ON THE SCHREIER THEORY OF NONABELIAN EXTENSIONS: GENERALISATIONS AND COMPUTATIONS

Ronald Brown and Timothy Porter
School of Mathematics
University of Wales, Bangor
Gwynedd LL57 1UT

Proceedings Royal Irish Academy 96A (1996) 213-227.

Abstract

We use presentations and identities among relations to give a generalisation of the Schreier theory of nonabelian extensions of groups. This replaces the usual multiplication table for the extension group by more efficient, and often geometric, data. The methods utilise crossed modules and crossed resolutions.

Introduction

The classification of nonabelian extensions

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1 \tag{1}$$

of a group A by a group G may be found in many texts. Nicholson explains in [18] that the original impetus was work on many examples by Hölder, and that the now standard exposition goes back to Schreier's 1926 paper [20].

The standard exposition in essence assumes given the multiplication table of G and then describes in terms of a 'factor system' a multiplication on the set $E = G \times A$. The 2-cocycle condition imposed on a factor system is related to the associativity law. However, this approach does not lend itself to computation, particularly if G is infinite, nor to the use of combinatorial information on the group G. In any case, the multiplication table of a group is rarely used in group theory as a means of defining or describing a group.

There are other well-known approaches to the case when A is abelian. Our emphasis is on the nonabelian case.

Descriptions of E in terms of presentations of the groups A, G are given by several authors. Marshall Hall Jr. in [13, section 15.4], ends by saying it may be difficult to determine the identities leading to conditions for an extension. Cockcroft [11] gives a presentation of an extension, and this may also be found in Johnson [16]. A full account is given in Turing's not so well known paper [21], using Reidemeister's account in 1932 of identities among relations.

Our approach was developed prior to a reading of Turing's paper, but can be seen as giving a modern version and a generalisation of the results of Turing, using the notions of *crossed module*, the *module of identities among relations* (which are recalled in section 1: see the expositions in [5, 19]), and of crossed resolution (see section 3). Further, we examine in detail the construction that results when the transcription to this situation of the usual 2-cocycle condition fails. This leads not to an extension as in (1), in which A is a normal subgroup of E, but to a crossed sequence

$$1 \to K \to A \to E \to G \to 1 \tag{2}$$

in which the part $A \to E$ is a crossed module. We give results on the classification of crossed sequences which arise in this way. This construction also yields many finite crossed modules. Since any crossed module represents a homotopy 2-type, this also gives a construction relevant to homotopy theory. Other constructions of finite crossed modules relevant to homotopy theory are given by Brown and Wensley in [8].

Our results imply the following. Given a presentation $\langle X; R\rangle$ of G, the data for an extension may be given by elements $\gamma_x \in \mathrm{Aut}\, A$ for $x \in X$, $a_r \in A$ for $r \in R$, such that
(i) inner automorphism by a_r may be written in terms of the γ_x in the same form as the relator r is written in terms of the generators $x \in X$,
(ii) in a similar spirit, the elements $a_r \in A$ satisfy further identities corresponding to generators of the G-module π_2 of identities among the relations R.
Condition (i) is standard. Condition (ii) replaces the usual 2-cocycle condition, and is closely related to conditions given by Turing. Note that if G is FP_3, then this extension data is finite. The extension E is obtained from this data as follows. The elements $\gamma_x \in \mathrm{Aut}\, A, x \in X$, determine an action of the free group $F(X)$ on A and so the semidirect product group $C = F(X) \ltimes A$ is defined. Then E is the quotient of C by the elements $(r, a_r^{-1}), r \in R$. The condition (ii) on identities among the a_r is required to make the composite morphism $A \to C \to E$ injective. Equivalences of such data corresponding to equivalences of extensions are also described.

We give a generalisation and clarification of the above data for an extension, as follows.

We recall below that the G-module π_2 of identities among the relations R is the kernel of the morphism $\partial : F_C(R) \to F(X)$ of the free crossed $F(X)$-module on the relators R [5, 14]. We first replace this free crossed module by a general crossed module $\mu : M \to P$.

Second, the automorphism crossed module $A \to \mathrm{Aut}\, A$ is also replaced by a general crossed module, say $\alpha : A \to Q$, so that, following Dedecker [12], we consider extensions of A by G 'of the type of the crossed module $\alpha : A \to Q$'. Dedecker's motivation came from the consideration of the formalism of nonabelian cohomology, and in particular, coefficient morphisms. Brown and Mucuk [6] give a geometric situation in which such extensions arise. The interest in extensions of this more general type is increased by the constructions of many finite crossed modules in [8].

Our main result is Theorem 1.2.

1 Crossed modules and extensions

We recall the elements of the theory of crossed modules and of extensions of the type of a crossed module.

A *crossed module* $\mu : M \to P$ is a morphism μ of groups together with an action of P on M, here written $(m, p) \mapsto m^p$, such that

- CM1) $\mu(m^p) = p^{-1}(\mu m)p$,
- CM2) $m^{-1}m_1 m = m_1^{\mu m}$,

for all $m, m_1 \in M, p \in P$. A *morphism* (f, g) of crossed modules $\mu : M \to P, \alpha : A \to Q$ is a commutative diagram of morphisms of groups

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "f"'] & P \arrow[d, "g"] \\ A \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

such that $f(m^p) = (fm)^{gp}$ for all $m \in M, p \in P$.

If A is a group, its *automorphism* crossed module is $\chi_A : A \to \mathrm{Aut}A$, where $\chi_A(a)$ is conjugation by a, i.e. $b \mapsto a^{-1}ba$.

Definition 1.1 [12] An *extension* of A by G *of the type of the crossed module* $\alpha : A \to Q$ is given by a diagram of morphisms of groups

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] \arrow[d, "1"'] & E \arrow[r, "p"] \arrow[d, "\omega"] & G \arrow[r] & 1 \\ & A \arrow[r, "q"'] & Q & & \end{tikzcd}$$

where the top row is exact, and the square is a morphism of crossed modules, with the operation of E on A given essentially by conjugation. Note that if $A \xrightarrow{q} Q$ is the automorphism crossed module χ_A of A then conjugation within E defines the morphism ω for an arbitrary extension.

An *equivalence* of such extensions

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1, \ \omega : E \longrightarrow Q$$

$$1 \longrightarrow A \xrightarrow{i'} E' \xrightarrow{p'} G \longrightarrow 1, \ \omega' : E' \longrightarrow Q$$

of type $\alpha : A \to Q$ is an isomorphism $h : E \to E'$ such that $hi = i', p'h = p, \omega' h = \omega$. The set of equivalence classes of such extensions is written $\mathrm{Ext}_{A \to Q}(G, A)$. Again, if $A \xrightarrow{q} Q$ is χ_A, then this reduces to the standard form of equivalence.

Let $\mu : M \to P, \alpha : A \to Q$ be crossed modules. Let $\mathbf{k} = (k^2, k^1)$, $\mathbf{l} = (l^2, l^1)$ be morphisms $(M \to P) \to (A \to Q)$ of crossed modules. A *homotopy* $h : \mathbf{k} \simeq \mathbf{l}$ is a function $h : P \to A$ such that for all $p_1, p \in P, m \in M$,

- CMH1) $h(p_1 p) = (hp_1)^{l^1 p}\ (hp)$,
- CMH2) $(k^1 p) = (l^1 p)(\alpha h p)$,
- CMH3) $k^2 m = (l^2 m)(h \mu m)$.

The condition CMH1) is also described by saying h is an l^1-*derivation*. These conditions are consistent with k^1, k^2 being morphisms of groups, given that $\mathbf{k}, \mathbf{l}$ are morphisms of crossed modules. These homotopies define an equivalence relation on morphisms $(M \to P) \to (A \to Q)$ of crossed modules. This definition of homotopy is more special than that given in [22, 3]. See [2] for a discussion of the use of the more general homotopies in the classification of extensions.

Our main theorem is the following.

Theorem 1.2 *Let $\mu : M \to P, \alpha : A \to Q$ be crossed modules. Let $[M \to P, A \to Q]^0$ denote the set of homotopy classes of morphisms $\mathbf{k} = (k^2, k^1) : (M \to P) \to (A \to Q)$ of crossed modules, such that $k^2(Ker\ \mu) = 1$. Then there is a natural injection*

$$\mathbf{E} : [M \to P, A \to Q]^0 \to Ext_{A \to Q}(G, A)$$

sending the class of a morphism $\mathbf{k}$ *to the extension*

$$1 \to A \to E(\mathbf{k}) \to G \to 1$$

where $E(\mathbf{k})$ *is the quotient of the semidirect product group* $P \ltimes A$*, in which* P *acts on* A *via* Q*, by the elements* $(\mu m, (k^2 m)^{-1})$*,* $m \in M$*. The function* $\mathbf{E}$ *is surjective if* P *is a free group.*

2 Relation with the classical Schreier theory

In order to explain the use of Theorem 1.2, we shall require J.H.C. Whitehead's notion of the *free crossed module* on a function $w : R \to P$, where R is a set and P is a group [22]. This is a crossed module $\partial : F_C(R) \to P$ together with a function $\bar{w} : R \to F_C(R)$ such that $\partial\bar{w} = w$, and which is universal for this property. That is, if $\gamma : C \to P$ is a crossed module and $v : R \to C$ is a function such that $\gamma v = w$, then there is a unique morphism $(v', 1_P)$ of crossed modules, where $v' : F_C(R) \to C$, such that $v'\bar{w} = v$. The group $F_C(R)$ may be constructed as the quotient of $F(R \times P)$, the free group on the set $R \times P$, by the so called Peiffer relations $(r,p)^{-1}(s,p_1)^{-1}(r,p)(s,p_1 p^{-1}(wr)p)$ for all $r, s \in R, p, p_1 \in P$, and where $\partial(r,p) = p^{-1}(wr)p$. The *module of identities* for $w : R \to P$ is the G-module $\pi_2(w) = \text{Ker }\partial$, where $G = \text{Coker }\partial$. In the case $P = F(X)$, the free group on X, there is considerable work on $\pi_2(w)$, see for example [5, 14, 7].

Theorem 1.2 does not require that M be a free crossed P-module. However, the advantage of having P and M free is that a morphism $\mathbf{k} : (M \to P) \to (A \to Q)$ can be specified by arbitrary values $k^1(x), x \in X,\ k^2(r), r \in R$, where X, R are the free basis sets for P, M respectively, provided only that $\alpha k^2(r) = k^1(\mu r)$, for all $r \in R$. In particular, if X, R are finite, then the required data for a morphism $\mathbf{k}$ is finite. Further, if there is a finite set of generators for the module of identities $\pi_2 = \text{Ker }\mu$, then the data for the extra identities among the elements $k^2(r), r \in R$, for $k^2(\text{Ker }\mu)$ to be 0, is also finite. For example, the presentation might be aspherical, that is the module of identities is trivial, in which case there need be no identities among the $k^2(r), r \in R$. This occurs for example with certain presentations of knot groups, and with one relator presentations where the relator is not a proper power [17].

The Schreier theory of factor systems has been expressed in terms of the standard crossed resolution $C(G)$ of a group G by Brown and Higgins [2], following ideas of Huebschmann [15]. Here $C(G)$ is defined in [15] and is shown in [2] to be the fundamental crossed complex of the classifying space of the group G, with its skeletal filtration. Thus $C_1(G)$ is the free group on G, with generators written, say, $[g], g \in G; C_2(G)$ is the free crossed $C_1(G)$-module on the function $\delta_2 : G \times G \to C_1(G), (g,h) \mapsto [g][h][gh]^{-1}$; for $n \geqslant 3, C_n(G)$ is the free G-module on $G^{(n)}$, the n-fold product of the set G with itself, with generators written $[g_1, g_2, \ldots, g_n], g_i \in G$. The elements of the $C_1(G)$-basis for $C_2(G)$ are similarly written $[g_1, g_2], g_i \in G$.

The boundary $\delta_3 : C_3(G) \to C_2(G)$ is given by

$$\delta_3[g,h,k] = [h,k]^{[g]^{-1}}[g,hk][gh,k]^{-1}[g,h]^{-1},$$

and there are formulae for $\delta_n, n \geqslant 4$, which do not concern us. Since $C(G)$ is a free crossed resolution of G, we have $\text{Im }\delta_3 = \text{ Ker }\delta_2$. Thus the elements $\delta_3[g,h,k], g,h,k \in G$, give a set of $C_1(G)$-generators of $\text{Ker }\delta_2$. A morphism of groups $k^1 : C_1(G) \to Q$ is determined by its values on the free generators $[g], g \in G$, and so is equivalent to a function $k_1 : G \to Q$. The morphism $k^2 : C_2(G) \to A$ over k^1 is determined by its values on the free generators $[g,h] \in C_2(G)$, and so is equivalent to a function $k_2 : G \times G \to A$ such that $\alpha k_2(g,h) = k_1(g)k_1(h)k_1(gh)^{-1}, g,h \in G$. The condition $k^2\delta_3[g,h,k] = 1$ for all $g,h,k \in G$ is equivalent to the usual factor set condition on the function k_2. It is equivalent to $k^2\delta_3 = 0$, since the $[g,h,k]$ generate $C_3(G)$ as a G-module. Since $C(G)$ is a resolution, the condition $k^2\delta_3 = 0$ is equivalent to the condition $k^2(\text{ Ker }\delta_2) = 0$, and $\text{Ker }\delta_2$ is the module of identities for the presentation of G given by its multiplication table. The fact that $C(G)$ is a free crossed resolution, and $C_3(G)$ has a known set of generators, thus gives a convenient set of generators of this module of identities.

Later, we shall give other examples of this procedure for obtaining generators of the module of identities among relations, and so for specifying conveniently the 2-cocycles. Yet other examples, involving 3-cocycles and a third cohomology group (with twisted coefficients), are given in [8, 9]. These examples illustrate the computational utility of free crossed resolutions.

The extension $E(\mathbf{k})$ corresponding to $\mathbf{k}$ has elements written $[\![p, a]\!], p \in C_1(G), a \in A$ and is the quotient of $C_1(G) \ltimes A$ by the elements

$$(\delta_2[g,h], (k^2[g,h])^{-1}), g, h \in G.$$

There is a bijection

$$\begin{aligned} G \times A &\to E(\mathbf{k}) \\ (g, a) &\mapsto [\![[g], a]\!] \end{aligned}$$

Then

$$\begin{aligned} [\![[g], a]\!][\![[h], b]\!] &= [\![[g][h], a^{[h]}b]\!] \\ &= [\![\delta_2[g,h][gh], a^{[h]}b]\!] \\ &= [\![[gh], k^2[g,h]a^{[h]}b]\!]. \end{aligned}$$

This is the usual multiplication on the extension determined by a factor system.

However our methods allow for the calculation of extensions using any free crossed resolution of G. In cohomological terms, this is not surprising. The cohomology set $H^2(G, A \to Q)$ of [12] is shown in [2] to be expressable in terms of homotopy classes of morphisms from $C(G)$, and so is bijective with the homotopy classes of morphisms from any free crossed resolution of G, since any two such are homotopy equivalent, by standard methods of homological algebra adapted to the crossed context.

3 Examples

In this section we give three examples which illustrate different aspects of the use of these results.

The first recovers a classical description of extensions, in this case the classification of extensions by a cyclic group, see for example [23].

The second example gives the classification of extensions by the trefoil group, illustrating the fact that no cocycle condition whatsoever is required in the case of an aspherical presentation.

The third example shows how to describe extensions by a product of groups, using the tensor product of crossed complexes defined by Brown and Higgins in [3]. This is another illustration of the use of the non abelian methods of crossed complexes.

Example 3.1 Extensions by a cyclic group Let C_n denote the cyclic group of order n, written multiplicatively, and generated by an element t, so that the infinite cyclic group is C_∞. The presentation $\langle t : t^n \rangle$ for C_n gives rise to the free crossed module $\delta : (C_\infty)^n \to C_\infty$ where $(C_\infty)^n$ is generated as a group by $t_0, t_1, \ldots, t_{n-1}$ and as a crossed C_∞-module by t_0; here $t \in C_\infty$ operates on $(C_\infty)^n$ by $(t_i)^t = t_{i+1}, i = 0, \ldots, n-1 \pmod n$; and for all i, $\delta(t_i) = t^n$. This description of this free crossed module is given in [8]. A morphism

$$\begin{tikzcd} (C_\infty)^n \arrow[r, "\delta"] \arrow[d, "k^2"'] & C_\infty \arrow[d, "k^1"] \\ A \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

of crossed modules is thus specified by elements $q = k^1(t) \in Q,\ a = k^2(t_0) \in A$ such that $\alpha a = q^n$. Further, Ker δ, the module of identities for the presentation, is the submodule generated by the element $t_0 t_1^{-1}$. Hence the condition $k^2(\mathrm{Ker}\ \delta) = 0$ is equivalent to $k^2(t_0 t_1^{-1}) = a(a^q)^{-1} = 1$, that is $a = a^q$. An equivalence $(a' : q') \simeq (a : q)$ of such data is given by a derivation $h : C_\infty \to A$, and so by an element $b = h(t) \in A$, such that $q' = q(\alpha b)$ and

$$a' = ah(t^n) = ab^{q^{n-1}} b^{q^{n-2}} \dots b^2 b.$$

This result is given in [23] ChIII, section 7. The extension group E determined by the data $(a : q)$ is the quotient of $C_\infty \ltimes A$ by the element (t^n, a^{-1}).

Example 3.2 The trefoil group Let G be the trefoil group with presentation $\langle x, y : x^2 = y^3 \rangle$. This is a 1-relator presentation whose relator is not a proper power, and so there are no identities among the relations, as proved by Lyndon in [17] (see [5] for more information). Therefore the extension data of A by G of type $\alpha : A \to Q$ is given by elements $q_x, q_y \in Q, a_r \in A$, such that $\alpha a_r = (q_x)^2 (q_y)^{-3}$. An equivalence $(a'_r : q'_x, q'_y) \simeq (a_r : q_x, q_y)$ of such data is given by elements $b, c \in A$ such that $q'_x = q_x(\alpha b), q'_y = (q_y)(\alpha c)$ and $a'_r = a_r h(x^2 y^{-3})$ where h is the derivation $F\{x, y\} \to A$ given by $hx = b, hy = c$. Thus

$$\begin{aligned} h(x^2 y^{-3}) &= h(x^2)^{q_y^{-3}} h(y^{-3}) \\ &= (b^{q_x} b)^{q_y^{-3}} (c^{-1})^{q_y^{-3}} (c^{-1})^{q_y^{-2}} (c^{-1})^{q_y^{-1}}. \end{aligned}$$

The group E determined by the extension data $(a_r : q_y, q_y)$ is the quotient of the semidirect product $F\{x, y\} \ltimes A$ by the element $(x^2 y^{-3}, a_r^{-1})$. Here $F\{x, y\}$ acts on A by $a^x = a^{q_x}, a^y = a^{q_y}, a \in A$.

Example 3.3 Extensions by a product The tensor product of crossed complexes as defined in [3] may be used to describe extensions by a product $G \times H$ of groups. Let $F_*(G),\ F_*(H)$ be free crossed resolutions of groups G, H respectively. The tensor product $F_*(G) \otimes F_*(H)$ is then a free crossed resolution of $G \times H$. A proof of asphericity will be given in [7]. It is proved in [4] that the tensor product of free crossed complexes is free on the tensor product of the free generators, so that in particular $F_*(G) \otimes F_*(H)$ is freely generated as a crossed complex by $a_i \otimes b_j$, where the a_i, b_j run over sets of free generators of $F_*(G),\ F_*(H)$ respectively. Thus it is easy to specify morphisms from $F_*(G) \otimes F_*(H)$ to a crossed module or crossed complex. Further, generators for the module of identities for a presentation of the product $G \times H$ are the images under δ_3 of free generators of $(F_*(G) \otimes F_*(H))_3$, by asphericity. Such free generators are of the form $a_3 \otimes *, * \otimes b_3, a_2 \otimes b_1, a_1 \otimes b_2$ where a_i, b_j run over free generators of $F_i(G), F_j(H)$ respectively.

This implies the following. Let $\langle X; R \rangle, \langle Y, S \rangle$ be presentations of G, H respectively, and let I, J be generating sets for the modules of identities for these presentations. Then a free crossed resolution $F_*(G)$ corresponding to X, R, I is in dimensions $\leqslant 3$ of the form

$$C_3(I) \xrightarrow{\delta_3} F_C(R) \xrightarrow{\delta_2} F(X)$$

where $C_3(I)$ is the free G-module on I, and similarly for $F_*(H)$. Thus in dimensions $\leqslant 3$, $F_*(G) \otimes F_*(H)$ has generators as follows, where for Z any set, $\bar{Z}$ denotes a set of formal generators $\bar{z}, z \in Z$:

- *dimension* 1: X, Y,
- *dimension* 2: $\bar{R}, \bar{S}, \{x \otimes y : x \in X, y \in Y\}$,
- *dimension* 3: $\bar{I}, \bar{J}, \{x \otimes \bar{s}, \bar{r} \otimes y : x \in X, y \in Y, r \in R, s \in S\}$.

The boundaries are given by:

$$\delta_2\bar{r} = r,\ \ \delta_2\bar{s} = s,\ \ \delta_2(x \otimes y) = y^{-1}x^{-1}yx,$$

$$\delta_3\bar{i} = i, \delta_3\bar{j} = j, \delta_3(x \otimes \bar{s}) = \bar{s}^{-1}\bar{s}^{x}(x \otimes s)^{-1},$$

$$\delta_3(\bar{r} \otimes y) = (r \otimes y)\bar{r}^{-1}\bar{r}^{y}.$$

Now the elements $x \otimes s, r \otimes y$ have to be expressed in terms of the free generators in dimension 2. This is done by using the biderivation rules

$$x \otimes uv = (x \otimes u)^{v}\,(x \otimes v),$$

$$\omega z \otimes y = (z \otimes y)(\omega \otimes y)^{z},$$

which are part of the crossed complex structure of the tensor product.

Note that in this example, we obtain nice generators of the module of identities for the product, by applying the boundary to free generators in dimension 3 of a crossed resolution.

The above description explains the determination of extensions by a product of cyclic groups given in [23]. Different conventions for the tensor product have been adopted by Baues in [1].

4 The construction and classification of extensions

The main point of this section is the proof of Theorem 1.2, the classification of extensions of A by G of a given type $\alpha : A \to Q$. A major feature of the proof is the construction, from a morphism of crossed modules

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "k^2"'] & P \arrow[d, "k^1"'] \\ A \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

such that $k^2(\text{Ker}\ \mu) = 1$, of a "crossed pushout"

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "k^2"'] & P \arrow[d, "k^1"] \\ A \arrow[r, "i_{\mathbf{k}}"] \arrow[d, "1"'] & E(\mathbf{k}) \arrow[d, "\omega_{\mathbf{k}}"] \\ A \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

where $i_{\mathbf{k}}$ is a crossed normal inclusion. We present this construction in greater generality, by dropping the condition $k^2(\text{Ker}\ \mu) = 1$. Then $i_{\mathbf{k}}$ becomes a crossed module with kernel $k^2(\text{Ker}\ \mu)$. This generality is not necessary for the extension problem, but it does reveal more clearly the properties of the construction, and could be useful elsewhere. For example, when M, P, A are finite, this gives, in the spirit of [8], another method for constructing finite crossed modules.

Proposition 4.1 *Let* $\mathbf{k} = (k^2, k^1)$ *be a morphism of crossed modules*

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "k^2"'] & P \arrow[d, "k^1"] \\ A \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

and let $K = k^2(\text{Ker } \mu)$. Let $G = \text{Coker } \mu$, and let $\phi : P \to G$ be the quotient morphism. Then there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \text{Ker } \mu \arrow[r] \arrow[d] & M \arrow[r, "\mu"] \arrow[d, "k^2"'] & P \arrow[r, "\phi"] \arrow[d, "k_1"] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K \arrow[r] & A \arrow[r, "i_{\mathbf{k}}"] \arrow[d, "1"'] & E(\mathbf{k}) \arrow[r, "\pi_{\mathbf{k}}"] \arrow[d, "\omega_{\mathbf{k}}"] & G \arrow[r] & 1 \\
 & & A \arrow[r, "\alpha"'] & Q & &
\end{tikzcd}$$

such that
(i) the rows are exact,
(ii) $E(\mathbf{k})$ acts on A so that $i_{\mathbf{k}}$ is a crossed module,
(iii) $(k^2, k_1), (1, \omega_{\mathbf{k}})$ are morphisms of crossed modules and $\omega_{\mathbf{k}} k_1 = k^1$.
(iv) Given any other commutative diagram in which $i : A \to E$ is a crossed module and $(k^2, f), (1, \omega)$ are morphisms of crossed modules,

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[d, "k^2"'] & P \arrow[r, "\phi"] \arrow[d, "f"] & G \arrow[r] \arrow[d, "1"] & 1 \\
A \arrow[r, "i"] \arrow[d] & E \arrow[r, "\pi"] \arrow[d, "\omega"] & G \arrow[r] & 1 \\
A \arrow[r, "\alpha"'] & Q & &
\end{tikzcd}$$

with exact second row, and such that $\omega f = k^1$, then there is a unique morphism $f_{\mathbf{k}} : E(\mathbf{k}) \to E$ such that $f_{\mathbf{k}} k_1 = f, f_{\mathbf{k}} i_{\mathbf{k}} = i$. Further $f_{\mathbf{k}}$ is an epimorphism, $\omega f_{\mathbf{k}} = \omega_{\mathbf{k}}$, and $\text{Ker } f_{\mathbf{k}} \cong (\text{Ker } i)/K$.

PROOF The group P acts on A via $k^1 : P \to Q$ and the given action of Q on A. Form the semidirect product group $C(\mathbf{k}) = P \ltimes A$, and let $\xi : M \to C(\mathbf{k})$ be the function $m \mapsto (\mu m, (k^2 m)^{-1})$. Let $C(\mathbf{k})$ act on M by $m^{(p,a)} = m^p$. Then ξ is a crossed module. Let $E(\mathbf{k}) = \text{Coker } \xi$, and write $[\![p, a]\!]$ for the image of $(p, a) \in C(\mathbf{k})$ in $E(\mathbf{k})$. Thus if $m \in M$, then in $E(\mathbf{k}), [\![\mu m, 1]\!] = [\![1, k^2 m]\!]$. Let $i_{\mathbf{k}} : A \to E(\mathbf{k}), \pi_{\mathbf{k}} : E(\mathbf{k}) \to G$ be defined by $a \mapsto [\![1, a]\!], [\![p, a]\!] \mapsto \phi p$ respectively. These are well defined morphisms, and $\text{Ker } i_{\mathbf{k}} = K$. Clearly $\pi_{\mathbf{k}}$ is surjective, and $\pi_{\mathbf{k}} i_{\mathbf{k}}$ is trivial. Suppose $\pi_{\mathbf{k}}[\![p, a]\!] = 1$. Then $p = \mu m$ for some $m \in M$, and so $[\![p, a]\!] = [\![\mu m, a]\!] = [\![1, (k^2 m)a]\!] \in \text{Im } i_{\mathbf{k}}$. This proves exactness.

Let $E(\mathbf{k})$ act on A by

$$a^{[\![p,b]\!]} = b^{-1} a^p b.$$

This action is well defined since

$$\begin{aligned}
a^{[\![\mu m, (k^2 m)^{-1}]\!]} &= (k^2 m) a^{\mu m} (k^2 m)^{-1} \\
&= (k^2 m) a^{\alpha k^2 m} (k^2 m)^{-1} \\
&= a.
\end{aligned}$$

This action makes $i_{\mathbf{k}} : A \to E(\mathbf{k})$ a crossed module.

Let $k_1(p) = [\![p, 1]\!], p \in P$. Then $k_1 \mu(m) = [\![\mu m, 1]\!] = [\![1, k^2 m]\!] = i_{\mathbf{k}} k^2(m), m \in M$. Clearly $k^2(m^p) = (k^2 m)^{k_1 p}$.

Lemma 4.2 *Suppose given a diagram of morphisms of crossed modules*

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "k^2"'] & P \arrow[d, "f'"] \\ A \arrow[r, "i'"] \arrow[d, "1"'] & E' \arrow[d, "\omega'"] \\ A \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

such that $\omega' f' = k'$. Then there is a unique morphism $f'_{\mathbf{k}} : E(\mathbf{k}) \to E'$ of groups such that $f'_{\mathbf{k}} k_1 = f', f'_{\mathbf{k}} i_{\mathbf{k}} = i'$ and $(1, f'_{\mathbf{k}}) : (A \to E(\mathbf{k})) \to (A \to E')$ is a morphism of crossed modules.

Proof If $f'_{\mathbf{k}}$ is a morphism satisfying $f'_{\mathbf{k}} k_1 = f', f'_{\mathbf{k}} i_{\mathbf{k}} = i'$, then

$$\begin{aligned} f'_{\mathbf{k}}[\![p, a]\!] &= f'_{\mathbf{k}}([\![p, 1]\!][\![1, a]\!]) \\ &= (f'p)(i'a). \end{aligned}$$

Note also that $i'a^p = i'a^{k^1 p} = i'a^{\omega' f' p} = i'a^{f'p} = (f'p)^{-1}(i'a)(f'p)$. Hence this definition of $f'_{\mathbf{k}}$ gives a morphism, since if $p_1, p \in P, a_1, a \in A$ then

$$\begin{aligned} f'(p_1 p) i'(a_1^p a_1) &= (f'p_1)(f'p)(f'p)^{-1}(i'a_1)(f'p)(i'a) \\ &= f'_{\mathbf{k}}[\![p_1, a_1]\!] f'_{\mathbf{k}}[\![p, a]\!]. \end{aligned}$$

□

The lemma can be applied with $i' : A \to E', f', \omega'$ replaced by $\alpha : A \to Q, k', 1_Q$ respectively. This yields a morphism of crossed modules $(1, \omega_{\mathbf{k}}) : (A \to E(\mathbf{k})) \to (A \to Q)$ such that $\omega_{\mathbf{k}} k_1 = k'$, and $\omega_{\mathbf{k}} : [\![p, a]\!] \mapsto (k^1 p)(\alpha a)$. In the circumstances of the Lemma, we now have $\omega' f'_{\mathbf{k}} = \omega_{\mathbf{k}}$.

The Lemma can be further applied to the situation of (iv) of the Proposition. This yields a morphism $f_{\mathbf{k}} : E(\mathbf{k}) \to E$ such that $f_{\mathbf{k}} k_1 = f_1 f_{\mathbf{k}} i_{\mathbf{k}} = i, \omega f_{\mathbf{k}} = \omega_{\mathbf{k}}$.

That $f_{\mathbf{k}}$ is an epimorphism follows from the 4-lemma. Suppose $[\![p, a]\!] \in \text{Ker } f_{\mathbf{k}}$. Then $(fp)(ia) = 1$. Hence $\phi p = \pi f p = 1$, and so $p = \mu m$ for some $m \in M$. Hence $[\![p, a]\!] = [\![1, (k^2 m) a]\!] = i_{\mathbf{k}}((k^2 m) a)$, and $i((k^2 m) a) = 1$. Hence $i_{\mathbf{k}}$ induces an isomorphism $(\text{Ker } i)/K \cong \text{Ker } f_{\mathbf{k}}$. □

Next we introduce homotopies

$$\begin{tikzcd} \text{Ker } \mu \arrow[r] & M \arrow[r, "\mu"] \arrow[dl, "h_2"] \arrow[d] & P \arrow[dl, "h_1"] \arrow[d] \\ K \arrow[r, "i"'] & A \arrow[r, "\alpha"'] & Q. \end{tikzcd}$$

Definition 4.3 *Let $\mathbf{k} = (k^2, k^1), \mathbf{l} = (l^2, l^1)$ be morphisms $(M \to P) \to (A \to Q)$ such that $k^2($* Ker $\mu) = l^2($ Ker $\mu) = K$. *A homotopy $\mathbf{h} : \mathbf{k} \simeq \mathbf{l}$ is a pair of functions $h_1 : P \to A, h_2 : M \to K$ such that*

CSH1) h_1 is an l^1-derivation, and h_2 is an l^1-operator morphism, i.e. if $p_1, p \in P, m, m_1 \in M$

$$h_1(p_1 p) = (h_1 p_1)^{l^1 p}(h_1 p)_1$$

$$h_2(m^p m_1) = (h_2 m)^{l^1 p}(h_2 m).$$

CSH2) if $p \in P$, then

$$k^1 p = (l^1 p)(\alpha_1 p),$$

CSH3) if $m \in M$, then

$$k^2 m = (l^2 m)(h_1 \mu m)(i h_2 m).$$

Proposition 4.4 *A homotopy $\mathbf{h} : \mathbf{k} \simeq \mathbf{l}$ of morphisms as above induces an isomorphism $\theta_{\mathbf{h}} : E(\mathbf{k}) \to E(\mathbf{l})$ such that $\theta_{\mathbf{h}} i_{\mathbf{k}} = i_{\mathbf{l}}, \pi_{\mathbf{l}}\theta_{\mathbf{h}} = \pi_{\mathbf{k}}, \omega_{\mathbf{l}}\theta_{\mathbf{h}} = \omega_{\mathbf{k}}$.*

Proof Consider the diagram

$$\begin{tikzcd}
\text{Ker } \mu \arrow[r] \arrow[d] & M \arrow[r, "\mu"] \arrow[d, "k^2"'] & P \arrow[r, "\phi"] \arrow[d, "f"] & G \arrow[d, "1"] \\
K \arrow[r, "i"] & A \arrow[r, "i_{\mathbf{l}}"] \arrow[d, "1"'] & E(\mathbf{l}) \arrow[r, "\pi_{\mathbf{l}}"] \arrow[d, "\omega_{\mathbf{l}}"] & G \\
 & A \arrow[r, "\alpha"'] & Q &
\end{tikzcd}$$

in which $f : p \mapsto [\![p, h_1 p]\!]_{\mathbf{l}}$, where $[\![,]\!]_{\mathbf{l}}$ denotes the class in $E(\mathbf{l})$, and similarly for $E(\mathbf{k})$. We first prove that (k^2, f) is a morphism of crossed modules.

That f is a morphism of groups is clear since h_1 is an l^1-derivation. Let $m \in M$. Then

$$\begin{aligned} f\mu m &= [\![\mu m, h_1 \mu m]\!]_{\mathbf{l}} \\ &= [\![1, (l^2 \mu m)(h_1 \mu m)]\!]_{\mathbf{l}} \\ &= [\![1, (k^2 m)(i h_2 m)^{-1}]\!]_{\mathbf{l}} \\ &= i_{\mathbf{l}} k^2 m. \end{aligned}$$

Also, if $p \in P$, then

$$\begin{aligned} \omega_{\mathbf{l}} f(p) &= (l^1 p)(\alpha h_1 p) \\ &= k^1 p. \end{aligned}$$

Finally, (k^2, f) preserves the action of P on M given by k^1 since if $m \in M, p \in P$, then

$$\begin{aligned} (k^2 m)^{fp} &= (h_1 p)^{-1} (k^2 m)^{l^1 p} (h_1 p) \\ &= (k^2 m)^{k^1 p}. \end{aligned}$$

Now (iv) of Proposition 4.1 gives the required isomorphism $\theta_{\mathbf{h}}$. □

There is a converse to the previous proposition.

Proposition 4.5 *Let $\mathbf{k}, \mathbf{l}$ be morphisms $(M \to P) \to (A \to Q)$ such that $k^2($ Ker $\mu) = l^2($ Ker $\mu) = K$, and let $\theta : E(\mathbf{k}) \to E(\mathbf{l})$ be an isomorphism such that $\theta i_{\mathbf{k}} = i_{\mathbf{l}}, \pi_{\mathbf{l}}\theta = \pi_{\mathbf{k}}, \omega_{\mathbf{l}}\theta = \omega_{\mathbf{k}}$. Suppose either*
(i) $K = 1$, or
(ii) P is a free group.
Then there is a homotopy $\mathbf{h} : \mathbf{k} \simeq \mathbf{l}$ such that $\theta = \theta_{\mathbf{h}}$.

Proof Let $p \in P$. Then

$$\pi_{\mathbf{l}}([\![p, 1]\!]_{\mathbf{l}}^{-1} \theta([\![p, 1]\!]_{\mathbf{k}})) = (\phi p)^{-1}(\phi p) = 1$$

Hence there is an element $h_1p \in A$ such that

$$i_{\mathbf{l}}h_1p = [\![p,1]\!]_{\mathbf{l}}^{-1}\theta([\![p,1]\!])_{\mathbf{k}},$$

i.e.

$$\theta[\![p,1]\!]_{\mathbf{k}} = [\![p,h_1p]\!]_{\mathbf{l}}. \tag{3}$$

(i) Suppose $K = 1$, so that $i_{\mathbf{k}}, i_{\mathbf{l}}$ are injective. Then the element h_1p is uniquely determined, and in particular $h_1(1) = 1$. That θ is a morphism implies that h_1 is an l^1-derivation.

Let $m \in M$. Then

$$\begin{aligned}
[\![1,k^2m]\!]_{\mathbf{l}} &= \theta[\![1,k^2m]\!]_{\mathbf{k}} \\
&= \theta[\![\mu m,1]\!]_{\mathbf{k}} \\
&= [\![\mu m,h_1\mu m]\!]_{\mathbf{l}} \\
&= [\![1,(l^2m)(h_1\mu m)]\!]_{\mathbf{l}}.
\end{aligned}$$

Since $i_{\mathbf{l}}$ is in-jective, $k^2m = (l^2m)(h_1\mu m)$.

Let $p \in P$. Then

$$\begin{aligned}
k^1p &= \omega_{\mathbf{k}}[\![p,1]\!]_{\mathbf{k}} \\
&= \omega_{\mathbf{l}}\theta[\![p,1]\!]_{\mathbf{k}} \\
&= \omega_{\mathbf{l}}[\![p,h_1p]\!]_{\mathbf{l}} \\
&= (l^1p)(\alpha h_1p).
\end{aligned}$$

This verifies that h_1 is a homotopy $\mathbf{k} \simeq \mathbf{l}$.

(ii) Suppose that P is free on the set X. We start the proof as before, but simply choosing an element $h_1x \in A$ for each $x \in X$ such that (3) holds with $p = x$. This function h_1 extends uniquely to a derivation h_1 such that (3) holds for all $p \in P$. Let $m \in M$. As before, it follows that

$$[\![1,k^2m]\!]_{\mathbf{l}} = [\![1,(l^2m)(h_1\mu m)]\!]_{\mathbf{l}}.$$

Since $K = \mathrm{Ker}\ i_{\mathbf{k}}$, it follows that there is a unique element $h_2m \in K$ such that

$$k^2m = (l^2m)(h_1\mu m)(\iota h_2m).$$

Using the fact that ι is injective, it can be proved that h_2 is an l_1-operator morphism. So we have a homotopy $\mathbf{k} \simeq \mathbf{l}$. $\square$

Now we come to the general classification theorem. We write

$$[M \xrightarrow{\mu} P, A \xrightarrow{\alpha} Q]^K$$

for the set of homotopy classes of morphisms $\mathbf{k} : (M \to P) \to (A \to Q)$ such that $k^2($ Ker $\mu) = K$. We write $XS_{A\to Q}(G, K \to A)$ for the equivalence classes of crossed sequences (4) of type $\alpha : A \to Q$, where an equivalence

$$1 \to K \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1, \qquad \omega : E \to Q \tag{4}$$

$$1 \to K \to A \xrightarrow{i'} E' \xrightarrow{\pi'} G \to 1, \qquad \omega' : E' \to Q$$

is an isomorphism $\theta : E \to E'$ such that $\theta i = i', \pi\theta = \pi, \omega'\theta = \omega$.

Theorem 4.6 *There is a natural function*

$$\mathbf{E}_K : [M \xrightarrow{\mu} P, A \xrightarrow{\alpha} Q]^K \to XS_{A \to Q}(G, K \to A)$$

such that:
(i) $\mathbf{E}_K$ *is injective if (a)* $K = 1$, *or (b)* P *is free;*
(ii) $\mathbf{E}_K$ *is surjective if* $K = 1$ *and* P *is free.*

Proof The proof of this has been given except for the proof of surjectivity, that is the construction from a crossed sequence (4) of type $\alpha : A \to Q$ of a morphism $\mathbf{k} : (M \to P) \to (A \to Q)$.

We refer to the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Ker}\ \mu \arrow[r] \arrow[d, dashed, "k^3"] & M \arrow[r, "\mu"] \arrow[d, dashed, "k^2"] & P \arrow[r, "\phi"] \arrow[d, dashed, "k^1"] & G \arrow[r] \arrow[d, "1"] & 1 \\
1 \arrow[r] & K \arrow[r] & A \arrow[r, "i"'] \arrow[d, "1"'] & E \arrow[r, "\pi"'] \arrow[d, "\omega"] & G \arrow[r] & 1 \\
 & & A \arrow[r, "\alpha"'] & Q & &
\end{tikzcd}$$

If P is free, then the morphism k_1 exists such that $\pi k_1 = \phi$, and we set $k^1 = \omega k_1$. The morphism k^2 is then uniquely determined if i is injective, i.e. if $K = 1$. □

In the general case $K \neq 1$, there seems no reason for the function $\mathbf{E}_K$ of the theorem to be surjective, even if P and M are free. The problem is that while one can in the latter case produce a morphism of crossed modules $\mathbf{k} = (k^2, k^1) : (M \to P) \to (\alpha : A \to Q)$ such that $k^2(\mathrm{Ker}\ \mu) \subseteq K$, it is difficult to ensure that $k^2(\mathrm{Ker}\ \mu) = K$, except in the case $K = 1$. This problem is addressed in [10]. Thus this theorem leaves us with the problem of determining or characterising the image and kernel of these $\mathbf{E}_K$ in terms of properties of $M \xrightarrow{\mu} P$, $A \xrightarrow{\alpha} Q$, and K.

References

[1] Baues, H. 1991 *Combinatorial homotopy and 4-dimensional complexes.* Berlin, New York. Walter de Gruyter.

[2] Brown, R. and Higgins, P.J. 1982, Crossed complexes and non-abelian extensions. In K.H.Kamps *et al* (eds), *Category Theory Proceedings, Gummersbach, 1981*, 39-50. Lecture Notes in Mathematics. Berlin. Springer.

[3] Brown, R. and Higgins, P.J. 1987 Homotopies and tensor products for ω-groupoids and crossed complexes. *Journal of Pure and Applied Algebra.* **47** 1-33.

[4] Brown, R. and Higgins, P.J. 1992 The classifying space of a crossed complex. *Mathematical Proceedings of the Cambridge Philosophical Society* **110** 95-120.

[5] Brown, R. and Huebschmann, J. 1982 Identities among relations. In R. Brown and T. L. Thickstun (eds) *Low-Dimensional Topology* 153-202. London Mathematical Society Lecture Notes, 48. Cambridge University Press.

[6] Brown, R. and Mucuk, O. 1994 Covering groups of non connected topological groups revisited. *Mathematical Proceedings of the Cambridge Philosophical Society* **115** 97-110.

[7] Brown, R. and Razak Salleh, A. 1999 Free crossed crossed resolutions of groups and presentations of modules of identities among relations, *LMS J. Comp. and Math.* **2** 28-61.

[8] Brown, R. and Wensley, C.D. 1995 Finite induced crossed modules and the homotopy 2-type of mapping cones. *Theory and Applications of Categories* **1** 51-74.

[9] Brown, R. and Wensley, C.D. 1996 Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy 2-types. *Theory and Applications of Categories* **2** 3-16.

[10] Cobbe, Anne 1951 Some algebraic properties of crossed modules. *Quarterly Journal of Mathematics Oxford* **(2)** **2** 269-285.

[11] Cockcroft, W.H. 1951 The word problem in a group extension. *Quarterly Journal of Mathematics Oxford* **(2)** **2**123-134.

[12] Dedecker, P. 1964 Les foncteurs Ext_Π, H^2_Π et H^2_Π non abéliens. *Compte Rendues Academie Science Paris* **258** 4891-4894.

[13] Hall, Marshall Jr. 1959 *The theory of groups*. Macmillan. New York.

[14] Hog-Angeloni, C., Metzler, W. and Sieradski, A.J. (eds) 1993 *Two-dimensional homotopy and combinatorial group theory*, London Math. Society Lecture Note Series 197, Cambridge University Press, Cambridge.

[15] Huebschmann, J. 1980 Crossed n-fold extensions and cohomology. *Commentarii Mathematii Helvetii* **55** 302-314.

[16] Johnson, D.L. 1990 *Presentations of groups*. London Mathematical Society Texts. Cambridge University Press.

[17] Lyndon, R.C. 1950, Cohomology of groups with a single defining relation, *Annals of mathematics* **52** 650-665.

[18] Nicholson, Julia 1993 *Otto Hölder and the development of group theory and Galois theory*. DPhil thesis, Oxford.

[19] Pride, S.J. 1991 Identities among relations. In *Proc. Workshop on Group Theory from a Geometrical Viewpoint, International Centre of Theoretical Physics, Trieste, 1990* E.Ghys, A.Haefliger and A. Verjodsky (eds) 687-716. Singapore. World Scientific.

[20] Schreier, O. 1926 Über die Erweiterung von Gruppen, I. *Monatschefte für Mathematik und Physik* **34** 165-180.

[21] Turing, A. 1938. The extensions of a group. *Compositio Mathematica.* **5** 357-367.

[22] Whitehead, J.H.C. 1949 Combinatorial homotopy II. *Bulletin American Mathematical Society* **55** 453-496.

[23] Zassenhaus, H. 1949 *The theory of groups* (trans. by S. Kraverts). New York, Dover.

Additional references

[A] Brown, R., Bullejos, M. and Porter, T. 2002 Crossed complexes, free crossed resolutions and graph products of groups, *Proceedings Workshop Korea 2000*, J. Mennicke, Moo Ha Woo (eds.) Recent Advances in Group Theory, Heldermann Verlag Research and Exposition in Mathematics **27** 8-23.

[B] Brown, R., Moore, E., Porter, T. 2002 Crossed complexes, and free crossed resolutions for amalgamated sums and HNN-extensions of groups, *Georgian Math. J.* **9** 623-644.
[C] Brown, R., Multiple groupoids as a non commutative tool for higher dimensional local-to-global problems
http://www.bangor.ac.uk/~mas010/fields-art3.pdf
[D] Brown, R. and Wensley, C.D. 2003 Computation and homotopical applications of induced crossed modules, *J. Symbolic Computation* **35** 59-72.