HOMOTOPICAL EXCISION, AND HUREWICZ THEOREMS, FOR n-CUBES OF SPACES *

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Introduction

The fact that the relative homotopy groups do not satisfy excision makes the computation of absolute homotopy groups difficult in comparison with homology groups. The failure of excision is measured by triad homotopy groups $\pi_n(X; A, B)$, with $n \geqslant 3$ (for $n = 2$, this gives a based set), which fit into an exact sequence .

$$\begin{aligned}\pi_{n+1}(A, A \cap B) \xrightarrow{e} \pi_{n+1}(X, B) \to \pi_{n+1}(X; A, B)\\ \to \pi_n(A, A \cap B) \to \pi_n(X, B) \to \cdots\end{aligned}$$

where e is induced by the 'excision' inclusion. That e is an isomorphism in a range of dimensions is shown by the classical Blakers-Massey triad connectivity theorem: if A, B, and $A \cap B$ are connected, $\{A, B\}$ is an open cover of X and $(A, A \cap B)$ is p-connected, $(B, A \cap B)$ is q-connected, then the triad $(X; A, B)$ is $(p + q)$-connected. (See, for example, [11, p. 211].) Further, if $p, q \geqslant 2$ and $\pi_1(A \cap B) = 0$, the critical group $\pi_{p+q+1}(X; A, B)$ is described in [2] as a tensor product of abelian groups $\pi_{p+1}(A, A \cap B) \otimes \pi_{q+1}(B, A \cap B)$.

One of our main results (Theorem 4.2) extends this description of the critical group to the cases where $p, q \geqslant 1, \pi_1(A \cap B) = 1 = 0$. Note that if p or q is 1, then one at least of the groups $\pi_{p+1}(A, A \cap B), \pi_{q+1}(B, A \cap B)$ may be non-abelian, and acts on the other group. In the description of the critical group, the usual tensor product must be replaced by the tensor product $G \otimes H$ defined in [5, 6], which involves actions of G on H and H on G. This description of $\pi_{p+q+1}(X; A, B)$ is a special case of a description of the hyper-relative group $\pi_{n+1}(X; A_1, \ldots, A_n)$ of a 'connected' excisive $(n+1)$-ad as determined by the lower dimensional information involved in the $(n+1)$-ad; a precise description is given in Theorem 4.1. As another consequence of Theorem 4.1 we obtain an exact sequence for a connected space

$$\pi_2 X \xrightarrow{E^2} \pi_4 S^2 X \xrightarrow{H^2} \pi_1 X \tilde{\wedge} \pi_1 X \xrightarrow{P} [\pi_1 X, \pi_1 X] \to 1,$$

*A.M.S. (1980) subject classification: 55Q30, 55Q15, 55Q52. Proc. London Math. Soc. (3) **5**4 (1987) 176-192 [Received 29 August 1985]

Acknowledgements. We are indebted to the British Council for support of visits by both of us in 1982 and 1983 under their Academic Exchange Scheme, to the Université Louis Pasteur for support of visits by R. Brown in 1984, and to the S.E.R.C. (U.K.) for support of visits by J.-L. Loday as Visiting Fellow in 1983 and 1985 (under grant GR/C/71675). We thank the referee for suggestions for shortening the proofs of Propositions 3.2 and .6.

where for a group G, the group $G \tilde{\wedge} G$ is obtained from the tensor product $G \otimes G$, with action of G on itself given by conjugation, by factoring by the relations

$$(x \otimes y)(y \otimes x)^{-1} = 1 \quad \text{for all } x, y \in G.$$

Our other results involve a hyper-relative form of homotopical excision. Let the space Y be covered by open sets $X, B_1, \dots B_n$ and let $A_i = X \cap B_i$, for $i = 1 \dots n$. We regard the excision map as the inclusion of $(n+1)$-ads

$$\mathbf{X} = (X; A_1, \dots, A_n) \to \mathbf{Y} = (Y; B_1, \dots, B_n).$$

Theorem 6.1 can be interpreted as describing, under appropriate connectivity assumptions, the group $\pi_{n+1}\mathbf{Y}$ as 'induced' from the group $\pi_{n+1}\mathbf{X}$ by the morphisms of 'lower dimensional information' . As a consequence, on taking $Y = X \cup C(A_1 \cup \dots \cup A_n), B_i = CA_i$, we obtain two forms of a Hurewicz theorem for $\pi_{n+1}(X \cup A_1, \cdots, A_n)$. We also obtain an explicit description of the triad homotopy group $\pi_3(X \cup \{e^3_\lambda\}; A, B)$ in terms of information on the triad $(X; A, B)$ and the attaching maps of the 3-cells. This result is analogous to Whitehead's description of the relative homotopy group $\pi_2(X \cup \{e^2_\lambda\}, X)$ as a free crossed $\pi_1 X$-module on the characteristic maps of the 2-cells e^2_λ.

All of our homotopical results are deductions from the Van Kampen theorem for n-cubes of spaces proved in [7]. This theorem states that the fundamental cat^n-group functor Π from n-cubes of spaces to cat^n-groups preserves certain colimits. The main trick for the applications is to choose the colimit of n-cubes in a way allied to the given structure of n-cubes. For example, if $X = U_1 \cup \dots \cup U_n$ then the sets U_i determine, by inclusion of intersections, an n-cube of spaces X; they also, as a covering of X, determine by intersection with X, a Van Kampen theorem for ΠX. The analysis of this kind of situation is accomplished in §1, *Formalities on n-cubes*. The notion of *n-pushout* is crucial throughout this paper. In §2, we recall the main facts on cat^n-groups, the functor Π, and the Van Kampen theorem. In §3 we discuss a particular kind of colimit of cat^n-groups, the *universal* cat^n-groups. This notion is applied in §4 to the Blakers-Massey theorem. In §5 we discuss *induced cat^n-groups* –this notion generalizes the notions of induced module and induced crossed module. Induced cat^n -groups are applied in §6 to give an 'excision theorem' and Hurewicz theorems for n-cubes of maps.

1 Formalities on n-cubes

The set $\{0,1\}$ is given its usual partial order with $0 < 1$. The product set $\{0,1\}^n$ is given the product partial order. This poset (partially ordered set) is a lattice, with meet operation $\wedge$. That is $a \wedge b$ is the greatest lower bound of a and b. The bottom element of $\{0,1\}^n$ is $\mathbf{0} = (0, \dots, 0)$, and the top element is $\mathbf{1} = (1, \dots, 1)$.

Let $\mathcal{C}$ be a category. An n-cube in $\mathcal{C}$ is a functor $\{0,1\}^n \to \mathcal{C}$, where the poset $\{0,1\}^n$ is regarded as a category in the usual way. We write $N_n\mathcal{C}$ for the class of all these n-cubes. The family $\{N_n\mathcal{C}\}_{n \geqslant 0}$ has also the standard cubical face operators ∂^α_i and degeneracy operators ε_i giving a cubical complex $N\mathcal{C}$, the *cubical nerve* of $\mathcal{C}$. For instance, a 1-cube in $\mathcal{C}$ is a map $f : A \to B$ and $\partial^0_1(f) = A, \partial^1_1(f) = B$, and $\varepsilon_1(A) = 1_A : A \to A$.

Let $\mathbf{X}$ be an n-cube in $\mathcal{C}$, and let $\alpha \in \{0,1\}^n$ be such that $i_i < \dots < i_r$ are exactly those indices i for which $\alpha_1 = 0$. The $(n-r)$-cube $R_\alpha\mathbf{X}$ is defined to be $\partial^0_{i_1} \dots \partial^0_{i_r}\mathbf{X}$.

There is an important formalism of passing from an n-cube in $\mathcal{C}$ to an n-cube of n-cubes in $\mathcal{C}$. Let $\mathbf{X}$ be an n-cube in $\mathcal{C}$. The n-cube $\mathbf{X}^\square$ of n-cubes in $\mathcal{C}$ is defined by

$$\mathbf{X}^{\square}(\alpha)(\beta) = \mathbf{X}(\alpha \wedge \beta), \quad \text{where } \alpha, \beta \in \{0,1\}^n,$$

where $\alpha \wedge \beta$ is the meet operation. For example, if $n = 2$, the square

$$\mathbf{X}: \quad \begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

which is often abbreviated to

$$\begin{matrix} C & A \\ B & X \end{matrix}$$

determines the square of squares

$$\mathbf{X}^{\square}: \quad \begin{tikzcd} \begin{matrix} C & C \\ C & C \end{matrix} \arrow[r] \arrow[d] & \begin{matrix} C & C \\ B & B \end{matrix} \arrow[d] \\ \begin{matrix} C & A \\ C & A \end{matrix} \arrow[r] & \begin{matrix} C & A \\ B & X \end{matrix} \end{tikzcd}$$

Note that, in the general case, if $\alpha \neq \mathbf{1}$, then $\mathbf{X}^{\square}(\alpha)$ is a degeneracy of R_α, X.

We now define the notion of *excision* for n-cubes. Recall that if a space X is a union of subspaces A, B, then the 'excision map' is the map of pairs

$$(A, A \cap B) \to (X, B).$$

Alternatively, we have a square of spaces

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

and the excision map regards this as a 'map of maps'

$$(A \cap B \to A) \to (B \to X).$$

Suppose now that $\mathbf{X}$ is an n-cube in a category $\mathcal{C}$. Then $\mathbf{X}$ defines a map of $(n-1)$-cubes

$$e_{\mathbf{X}} : \partial_n^0 \mathbf{X} \to \partial_n^1 \mathbf{X};$$

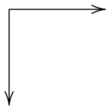
this we call the *excision map* of $\mathbf{X}$. This map determines also

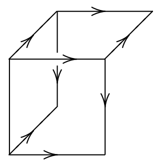
$$(e_{\mathbf{X}})^{\square} : (\partial_n^0 \mathbf{X})^{\square} \to (\partial_n^1 \mathbf{X})^{\square}$$

an 'excision map' of $(n-1)$-cubes of $(n-1)$-cubes, or if convenient, an n-cube of $(n-1)$-cubes.

In the applications we make essential use of a generalization of pushout squares to pushout n-cubes. An n-*corner* in a category $\mathcal{C}$ is a functor $\mathbf{Y}$ from $\{0,1\}^n \backslash \{\mathbf{1}\}$ to $\mathcal{C}$. The colimit of such an

n-comer $\mathbf{Y}$, if it exists, is called the *pushout* of $\mathbf{Y}$. Then $\mathbf{Y}$ and the universal natural transformation $\mathbf{Y} \to \mathrm{colim}\ \mathbf{Y}$ define an n-cube $\mathbf{X}$ in $\mathcal{C}$ with $\mathbf{X}(\mathbf{1}) = \mathrm{colim}\ \mathbf{Y}$ and for $\alpha \neq 1, \mathbf{X}(\alpha) = \mathbf{Y}(\alpha)$. Such an $\mathbf{X}$ we call an n-*pushout*, or *pushout* n-*cube*, in $\mathcal{C}$. Note that the diagrams for 2-corners and 3-corners are of the following forms respectively:





We mention for future reference the following result.

Proposition 1.1 *A degenerate n-cube is a pushout n-cube.*

2 The fundamental cat^n-group functor

We recall the notions of cat^n-groups, crossed squares, fundamental cat^n-group functor, and also the Van Kampen theorem for n-cubes which we use extensively.

First, a *cat*^{1}-*group* $\mathbf{G} = (G; s, b)$ is a group G with two endomorphisms s, b of G satisfying

(i) $sb = b, bs = s$,

(ii) $[\mathrm{Ker}\ s, \mathrm{Ker}\ b] = 1$.

From (i) it is easily deduced that $s^2 = s, b^2 = b, \mathrm{Im}\, b = \mathrm{Im}\, s$ and that s, b are the identity on $\mathrm{Im}\, s$. The cat^1-group G determines, and in fact is equivalent to, the crossed module $b| : \mathrm{Ker}\ s \to \mathrm{Im}\, s$ (cf. [**10**, Lemma 2.1]).

Next, a *cat*^{n}-*group* $\mathbf{G} = (G; s_1, b_1 \dots, s_n, b_n)$ consists of a group G with n cat^n-group structures $s_1; b_1, \dots, s_n, b_n$ which commute in the sense that

(iii) $s_i b_j = b_j s_i; \quad s_i s_j = s_j s_i \quad b_i b_j = b_j b_i, \quad$ if $i \neq j$.

The group G is called the *big group* of the cat^n-group $\mathbf{G}$ and will be written $\mathbf{B}(\mathbf{G})$; the intersection $\mathbf{L}(\mathbf{G}) = \bigcap_{i=1}^{n} \mathrm{Ker}\ s_i$ is also important in applications and is emphasized in [10, 7].

A *morphism* $f : \mathbf{G} \to \mathbf{H}$ of cat^n-groups is a homomorphism $f : G \to H$ of groups which commutes with the s_i, b_i, for $i = 1, \dots, n$. So we have a category (cat^n-groups) of cat^n -groups.

For explicit calculations with cat^n -groups it is useful to have an equivalent formulation in terms of 'crossed n-cubes'. A crossed 2-cube (or crossed square) consists of a commutative square of homomorphisms of groups

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

actions of P on L, M, N, and a function $h : M \times N \to P$ satisfying a number of rules which are given in [10] and, in slightly different formulation, in [7]. Given a cat^2-group $\mathbf{G} = (G; s_1, b_1, s_2, , b_2)$ its associated crossed square has

$$P = \operatorname{Im} s_1 \cap \operatorname{Im} s_2, \quad M = \operatorname{Ker} s_1 \cap \operatorname{Im} s_2,$$
$$N = \operatorname{Im} s_1 \cap \operatorname{Ker} s_2, \quad L = \operatorname{Ker} s_1 \cap \operatorname{Ker} s_2;$$

the maps λ', μ are induced by b_1; the maps λ, ν are induced by b_2; the actions of P and the h-function are determined by conjugation and commutators respectively. It is a theorem of D. Guin-Waléry and J.-L. Loday that this association gives an equivalence between cat^2-groups and crossed squares [**10**, §5]. *Crossed n-cubes* are defined in [**8,9**], and an equivalence between them and cat^n-groups is proved there. We shall use some information on crossed 3-cubes in §4.

In [**10**, Theorem 1.4] there is defined a functor from n-cubes of fibrations to cat^n-groups. This functor may be composed with a standard functor $\mathbf{X} \mapsto \bar{\mathbf{X}}$ from n-cubes of spaces to n-cubes of fibrations to give the *fundamental cat^n-group functor* Π from n-cubes of spaces to cat^n-groups [**7**].

We will use this functor in the case where the n-cube of spaces $\mathbf{X}$ is constructed from n subspaces $(A_1, \ldots, A_n)$ of the space $\mathbf{X}(\mathbf{1}) = X$ by taking intersections. Explicitly, one has that $\mathbf{X}(\alpha)$ is the intersection of the A_j such that $\alpha_j = 0$. In this case the big group G of the cat^n-group $\Pi\mathbf{X}$ is $\pi_1(\Phi)$ where Φ is the function space of base-pointed maps from I^n to X which send the faces $\partial_i^0 I^n$ and $\partial_i^1 I^n$ into A_i for $i = 1, \ldots, n$. In this framework the group $\mathbf{L}(\Pi\mathbf{X})$, which is the intersection $\bigcap_{i=1}^n \operatorname{Ker} s_i$, has the following interpretation. It is the fundamental group of the function space Ψ of base-pointed maps from I^n to X which send the faces $\partial_i^0 I^n$ to the base-point and the faces $\partial_i^1 I^n$ into A_i for $i = 1, \ldots, n$. In other words $\mathbf{L}(\Pi\mathbf{X})$ is the hyper-relative homotopy group $\pi_{n+1}(X; A_1 \ldots, A_n)$.

We recall now the Van Kampen theorem for n-cubes of spaces proved in [**7**, Theorem 5.4]. An n-cube $\mathbf{X}$ is called connected if its associated n-cube of fibrations $\bar{\mathbf{X}}$ has all its spaces $\bar{\mathbf{X}}(\beta)$ (path-) connected, $\beta \in \{-1, 0, 1\}^n$; thus not only is each of the original spaces $\mathbf{X}(\alpha)$ connected, for $\alpha \in \{0, 1\}^n$, but also the various homotopy fibres constructed from $\mathbf{X}$ are connected. For example, if $\mathbf{X}$ is the square

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "f'"'] & A \arrow[d, "a"] \\ B \arrow[r, "b"'] & X \end{tikzcd}$$

then $\mathbf{X}$ is connected if C, A, B, X and the homotopy fibres $F(a), F(b), F(f), F(f')$, and $F(\mathbf{X}) = F(F(f') \to F(a))$ are all connected.

Theorem 2.1 *Let $\mathbf{X}$ be an n-cube of spaces and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of $\mathbf{X}(\mathbf{1})$. If σ is a finite, non-empty subset of Λ, then U_σ, the intersection of the sets U_λ for $\lambda \in \sigma$, determines by inverse image an n-cube of spaces $\mathbf{U}_\sigma$. Suppose that each such $\mathbf{U}_\sigma$ is connected. Then the n-cube $\mathbf{X}$ is connected and the natural homomorphism of cat^n-groups* $\operatorname{colim} \Pi\mathbf{U}_\sigma \to \Pi\mathbf{X} = \Pi \operatorname{colim} \mathbf{U}_\sigma$ *is an isomorphism.*

In this theorem the cube structure and the covering of the bottom space $X = \mathbf{X}(\mathbf{1})$ are independent. However in the following we are dealing with a particular situation. We start with an n-covering of X and we consider the cube which is determined by this covering (see above). As a consequence, the cubes $\mathbf{U}_\sigma$ are degenerate cubes except for $\mathbf{X}$ itself (cf. §1). This leads to a particular type of cat^n-group which is discussed in the next section.

3 Universal cat^n-groups

Colimits of cat^n-groups are easily described. To form $\mathrm{colim}_\lambda\,\mathbf{G}_\lambda$, one first forms the group $G = \mathrm{colim}\,G_\lambda$. The structure maps s_i, b_i on each G_λ determine homomorphisms $s_i b_i$ on G satisfying conditions (i) and (iii) above. However, there is no reason for (ii) to be satisfied, and so $\mathbf{G}$ becomes only a 'pre-cat^n-group'. The cat^n-group $\mathrm{colim}_\lambda\,\mathbf{G}$ is $(G/N; s_1, b_1, \dots, s_n, b_n)$ where the normal subgroup N is generated by the subgroups $[\mathrm{Ker}\ s_i, \mathrm{Ker}\ b_i]$, for $i = 1, \dots, n$ and the structure maps are induced by those on G (cf. [**6**]).

We shall also need to consider n-cubes of cat^n-groups. They were used already in [**10**], and for $n = 2$ in [**7**].

Definition 3.1 Let $\mathbf{G} = (G; s_1, b_1, \dots, s_n, b_n)$ be a cat^n-group. The *associated* n*-cube* $\mathbf{G}^\square$ of cat^n-groups has $\mathbf{G}^\square(\mathbf{1}) = G$, and for other $\alpha \in \{0,1\}^n, \mathbf{G}^\square(\alpha)$ has big group the intersection of the groups $\mathrm{Im}\, s_i (= \mathrm{Im}\, b_i)$ for which $\alpha_i = 0$, with the restricted cat^n-structure. It is useful to think of the cat^n-groups $\mathbf{G}^\square(\alpha)$ for $\alpha \neq \mathbf{1}$ as representing the 'lower dimensional part' of $\mathbf{G}$.

Proposition 3.2 *Let the cat^n-group* $\mathbf{G}$ *be given as a colimit of cat^n-groups,* $\mathbf{G} = \mathrm{colim}_\lambda\,\mathbf{G}_\lambda$. *Let* $\alpha \in \{0,1\}^n$. *Then* $\mathbf{G}^\square_\lambda(\alpha) = \mathrm{colim}_\lambda\ \mathbf{G}^\square_\lambda(\alpha)$.

Proof We recall that $\mathbf{B}(\mathbf{G})$ denotes the big group of a cat^n-group G. Let

$$G = \mathrm{colim}_\lambda\ \mathbf{B}(\mathbf{G}_\lambda), \quad G(\alpha) = \mathrm{colim}_\lambda\ \mathbf{B}(\mathbf{G}^\square_\lambda(\alpha)).$$

Let N and $N(\alpha)$ be the normal subgroups of G and of $G(\alpha)$ generated by the $[\mathrm{Ker}\ s_i, \mathrm{Ker}\ b_i]$. Let S be the composite of the s_i for which $a_i = 0$. Since $S = SS, S$ preserves colimits. So

$$G(\alpha) = \mathrm{colim}_\lambda\ S\mathbf{B}(\mathbf{G}_\lambda) = S\,\mathrm{colim}_\lambda\ \mathbf{B}(\mathbf{G}_\lambda) = SG.$$

Thus $N(\alpha)$ is the normal subgroup of SG generated by the $[\mathrm{Ker}\ s_i \mathrm{Ker}\ b_i]$; one can check that this is SN. Thus

$$\mathbf{B}(\mathrm{colim}_\lambda(\mathbf{G}^\square_\lambda(\alpha))) = G(\alpha)/N(\alpha) = SG/SN = S(G/N) = S\mathbf{B}(\mathbf{G}) = \mathbf{B}(\mathbf{G}^\square(\alpha)).$$

This gives the result. □

If $\mathbf{G}$ is a cat^n-group, we write $\mathbf{G}^\ulcorner$ for the n-corner of cat^n-groups determined by $\mathbf{G}^\square$.

Proposition-Definition 3.3 *Let* $\mathbf{G}$ *be a cat^n-group. The following two conditions are equivalent and define* $\mathbf{G}$ *as a universal cat^n-group:*

(i) the n*-cube* $\mathbf{G}^\square$ *of cat^n-groups is a pushout cube in the category of cat^n-groups.*

(ii) if $\mathbf{H}$ *is a cat^n-group such that* $\mathbf{H}^\ulcorner = \mathbf{G}^\ulcorner$ *then there is a unique morphism* $\mathbf{G} \to \mathbf{H}$ *of cat^n-groups inducing the identity* $\mathbf{G}^\ulcorner \to \mathbf{H}^\ulcorner$.

Proof. That (i) implies (ii) is trivial. □

To prove that (ii) implies (i), let $\mathbf{K}$ be the cat^n-group $\mathrm{colim}\ \mathbf{G}^\ulcorner$. Propositions 1.1 and 3.2 show that $\mathbf{K}^\ulcorner = \mathbf{G}^\ulcorner$. Condition (ii) gives a morphism $G \to \mathbf{K}$, and the definition of $\mathbf{K}$ gives a morphism $\mathbf{K} \to \mathbf{G}$. By the two universal properties, these morphisms are the inverse of each other.

The implication of this definition is that if $\mathbf{G}$ is universal, then its big group G is determined by the family of subgroups $P_i = \mathrm{Im}\, s_i$ and their interrelated cat^n-group structures.

If $\mathbf{G}$ is a cat^2-group with associated crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

then the 2-cube $\mathbf{G}^{\square}$ of cat^2-groups has an associated square of crossed squares of the form

$$\begin{tikzcd} \begin{bmatrix} 1 & 1 \\ 1 & P \end{bmatrix} \arrow[r] \arrow[d] & \begin{bmatrix} 1 & M \\ 1 & P \end{bmatrix} \arrow[d] \\ \begin{bmatrix} 1 & 1 \\ N & P \end{bmatrix} \arrow[r] & \begin{bmatrix} L & M \\ N & P \end{bmatrix} \end{tikzcd}$$

where 1 denotes the trivial group. If this is a push out of crossed squares then the group L is canonically isomorphic to the tensor product $M \otimes N$ defined in [**6, 7**]. For the convenience of the reader we recall this definition.

Let the groups M and N be equipped with an action of M on the left of N, written ${}^m n$, where $m \in M, n \in N$, and an action of N on the left of M written ${}^n m$. It is always understood that a group acts on the left of itself by conjugation: ${}^x y = xyx^{-1}$. The *tensor product* $M \otimes N$ is the group generated by symbols $m \otimes n$ with the relations

(a) $mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n)$,

(a') $m \otimes nn' = (m \otimes n)({}^n m \otimes {}^n n')$,

for all $m, m' \in M, n, n' \in N$.

For the Blakers-Massey theorem in §4 we need to analyse particular kinds of cat^n-groups and their corresponding universal cat^n-groups.

Proposition 3.4 *The following are equivalent categories:*

(i) *the category of cat^1-groups* $(G; s, b)$ *in which* $s = b$*;*

(ii) *the category of crossed modules* $\mu : M \to P$ *in which* $\mu = 0$*;*

(iii) *the category of cat^n-groups* $(G; s_i, b_i)$*, with* $n \geqslant 2$*, in which* $s_i = s_j$ *for all* i, j.

Proof. The standard equivalence between cat^1-groups and crossed modules has $(G; s, b)$ corresponding to $\mu : \mathrm{Ker}\ s \to \mathrm{Im}\, s$ where μ is the restriction of b, and a crossed module $\mu : M \to P$ corresponding to the cat^2-group $(M \rtimes P; s, b)$ where $s(m, p) = (1, p), b(m, p) = (1, (\mu m)p)$. The equivalence between the categories given in (i), (ii) is automatic.

Now recall the rules $sb = b, bs = s$ for a cat^2-group. So $s = b$ is equivalent to $sb = bs$. The equivalence between the categories given in (i) and (iii) follows. □

Proposition 3.5 *Let* $\mathcal{C}_{p,q}$ *be the category of cat^{p+q}-groups*

$$(G; s_1, b_1, \dots, s_{p+q}, b_{p+q})$$

such that

$$(s_1, b_1) = \cdots = (s_q, b_q), (s_{q+1}, b_{q+1}) = \cdots = (s_{p+q}, b_{p+q}).$$

Then $\mathcal{C}_{p,q}$ is equivalent to the category of crossed squares

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

in which $\mu = 0$ *and* $\lambda' = 0$ *if* $q \geqslant 2, \nu = 0$ *and* $\lambda = 0$ *if* $P \geqslant 2$.

Proof The case in which $p = q = 1$ is the equivalence between cat^2-groups and crossed squares, referred to above. Clearly $\mathcal{C}_{p,q}$ is equivalent to $\mathcal{C}_{p+1,q}$ if $p \geqslant 2$, and to $\mathcal{C}_{p,q+1}$ if $q \geqslant 2$. Also by Proposition 3.4, $\mathcal{C}_{p,2}$ is equivalent to the category of cat^{p+1}-groups $(G; s_i, b_i)$ such that $s_1 = b_1$ and $\mathcal{C}_{q,2}$ is equivalent to the category of cat^{q+1}-groups such that $s_{q+1} = b_{q+1}$ So the required equivalence follows from Proposition 3.4 and the relation between cat^2-groups and crossed squares (§2). □

Proposition 3.6 . *The inclusion* $i : \mathcal{C}_{p,q} \to$ (cat^{p+q}*-groups) preserves colimits.*

Proof A colimit in cat^{p+q}-groups of objects of $\mathcal{C}_{p,q}$ is itself in $\mathcal{C}_{p,q}$ (This proof was pointed out by the referee.) □

Corollary 3.7 *Let* $\mathbf{G}$ *be an object of the category* $\mathcal{C}_{p,q}$*, with associated crossed square* $\begin{pmatrix} L & M \\ N & P \end{pmatrix}$. *Then* $\mathbf{G}$ *is a universal cat*$^{p+q}$*-group if and only if the canonical map* $M \otimes N \to L$ *is an isomorphism.*

4 The Blakers and Massey theorem

Let $\{U_1, \ldots, U_n\}$ be an open covering of a space X. Then we have an n-cube $\mathbf{X}$ such that $\mathbf{X}(\mathbf{1}) = X$ and for $\alpha \neq 1, \mathbf{X}(\alpha)$ is the intersection of the sets U_i for which $\alpha_i = 0$. We apply the generalized Van Kampen theorem to this n-cube X and to the covering $\{U_1, \ldots, U_n\}$ of X.

Theorem 4.1 *Let* $\{U_1 \ldots, U_n\}$ *be an open covering of the space* X *such that each face* $\partial_i^0 \mathbf{X}$ *of the associated cube* $\mathbf{X}$ *is a connected* $(n-1)$*-cube. Then* X *is a connected* n*-cube and its fundamental cat*^{n}*-group* $\Pi\mathbf{X}$ *is universal.*

Proof If σ is a non-empty finite subset of $\{1, \ldots, n\}$, the cube $\mathbf{X}_\sigma$ is defined to be $\mathbf{X} \cap U_\sigma$; it coincides with $\mathbf{X}^\square(\alpha)$ where $\alpha_i = 0$ if and only if $i \in \sigma$.

Since $\partial_i^0 \mathbf{X}$ is connected for $i = 1, \ldots, n$, each subcube $R_\alpha \mathbf{X}$ of $\mathbf{X}$ is connected for $\alpha \neq \mathbf{1}$. For $\alpha \neq \mathbf{1}, \mathbf{X}^\square(\alpha)$ is a degeneracy of $R_\alpha(\mathbf{X})$, and hence $\mathbf{X}^\square(\alpha)$ is connected. Hence each cube $\mathbf{X}_\sigma$ is connected. By the Van Kampen theorem for n-cubes (see §2) the cube $\mathbf{X}$ is connected and

$$\Pi\mathbf{X} = \operatorname{colim} \mathbf{X}_\sigma.$$

Hence $\Pi\mathbf{X} = \operatorname{colim} \Pi\mathbf{X}^\square(\alpha)$, where $\alpha \neq \mathbf{1}$. But the n-cube of cat^n-groups $\Pi(\mathbf{X}^\square)$ is isomorphic to $(\Pi\mathbf{X})^\square$, and hence $\Pi\mathbf{X}$ is universal. □

Remark 4.2 The particular case in which n = 2 was treated in [7].

We will now apply this theorem to a still more particular situation. Recall that if $(X; A, B)$ is a triad, that is, A and B are subspaces of X, containing the base-point, then the triad homotopy groups $\pi_i(X; A, B)$ fit into a long exact sequence $(C = A \cap B)$

$$\cdots \to \pi_{i+1}(A, C) \to \pi_{i+1}(X, B) \to \pi_{i+1}(X; A, B) \to \pi_i(A, C) \to \pi_i(X, B) \to \cdots .$$

As excision does not always hold in homotopy these triad homotopy groups are not always trivial in the excision situation $X = A \cup B$.

Theorem 4.3 (Blakers-Massey theorem) *Let X be the union of open subspaces A, B such that A, B and $C = A \cap B$ are path-connected, and the pairs $(A, C), (B, C)$ are respectively p-connected and q-connected. Then the triad $(X; A, B)$ is $(p + q)$-connected, and the generalized Whitehead product induces a map*

$$[\ ,\ ] : \pi_{p+1}(A, C) \otimes \pi_{q+1}(B, C) \to \pi_{p+q+1}(X; A, B)$$

which is an isomorphism.

Proof. Let $\mathbf{X}$ be the $(p + q + 1)$-ad $(X; A, \dots, A, B, \dots, B)$ with q copies of A and p copies of B. By symmetry, the fundamental cat^{p+q}-group $\Pi\mathbf{X}$ of $\mathbf{X}$ belongs to the category $\mathcal{C}_{p,q}$ (cf. §3). Consider the cover $U_1, \dots, U_{p+q}$ of X where $U_1 = \dots = U_q = A, U_{q+1} = \dots = U_{p+q} = B$. Theorem 4.1 shows first that $\mathbf{X}$ is connected, and hence that $(X; A, B)$ is $(p + q)$-connected. Theorem 4.1 also shows that $\Pi\mathbf{X}$ is universal. By Corollary 3.7, the theorem is proved once we have checked that the crossed square associated to $\Pi\mathbf{X}$ as an object of the category $\mathcal{C}_{p,q}$, is canonically isomorphic to the crossed square

$$\begin{tikzcd}
\pi_{p+q+1}(X; A, B) \arrow[r] \arrow[d] & \pi_{q+1}(B, C) \arrow[d] \\
\pi_{p+1}(A, C) \arrow[r] & \pi_1 C
\end{tikzcd}$$

with h-function given up to sign by the generalized Whitehead product.

The identification of the groups and actions in the crossed square of $\Pi\mathbf{X}$ follows from the construction of $\Pi\mathbf{X}$ (cf. §2). The universal example for the generalized Whitehead product is the triad $(E^{p+1} \vee E^{q+1}; E^{p+1} \vee S^q, S^p \vee E^{q+1})$. Let $\mathbf{W}$ be the $(p + q + 1)$-ad constructed from this triad in the same way as $\mathbf{X}$ is constructed from $(X; A, B)$. Then $\Pi\mathbf{W}$ belongs to the category $\mathcal{C}_{p,q}$ and its associated crossed square is of the above form with X, A, B, C replaced by

$$X' = E^{p+1} \vee E^{q+1}, \quad A' = E^{p+1} \vee S^q, \quad B' = S^p \vee E^{q+1}, \quad C' = S^p \vee S^q.$$

At this stage there arise questions of sign conventions and orientations, none of which affect the statement of the theorem. It is common in the literature to make conventions so that a square whose arrows are boundary maps is anticommutative (cf. [**1**, p. 105]). For our purposes, we want crossed squares to be commutative, and we assume the conventions are arranged in this way. Then the generalized Whitehead product $[t_{p+1}, t_{q+l}]$ in $\pi_r(X'; A', B')$, where $r = p + q + 1$, is defined using the following diagram (cf. [**1**, p. 108]):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi_r(X';A',B') \arrow[r, "d_2"] \arrow[d, "d_1"'] & \pi_{r-1}(A',C') \arrow[r, leftrightarrow] \arrow[d, "d_4"] & \pi_{r-1}(X',B') \\
 & \pi_{r-1}(B',C') \arrow[r, "d_3"'] \arrow[d, shift left] & \pi_{r-2}(C') \arrow[r, leftrightarrow] \arrow[d, shift left] & \pi_{r-2}(B') \\
 & \pi_{r-1}(X',A') \arrow[u, shift left] & \pi_{r-2}(A') \arrow[u, shift left] &
\end{tikzcd}$$

The element $[\iota_{p+1}, \iota_{q+1}]$ is the unique element which in the above central square maps as follows:

$$\begin{tikzcd}
{[\iota_{p+1}, \iota_{q+1}]} \arrow[r] \arrow[d] & \pm[\iota_{p+1}\iota_q] \arrow[d] \\
{[\iota_p, \iota_{q+1}]} \arrow[r] & {[\iota_p, \iota_q]}
\end{tikzcd}$$

where the signs depend on conventions, and the other Whitehead products are relative or absolute.

One now has to examine separately the cases where $p, q > 1$ and those in which p or q is 1. If $p, q > 1$, then $\pi_{p+q+1}(X'; A', B') \cong \mathbb{Z}$ generated by $[\iota_{p+1}, \iota_{q+1}]$ while the crossed square associated to $\Pi \mathbf{X}'$ is

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

This crossed square is universal, and so its h-function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is $(m, n) \mapsto \pm mn$. Hence $h(1, 1) = \pm 1$, as required. The case where $p = q = 1$ is dealt with in **[7]** (but without mentioning the point about sign conventions). Suppose $p > 1, q = 1$. Then d_1 and d_4 are isomorphisms, and $[\iota_p, \iota_1] \in \pi_p(S^p \vee S^1)$ is the element ${}^{\iota_1}(\iota_p) - \iota_p$. The element $[\iota_p, \iota_2] \in \pi_{p+1}(S^p \vee E^2, S^p \vee S^1)$ is of a similar form. Since d_1 is an isomorphism, and is part of the crossed square associated with $\Pi \mathbf{X}'$, the h-function of this crossed square sends (ι_p, ι_2) to $\pm[\iota_{p+1}, \iota_{q+1}]$, as required. The case where $p = 1, q > 1$ is similar. $\square$

We now give an application combining Theorem 4.1 with the case $n = 3$ of the equivalence between cat^n-groups and crossed n-cubes established in [8, 9]. We do not give definitions here since we need only one example.

Example 4.4 **[8,9]**. Consider a crossed 3-cube of the form

$$\begin{tikzcd}[row sep=small, column sep=small]
L \arrow[rr] \arrow[dr] \arrow[dd] & & G \arrow[dr] \arrow[dd] & \\
 & G \arrow[rr] \arrow[dd] & & G \arrow[dd] \\
G \arrow[rr] \arrow[dr] & & G \arrow[dr] & \\
 & G \arrow[rr] & & G
\end{tikzcd}$$

in which all the maps $G \to G$ are identities. If this crossed cube is universal, then L is canonically isomorphic to the group $G\tilde{\wedge}G$ defined in [**6,7**]; that is, L is generated by elements $x \tilde{\wedge} y$, with $x, y \in G$, with relations

$$\begin{aligned} x \tilde{\wedge} yz &= (x \tilde{\wedge} y)({}^y x \tilde{\wedge} {}^y z), \\ xy \tilde{\wedge} z &= ({}^x y \tilde{\wedge} {}^x z)(x \tilde{\wedge} z), \\ x \tilde{\wedge} y &= (y \tilde{\wedge} x)^{-1}, \end{aligned}$$

where ${}^y x = yxy^{-1}$, for all $x, y, z \in G$.

Proposition 4.5 *For any connected space X, there is an exact sequence*

$$\pi_2 X \xrightarrow{E^2} \pi_4 S^2 X \xrightarrow{H^2} \pi_1 X \tilde{\wedge} \pi_1 X \xrightarrow{P} [\pi_1 X, \pi_1 X] \to 1.$$

Proof We write the two vertices of the (unreduced) suspension SX of X as v_+, v_-, and write

$$C^+X = SX \setminus \{v_-\}, C^-X = SX \setminus \{v_+\}.$$

These two cones are contractible and have intersection which is homeomorphic to $X \times (-1, 1)$, and so is homotopy equivalent to X. We now cover S^2X by the open contractible spaces

$$U_1 = SC_1^+X,\ U_2 = C_2^+C_1^-X,\ U_3 = C_2^-C_1^-X,$$

where the suffices on the C's denote the two directions. Then

$$U_1 \cap U_2 = C_2^+C_1^-X, \quad U_2 \cap U_3 = (C_1^-X) \times (-1, 1), \quad U_3 \cap U_1 \cong C_2^-(X \times (-1, 1),$$

so that these double intersections are contractible. Also

$$U_1 \cap U_2 \cap U_3 \cong X \times (-1, 1)^2.$$

Thus in terms of homotopy types, the 3-cube over S^2X defined by this cover can be represented as

$$\mathbf{S}^2X : \quad \begin{tikzcd} X \arrow[rr] \arrow[d] \arrow[dr] & & * \arrow[d] \arrow[dr] & \\ * \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\ & * \arrow[rr] \arrow[d] & & * \arrow[d] \\ & * \arrow[rr] & & S^2X \end{tikzcd}$$

Since X is connected, each face $\partial_i^0 \mathbf{S}^2X$ is connected. By Theorem 4.1, the cube $\mathbf{S}X$ is connected, and its fundamental cat^2-group is universal. Equivalently, the fundamental crossed cube $\Pi\S^2X$ is universal. But this crossed cube is the cube of example 4.3 with $G = \pi_1 X$. The homotopy fibre $F(\mathbf{S}^2X)$ is equivalent to the homotopy fibre of$X \to \Omega^2 S^2 X$. The exact sequence follows. □

Remark 4.6 As a deduction from Proposition 4.4 we find the formula $\pi_2^S K(G, 1) \cong \mathrm{Ker}\,(G \tilde{\wedge} G \to G)$ obtained by other means in [**7**]. In particular, we obtain a new proof that $\pi_4(S^3) \cong \mathbb{Z}$.

5 Induced cat^n-groups

The notion of inducing cat^n-group is comparable to that of changing base rings in module theory, and also to that of inducing representations in group representation theory. In order to explain the intuitive basis of the construction, we recall from [**4**] the notion of induced crossed module.

Let Q be a group, and let $\mathcal{XM}_Q$ be the category of crossed Q-modules. Let $f : P \to Q$ be a homomorphism of groups. Then pullback (fibre product) defines a functor $f_* : \mathcal{XM}_Q \to \mathcal{XM}_P$. This functor has a left-adjoint $f_* : \mathcal{XM}_P \to \mathcal{XM}_Q$, giving for a crossed P-module M a crossed Q-module f_*M, the crossed Q-module *induced* by f.

Since f_* is a left adjoint, it preserves colimits. This allows one to give presentations of f_*M, given a presentation of M. Also f_*M can be defined without reference to f^* as the pushout crossed module

$$\begin{array}{ccc} (1\ P) & \xrightarrow{(1\ f)} & (1\ Q) \\ \downarrow & & \downarrow \\ (M\ P) & \longrightarrow & (f_*M\ Q) \end{array}$$

This description shows how induced crossed modules can arise as examples in a Van Kampen type theorem for the fundamental crossed module $\pi_2(X, Y) \to \pi_1 Y$ of a based pair, as in [**4**]. This example of a crossed module suggests that we think of a crossed module $\mu : M \to P$ as having P in dimension 1, M in dimension 2, and that an induced crossed module arises from changing the lower dimensional part by a morphism $f : P \to Q$.

We recall from [**4**] a presentation of the induced crossed module.

Proposition 5.1 (4) . *If* $\mu : M \to P$ *is a crossed* P*-module, and* $f : P \to Q$ *is a morphism of groups, then the induced crossed* Q*-module* $\theta : f_*M \to Q$ *has a group presentation with generators* $(q, m) \in Q \times M$ *and relations*

(i) $(q, m)(q, m') = (q, mm')$,

(ii) $(q,\ {}^p m) = (qf(p), m)$, *item* $(q, m)(q', m')(q, m)^{-1} = (qf\mu, (m)q^{-1}q', m')$,

for all $q, q' \in Q,\ m, m' \in M,\ p \in P$. *The action of* Q *on* f_*M *is determined by* ${}^q(q', m) = (qq', m)$, *and* θ *is determined by* $\theta(q, m) = qf\mu(m)q^{-1}$.

In examples, the element of f_*M determined by (q, m) is usually written ${}^q m$, so that the universal map $M \to f_*M$ is given by $m \mapsto {}^1 m$. Of course this map need not be injective.

We now discuss corresponding notions for cat^n-groups. For a cat^n-group $\mathbf{G}$, the 'lower dimensional part' of $\mathbf{G}$ is the corner $\mathbf{G}^\ulcorner$ of cat^n-groups determined by $\mathbf{G}$ (§3).

Proposition-Definition 5.2 *Let* $f : \mathbf{G} \to \mathbf{H}$ *be a morphism of cat*^{n}*-groups, and let* $f^\ulcorner : \mathbf{G}^\ulcorner \to \mathbf{H}^\ulcorner$, $(f^\square : \mathbf{G}^\square \to \mathbf{H}^\square)$ *be the morphisms of* n*-corners (*n*-cubes) of cat*^{n}*-groups determined by* f *Then the following two conditions are equivalent, and define the cat*^{n}*-group* $\mathbf{H}$ *(up to an isomorphism which is the identity on* $H^\ulcorner$*) as induced from the cat*^{n}*-group* $\mathbf{G}$ *by the morphism* $f^\ulcorner : \mathbf{G}^\ulcorner \to \mathbf{H}^\ulcorner$.

(i) *If* $\mathbf{T}$ *is the* $(n+1)$*-cube of cat*^{n}*-groups determined by the morphism* $f^\square : \mathbf{G}^\square \to \mathbf{H}^\square$*, so that* $\partial^0_{n+1}\mathbf{T} = \mathbf{G}^\square, \partial^1_{n+1}\mathbf{T} = \mathbf{H}^\square$*, then* $\mathbf{T}$ *is a pushout* $(n+1)$*-cube.*

(ii) *For any morphism* $g : \mathbf{G} \to \mathbf{K}$ *of cat*^{n}*-groups such that* $\mathbf{K}^\ulcorner = \mathbf{H}^\ulcorner$ *and* $g^\ulcorner = f^\ulcorner$*, there is a unique morphism* $h : \mathbf{H} \to \mathbf{K}$ *of cat*^{n}*-groups such that* $hg = f$.

Proof That (i) implies (ii) is clear. To prove that (ii) implies (i), let $\mathbf{K}$ be the cat^n-group which is the pushout of the $(n+1)$-corner determined by $\mathbf{T}$. For each $(n+1)$-multi-index $\alpha \neq \mathbf{1}$, the $(n+1)$-cube determined by the map of n-cubes $(\mathbf{G}^{\square}(\alpha))^{\square} \to (\mathbf{H}^{\ulcorner}(\alpha))^{\square}$ is degenerate, and so an $(n+1)$-pushout by Proposition 1.1. By Proposition 3.2, $\mathbf{K}^{\square}(\alpha) = \mathbf{H}^{\square}(\alpha)$. Hence $\mathbf{K}^{\ulcorner} = \mathbf{H}^{\ulcorner}$. The two universal properties give an isomorphism $\mathbf{H} \to \mathbf{K}$ which induces the identity $\mathbf{H}^{\ulcorner} \to \mathbf{K}^{\ulcorner}$. □

An example of inducing which will occur in connection with the Hurewicz theorem is when each vertex cat^n-group of $\mathbf{H}^{\ulcorner}$ is a trivial cat^n-group.

Proposition and Notation 5.3 *Suppose that the morphism $f : \mathbf{G} \to \mathbf{H}$ of cat^n-groups presents $\mathbf{H}$ as induced from $\mathbf{G}$ by $f^{\ulcorner}$, and that $\mathbf{H}$ is the cat^n-group $(H; 1, 1, \dots, 1)$ in which all s_i, b_i are trivial. Then H is the factor group of G, the big group of $\mathbf{G}$, by the normal subgroup generated by $s_1(G), \dots, s_n(G)$. This cat^n-group $\mathbf{H}$ will be denoted by $\mathsf{triv}(\mathbf{G})$.*

The proposition is clear from the universal property 5.2 (ii).

It will be useful to describe induced cat^n-groups more explicitly for $n = 2$, using in this case the equivalence between cat^2-groups and crossed squares given in [**9**] (see §2 above). To this end we first recall that the category $\mathcal{XM}_Q$ of crossed Q-modules has a coproduct which is described in [**3**] as follows. Let M, N be crossed Q-modules. Then N acts on M, and M acts on N, via the given actions of Q. Let $M \rtimes N$ denote the semidirect product with injections $i' : M \to M \rtimes N, m \mapsto (m, 1)$, and $j' : N \to M \rtimes N, n \mapsto (1, n)$; let $\{M, N\}$ denote the subgroup of $M \rtimes N$ generated by $({}^n m\, m^{-1},\ {}^m n\, n^{-1})$, for all $m \in M, n \in N$; let $q : M \rtimes N \to M \circ N = (M \rtimes N)/\{M, N\}$ be the projection and let $i = qi', j = qj'$. Then $M \circ N$, with the morphisms i, j, is the coproduct of M, N in the category $\mathcal{XM}_Q$.

Remark 5.4 The above construction of a group $M \circ N$ with morphisms $i : M \to M \circ N, j : N \to M \circ N$ requires only actions of M on N and N on M. If also these actions are compatible in the sense of [**7**], then $i : M \to M \circ N, j : N \to M \circ N$ may be given the structure of crossed modules.

Proposition 5.5 *In the following cubical diagram of homomorphisms of groups*

$$\begin{tikzcd}
L \arrow[rr, "\lambda'"] \arrow[d, "\lambda"'] \arrow[ddr, "\delta"] & & N \arrow[d, "\nu"'] \arrow[ddr, "\beta"] & \\
M \arrow[rr, "\mu"] \arrow[ddr, "\alpha"'] & & P \arrow[ddr, "\gamma"'] & \\
& T \arrow[rr, "\tau'"] \arrow[d, "\tau"] & & S \arrow[d, "\sigma"] \\
& R \arrow[rr, "\rho"'] & & Q
\end{tikzcd}$$

suppose that the front and back squares are crossed squares, that the cube is a morphism of these crossed squares, and that the front crossed square is induced from the back crossed square by α, β, γ. Consider $R \otimes S$ and $\gamma_* L$ as crossed Q-modules. Then T is isomorphic to the coproduct $(R \otimes S) \circ (\gamma_* L)$ factored by the relations

(i) $i(\alpha m \otimes \beta n) = jh(m, n)$,

(ii) $i({}^q \alpha\lambda l \otimes s) = j({}^q l) j({}^{sq} l^{-1})$,

(iii) $i(r \otimes {}^q \beta\lambda' l) = j({}^{rq} l) j({}^q l^{-1})$, *for all* $r \in R, s \in S, l \in L, m \in M, n \in N, q \in Q$. *The maps* τ, τ' *are determined by the usual maps* $R \otimes S \to R, R \otimes S \to S$, *and on* $j(\gamma_* L)$ *by* $\tau :\ {}^q l \mapsto {}^q(\alpha\lambda l), \tau' : {}^q l \mapsto {}^q(\beta\lambda' l)$, *while* $\delta : L \to T$ *is given by* $\delta(l) = j'(l)$. *The* h*-function of the induced crossed square is* $(r, s) \mapsto i(r \otimes s)$. □

The proof of Proposition 5.4 is a direct verification.

6 Applications of induced cat^n-groups: the excision and Hurewicz theorems

In this section we show that a Hurewicz type theorem for $\Pi\mathbf{X}$ follows from a description of the excision map, which is more general and has other applications. The general situation is given by the following theorem.

Theorem 6.1 (Excision Theorem) . *Let the space* Y *be the union of open sets* $X, B_1 \dots, B_n$, *and let* $A_i = X \cap B_i$, *for* $i = 1, \dots, n$. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be the* n*-cubes determined by the* $(n+1)$*-ads* $(X; A_1, \dots, A_n)$ *and* $(Y; B_1, \dots, B_n)$. *Assume that* $\mathbf{X}$ *and each* $\partial_i^0 \mathbf{Y}$ *is connected, for* $1 \leqslant i \leqslant n$. *Then* $\mathbf{Y}$ *is connected and the morphism* $f : \Pi\mathbf{X} \to \Pi\mathbf{Y}$ *determined by the inclusion* $\mathbf{X} \to \mathbf{Y}$ *presents* $\Pi\mathbf{Y}$ *as the cat*^{n}*-group induced from* $\Pi\mathbf{X}$ *by the corner morphism* $f^\ulcorner : (\Pi\mathbf{X})^\ulcorner \to (\Pi\mathbf{Y})^\ulcorner$.

Proof Let $\mathbf{T}$ be the $(n+1)$-cube of cat^n-groups determined by the morphism $f : \Pi\mathbf{X} \to \Pi\mathbf{Y}$ of fundamental cat^n-groups. We prove that $\mathbf{T}$ is a pushout $(n+1)$-cube by applying the Van Kampen theorem (Theorem 2.1).

The space Y is given to have the open cover $V = \{V_1, \dots, V_n, V_{n+1}\}$ where $V_{n+1} = X, V_i = B_i$ for $i \leqslant 1 \leqslant n$. Let $\mathbf{V}$ be the $(n+1)$-cube of n-cubes of spaces in which $\mathbf{V}(\mathbf{1}) = \mathbf{Y}$ and for $\mathbf{1} \neq \alpha \in \{0,1\}^{n+1}, \mathbf{V}(\alpha) = \mathbf{Y} \cap V \cap \dots V_i$ where $i_1 < \dots < i_r$ are exactly those indices i such that $\alpha_i = 0$. Then $\mathbf{T}$, the $(n+1)$-cube of cat^n-groups defined above, may be identified with the $(n+1)$-cube of cat^n-groups $\Pi\mathbf{V}(\alpha)$ for each $\alpha \in \{0,1\}^{n+1}$. Thus to apply Theorem 2.1 we have only to prove that each n-cube of spaces $\mathbf{V}(\alpha)$ is connected, for $\alpha \in \{0,1\}^{n+1}, \alpha \neq 1$.

Write $\alpha \in \{0,1\}^{n+1}$ as (β, γ) where $\beta \in \{0,1\}^n, Y \in \{0,1\}$. If $\gamma = 0$, then $\mathbf{V}(\alpha) = R_\beta \mathbf{X}$, and so $\mathbf{V}(\alpha)$ is connected, since $\mathbf{X}$ is connected. If $\gamma = 1$ then $\beta \neq \mathbf{1}$ (since $\alpha \neq \mathbf{1}$) and $\mathbf{V}(\alpha) = R_\beta \mathbf{Y}$ is connected since each $\partial_i^0 \mathbf{Y}$ is connected, for $i = 1, \dots, n$. The result follows. □

As a corollary we obtain our first version of the Hurewicz theorem for $(n+1)$-ads. Recall from §5 that a cat^n-group $\mathbf{G}$ determines an induced cat^n-group $\mathsf{triv}(\mathbf{G})$ with trivial corner.

Theorem 6.2 (Hurewicz Theorem, version 1) . *Let* $(X; A_1, \dots, A_n)$ *be an* $(n+1)$*-ad such that each* A_i *is closed in* X, *and the associated* n*-cube* $\mathbf{X}$ *is connected. Then the space*

$$\mathbf{Y} = X \cup C(A_1 \cup \dots \cup A_n)$$

is n*-connected and* $H_{n+1}(Y)$ *is canonically isomorphic to the big group of the cat*^{n}*-group* $\mathsf{triv}(\Pi\mathbf{X})$.

Proof Let ν be the vertex of the cone in $Y = X \cup C(A_1 \cup \dots \cup A_n)$. The cover of Y given by $V = \{CA_1, \dots, CA_n, X\}$ determines, by intersection with the n-cube of $(Y; CA_1, \dots, CA_n)$, as in the proof of Theorem 6.1, an $(n+1)$-cube of n-cubes which is homotopy equivalent to the $(n+1)$-cube of n-cubes determined by the open cover $\{CA_1, \dots, CA_n, Y \backslash \{\nu\}\}$ of Y. So the Van Kampen theorem may be applied to the cover V, even though it is not an open cover. Apart from openness, the hypotheses of Theorem 6.1 hold. Since the fundamental cat^n-group $\Pi\mathbf{Y}$ has trivial corner, the result follows. □

The n-connectivity of Y was proved directly by M. Zisman in 1981, in response to a question of Loday.

Let $\mathbf{G}$ be a cat^n-group. The *abelianization* of $\mathbf{G}$ is the cat^n-group $\mathbf{G}^{ab}$ obtained by the abelianization of the big group of $\mathbf{G}$. The maps s_i, b_i become s_i^{ab} and b_i^{ab} respectively. (Note that $[\mathrm{Ker}\ s_i^{ab}, \mathrm{Ker}\ b_i^{ab}] = 1$ since G^{ab} is abelian.)

Theorem 6.3 (Hurewicz Theorem, version II) . *Let* $(X; A_1, \dots, A_n)$ *be an* $(n+1)$*-ad such that each* A_i *is closed in* X *and the associated* n*-cube* $\mathbf{X}$ *is connected. Then the hyper-relative homology group* $H_n(X; A_1, \dots, A_n)$ *is isomorphic to the subgroup* $\bigcap_{i=1}^n \mathrm{Ker}\ s_i^{ab}$ *of the cat*^{n}*-group* $(\Pi\mathbf{X})^{ab}$.

Proof It is immediate that $H_{n+1}(Y) = H_{n+1}(X; Al, \dots, A_n)$ where Y is the space described in Theorem 6.2. With the notation of §1 we have $\bigcap_{i=1}^n \mathrm{Ker}\ s_i^{ab} = \mathbf{L}((\Pi\mathbf{X})^{ab})$. On the other hand, Theorem 6.2 says that

$$H_{n+1}(X; A_1 \dots, A_n) = \mathsf{triv}(\Pi\mathbf{X}) = \mathbf{L}(\mathsf{triv}(\Pi\mathbf{X})).$$

Hence Theorem 6.3 follows from the algebraic

Lemma 6.4 *Let* $\mathbf{G}$ *be a cat*^{n}*-group* $(n \geqslant 1)$. *Then* $\mathbf{L}(\mathbf{G}^{ab}) = \mathbf{L}(\mathsf{triv}(\mathbf{G}))$.

Proof. Let $\mathbf{H} = (H; 1, \dots, 1)$ be a trivial cat^n-group (with $n \geqslant 1$). Then H is abe1ian because $H \to 1$ is a crossed module. So H is equal to its abelianization. By universality any map $\mathbf{G} \to \mathbf{H}$ factors uniquely through $\mathbf{G}^{ab}$. As $\mathbf{H} = \mathsf{triv}(\mathbf{H})$ the map $\mathbf{G}^{ab} \to \mathbf{H}$ factors uniquely through $\mathsf{triv}(\mathbf{G}^{ab})$. As a result $\mathsf{triv}(\mathbf{G}) = \mathsf{triv}(\mathbf{G}^{ab})$ Therefore it suffices to prove the lemma when $\mathbf{G}$ is abelian (which means the big group of $\mathbf{G}$ is abelian).

When $\mathbf{G}$ is abelian the big group G splits as a cartesian product ΠG_α, where α runs over $\{0,1\}^n$. The group $\mathrm{Im}\ s_i$ is the sub-group of G consisting of all the factors G_α, such that the ith index of α is 1. Therefore

$$\mathbf{L}(\mathbf{G}) = \bigcap_{i=1}^{n} \mathrm{Ker}\ s_i = G_{\mathbf{0}} = G/\{\mathrm{Im}\ s_i\} = \mathsf{triv}(\mathbf{G}) = \mathbf{L}(\mathsf{triv}(\mathbf{G})).$$

□

Remark 6.5 Under the hypothesis of Theorem 6.2 there exist a cat^n-group $\Pi\mathbf{X}$ (such that $\pi_{n+1}(X; A_1 \dots A_n) = \mathbf{L}(\Pi\mathbf{X})$) and also an abelian cat^n-group $H\mathbf{X}$ (such that $H_{n+1}(X; A_1 \dots, A_n) = \mathbf{L}(H\mathbf{X})$). Then Theorem 6.3 is equivalent to

$$H\mathbf{X} = (\Pi\mathbf{X})^{ab}.$$

This shows that our result is a direct analogue of the classical Hurewicz theorem: if $\mathbf{X}$ is a connected space, then $H_1(X)$ is the abelianization of $\pi_1(X)$.

We now consider computations involving attaching 3-cells. We use the explicit description of induced crossed squares of §5.

Proposition 6.6 *Let* $(Z; U, V)$ *be a triad such that*

(i) *the interiors of* U, V *cover* Z*;*

(ii) U, V *and* $W = U \cap V$ *are connected; and*

(iii) *the pairs* $(U,W),(V,W)$ *are 1-connected.*

Let $f_k : (S^2; E^2_+ E^2_-) \to (Z,U,V)$*, where* $k \in K$*, be a family of pointed maps and let the space* $Z \cup \{e^3_k\}$ *be formed by attaching 3-cells to* Z *using* $f_k S^2 \to Z$*. Then the triad homotopy group*

$$T = \pi_3(Z \cup \{e^3_k\}; U, V)$$

is describable as follows. For each $k \in K$*, let*

$$\begin{aligned} u_k &= f_k(\iota_+) \in \pi_2(U,W), \\ v_k &= f_k(\iota_-) \in \pi_2(V,W), \\ w_k &= f_k(\iota) \in \pi_1(W) \end{aligned}$$

be the images of generators

$$\iota_+ \in \pi_2(E^2_+, S^1), \quad \iota_- \in \pi_2(E^2_-, S^1), \quad \iota \in \pi_1(S^1),$$

where $\partial(\iota_+) = \partial(\iota_-) = \iota$*. Let* C *be the free crossed* $\pi_1(W)$*-module on generators 1 such that* $\partial l_k = w_k$ *where* $k \in K$*. Then* T *is isomorphic as crossed* $\pi_1(W)$*-module to the coproduct* $(\pi_2(U,W) \otimes \pi_2(V,W)) \circ C$ *with relations*

$$\left.\begin{array}{rcl} u_k \otimes v_{k'} &=& [l_k, l_{k'}] \\ {}^q u_k \otimes s &=& {}^q l_k {}^{sq} l_k^{-1} \\ r \otimes {}^q v_k &=& {}^{rq} l_k {}^q l_k^{-1} \end{array}\right\} \qquad (*)$$

Proof. We replace the space $Z \cup \{e^3_k\}$ by the mapping cylinder $Y = M(f) \cup \bigvee_k E^3_k$ where $f : \bigvee_k S^2_k \to Z$ is determined by the f_k, for $k \in K$. Then Y can be covered by open sets X, B_1, B_2 such that the triad $(Y; B_1, B_2)$ is homotopy equivalent to $(Z \cup \{e^3_k\}; U, V)$ and $(X; X \cap B_1, X \cap B_2)$ is homotopy equivalent to $\mathbf{E} = (\bigvee_k E^3_k; \bigvee_k (E^2_+)_k, \bigvee_k (E^2_-)_k)$, The assumptions of the Excision Theorem 6.1 apply to Y with the cover X, B_1, B_2, and so it is sufficient to show that Proposition 5.4 gives for $\pi_3(Y; B_1, B_2)$ the presentation of our proposition.

Let F be the free group on the set K. The crossed square $\Pi\mathbf{E}$ is

$$\begin{array}{ccc} F & \xrightarrow{1} & F \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ F & \xrightarrow[1]{} & F \end{array}$$

with h-map the commutator. Thus the relations (*) are particular cases of the relations (i), (ii), (iii) of Proposition 5.4. The proof is completed by showing that these particular cases imply the general case. The details are left to the reader. □

Remark 6.7 The above methods are a development of those used in [5] to deduce the relative Hurewicz theorem from a Van Kampen theorem for filtered spaces.

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