This means that for example the type of ring can be used in setting up the type of polynomial ring.

3. *Coercion* This means the ability to transfer objects from one type, or concept, to another one. As an example, mathematicians have no difficulty in using the notation 1 to mean: the positive integer 1; the rational number 1; a unit square matrix over the complex numbers; the unit of an arbitrary field. Each use allows a different set of operations, but the convenience of using one notation to be interpreted according to context is important for concision and also for analogies, which is perhaps the most powerful importance of abstraction. As an example of coercion we need to be able to shift from a square matrix of polynomials such as

$$\begin{bmatrix} 1+t & 1-t^2 \\ 2-t & 1+3t^2 \end{bmatrix}$$

to the polynomial over the ring of square matrices

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & -1 \\ 0 & 3 \end{bmatrix} t^2$$

We all know the difficulties of communicating in society across different languages and social customs, which might be expressed as coercing a concept from one language to another.

As far as I know the only computer algebra system which does all this has been AXIOM, whose development has however had problems. These principles are however an essential aspect of the

[4]http://en.wikipedia.org/wiki/Axiom_(computer_ algebra_ system)

software ExprLib developed for MSSRC[5], by Larry Lambe. Notice that in these systems the axioms for a given type are not verified. Instead the axioms give instructions to the programmer; thus the records for objects of a commutative associative binary operation can be simpler than those for a possibly noncommutative associative operation.

The purpose of developing this comparison and argument is to indicate how much and how difficult further work there is to make some of the ideas of this book realistic. This in itself is not a criticism of a landmark book. But perhaps the analysis of communication between different functions of the brain, or of systems, will give an opening into the analysis of structure.

Contents

The list of contents which follows shows how the book tackles a range of famous problems in systems science, biology and neuroscience:

Contents Part A: Hierarchy, Emergence; Chapter 1: Net of Interactions and Categories; Chapter 2: The Binding Problem; Chapter 3: Hierarchy and Reductionism; Chapter 4: Complexifications and Emergence; Part B: Memory Evolutive Systems; Chapter 5: Evolutionary Systems; Chapter 6: Internal Organization and MES; Chapter 7: Robustness and Plasiticity; Chapter 8: Memory and Learning; Part C: Application to Cognition and Consciousness; Chapter 9: Cognitive Systems and MES of CAT-Neurons; Chapter 10: Semantics, Archetypal Core and Consciousness.

Conclusion

In conclusion, the book should become a foundation for work for decades to come. There are all sorts of possibilities, including for example the area of higher dimensional categories, to give language and tools for exploring the structure of the brain.

[5] Multidisciplinary Software Systems Research Corporation, www.mssrc.com