A MODEL STRUCTURE FOR THE HOMOTOPY THEORY OF CROSSED COMPLEXES

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Abstract

Les complexes croisés sont analogues à des complexes de chaîne mais avec les propriétés non-abéliennes des modules croisés de dimensions 1 et 2. lIs interviennent dans la théorie homotopique et la cohomologie des groupes. Ici on montre que la catégorie Crs des complexes croisés a une bonne structure de catégorie avec modele pour la théorie d'homotopie, en prenant les classes déjà connues pour les équivalence faibles et les fibrations, et une nouvelle notion de cofibration. Les preuves utilisent la structure monoïdale fermée sur les complexes croisés développée par Brown et Higgins, laquelle fournit des objets cylindre et cocylindre adéquats pour Crs.

INTRODUCTION.

The definition of *crossed complex* is motivated by the principal example, the *fundamental crossed complex* $\pi\mathbf{X}$ of a *filtered space*

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty$$

([8], 5). In this crossed complex, $\pi_0 X_*$ is the set $\pi_0 X_0$; $\pi_1 X_*$ is the fundamental groupoid $\pi_1(X_1, X_0)$; and for $n \geqslant 2, \pi_n X_*$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-l}, p)$, for all p in X_0. This structure is also equipped with boundary maps from $\pi_n X_*$ to $\pi_{n-1} X_*, n \geqslant 1$, and operations of $\pi_1 X_*$ on $\pi_n X_*$ for $n \geqslant 2$, all satisfying appropriate axioms (see Section 1). It is because of the widespread use of crossed complexes (summarised below) that it is necessary to discuss their appropriate homotopy theory, and this is our aim. Crossed complexes with a single vertex and satisfying a freeness condition were used by Whitehead in [26], under the name "homotopy systems", for discussing realisation problems and models of low-dimensional homotopy types. They were also used in his famous paper [27] on simple homotopy types, again for realisation problems, although this application has been neglected up to now.. These aspects are taken up in [1], where free, reduced crossed complexes are seen as constituting the first level in a tower of approximations to homotopy theory. The functor

$$\pi : \text{(filtered spaces)} \to \text{(crossed complexes)}$$

satisfies a Generalised Van Kampen Theorem; ie., it preserves certain colimits [8]. This result includes the usual Van Kampen Theorem, and other basic results in homotopy theory, for example the relative Hurewicz Theorem, and the result of Whitehead that $\pi_2(X \cup \{e^2_\lambda\}, X)$ is a free crossed $\pi_1(X)$-module. It also implies new results on second homotopy groups [6]. There is a *classifying filtered-space functor*

$$B_* : \text{(crossed complexes)} \to \text{(filtered spaces)}$$

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such that πB_* is naturally equivalent to the identity ([8], Section 9. and [l1]). For a crossed complex C, the space $(B_*C)_\infty$ is written BC and called simply the *classifying space* of the crossed complex C. The main result of [11] is the homotopy classification result, that if X_* is the filtered space of skeleta of a CWcomplex X, then there is a bijection of homotopy classes

$$[X, BC] \cong [\pi X_*, C] \qquad \text{(see [5] for a summary)}.$$

A crossed complex is of *rank n* if it is zero above dimension n. The crossed complexes of rank 1 are the groupoids, and these are well known to be models of homotopy 1-types. The crossed complexes of rank 2 are the *crossed modules (of groupoids)*. These are models of homotopy 2-types. Thus the Generalised Van Kampen Theorem enables the computation in some cases of the 2-type of a union of spaces[1].

In homological algebra, it is common to consider a *free resolution* of an algebraic object, for example of a module, and such a resolution is a chain complex of free modules. It is explained in [l3] how crossed modules arise in considering identities among relations for a presentation of a group, and the general idea of a *crossed resolution* is explained in the survey article [4]. From this point of view, it is not surprising that crossed complexes have been used to interpret the cohomology $H^n(G, A)$ of a group G with coefficients in a G-module A (cf. [15,17,20,21]). It now seems reasonable to regard the replacement of chain complexes by crossed complexes as the first step towards a non-abelian homological algebra. It is these twin relations of crossed complexes to homological algebra and to homotopy theory which make it essential to have a satisfactory homotopy theory for crossed complexes.

The definition of a homotopy of morphisms of crossed complexes is well known and due to Whitehead [26]. It is exploited in [17] for the representation of cohomology of a group and in [9] for the theory of extensions of groups. However the theory of chain complexes has another type of homotopy theory due to Quillen [22], which is important in homological and homotopical algebra, and which involves defining notions of *weak equivalence, fibration* and *cofibration*, to obtain the structure of *closed model category*. For crossed complexes, weak equivalences are defined in [7] and fibrations in [16], We use the methods of [22] to define *cofibrations* of crossed complexes and we prove in Theorem 2.12 that the weak equivalences, fibrations and cofibrations satisfy the axioms for a *closed model category* in the sense of [22], However, we do not know if one further axiom is satisfied: *is it true that a pushout of a weak equivalence by a cofibration is again a weak equivalence?*

The proof of Theorem 2.12 requires machinery on crossed complexes developed by R. Brown and P.J. Higgins in [10]. They use ω-groupoid methods from [7] to give the category Crs of crossed complexes a symmetric, monoidal closed structure, with internal hom functor $\mathsf{CRS}(-,-)$ and tensor product $- \otimes -$, analogous to corresponding functors on chain complexes. If B and C are crossed complexes, then $\mathsf{CRS}(B, C)$ is: in dimension 0, the morphisms $B \to C$; in dimension 1, the homotopies of morphisms; in higher dimensions, the higher homotopies. Thus the closed structure on Crs includes a satisfactory theory of homotopy equivalences. But, in a similar manner to chain complexes, the definitions and applications of fibrations and cofibrations are not so straightforward, and it is useful to take weak equivalences rather than equivalences as basic. It is this theory that we develop.

In §1 we recall some basic definitions of crossed complexes, and weak equivalences. In §2 we follow [16] in defining fibrations, and follow [22] in deriving a definition of cofibration. We then prove that, with respect to these classes of morphisms, Crs is a closed model category. In §3 we derive a Whitehead type Theorem: *If a morphism of cofibrant objects in* Crs *is a weak equivalence then it is also a homotopy equivalence*. In §4 we point out other homotopical results for crossed complexes which may be obtained using the methods of double categories with connections due to Spencer [23] and Spencer-Wong [24].

[1]See for example R.Brown and C.D. Wensley, 'Computation and homotopical applications of induced crossed modules', J. Symbolic Computation 35 (2003) 59-72.

A different approach to abstract homotopy theory is given by Kamps - Porter [19] in terms of homotopy and cohomotopy systems. In these, a category is enriched over the category of cubical sets, and certain Kan extension conditions are imposed to allow manipulation of homotopies. This approach is useful for crossed complexes because the equivalence of crossed complexes and ω-groupoids [7], which is used in [10] to obtain the monoidal closed structure on Crs, also allows the category Crs to be enriched over ω-groupoids. The latter, as cubical sets, satisfy a strong form of the Kan extension condition. The consequences of this will be developed elsewhere by the second author and Kamps.

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1 PRELIMINARIES.

A crossed complex C of groupoids [7] is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_0$$

satisfying the following conditions:

(i) C_1 is a groupoid with C_0 as its set of vertices and δ^0, δ^1 its initial and final maps. We write $C_1(p,q)$ for the set of arrows from p to $q(p,q$ in $C_0)$ and $C_1(p)$ for the group $C_1(p,p)$.

(ii) For $n \geqslant 2, C_n$ is a family of groups $\{C_n(P)\}_{p\in C}$ and for $n \geqslant 3$, the groups $C_n(p)$ are abelian.

(iii) The groupoid C_1 operates on the right on each $C_n($ for $n \geqslant 2)$ by an action denoted $(x,a) \mapsto x^a$. Here, if $x \in C_n(p)$ and $a \in C_1(p,q)$, then we have $x^a \in C_n(q)$. We use additive notation for all groups $C_n(p)$ and the groupoid C_1.

(iv) For $n \geqslant 2, \delta : C_n \to C_{n-l}$ is a morphism of groupoids over C_0 and preserves the action of C_1, where C_1 acts on the group $C_1(p)$ by conjugation: $x^a = -a + x + a$.

(v) $\delta\delta = 0 : C_n \to C_{n-2}$ for $n \geqslant 3$ (and $\delta^0\delta = \delta^1\delta : C_2 \to C_0$ as follows from (iv).

(vi) If $c \in C_2$, then δc operates trivially on C_n for $n \geqslant 3$ and operates on C_2 as conjugation by c, that is
$$x^{\delta c} = -c + x + c \qquad (x, c \in C(p)).$$

In any crossed complex C, βc denotes the *base point* of c, that is, if $C \in C_0$ then $\beta c = c$, if $C \in C_1(p,q)$ or $C \in C_n(q)$ for $n \geqslant 2$, then $\beta C = q$.

A *morphism of crossed complexes* $f : B \to C$ is a family of morphisms of groupoids $f_n : B_n \to C_n (n \geqslant 1)$ all inducing the same map of vertices $f_0 : B_0 \to C_0$ and compatible with the *boundary maps* $\delta : B_n \to B_{n-l}, C_n \to C_{n-l}$ and the actions of B_1, C_1 on B_n, C_n. We denote by Crs the resulting category of crossed complexes.

The basic example we have in mind is the fundamental crossed complex πX_* of a filtered space X_*, as described in the Introduction.

It follows from observations by Brown-Higgins in [7], p.238, that the category Crs is complete and cocomplete. The coproduct in Crs is just disjoint union $\bigsqcup$, while colimits in Crs are constructed in Section 6 of [8]. Moreover the paper [10] defines for any crossed complex B an internal hom functor

$\mathsf{CRS}(B,-)$ and its left adjoint, a tensor product $-\otimes B$. This gives Crs the structure of a symmetric, monoidal closed category.

Write $\mathbb{C}(n)$ for the crossed complex freely generated by one generator c_n in dimension n. So $\mathbb{C}(0)$ is $*$; $\mathbb{C}(1)$ is the groupoid $\mathcal{I}$ and for $n \geqslant 2$, $\mathbb{C}(n)$ is in dimensions n and $n-1$ an infinite cyclic group with generators c_n and δc_n respectively, and is otherwise trivial. Let $\mathbb{S}(n-l)$ be the $(n-1)$-skeleton of $\mathbb{C}(n)$, with inclusions $\mathbb{S}(n-1) \to \mathbb{C}(n)$. If $\mathbf{E}^{n-l}$ and $\mathbf{S}^{n-l}$ denote the skeletal filtrations of the standard n-ball

$$E^n = e^0 \cup e^{n-1} \cup e^n$$

and $(n-1)$–sphere $S^{n-l} = e^0 \cup e^{n-l}$, then it is clear that

$$\mathbb{C}(n) \cong \pi\mathbf{E}^n \quad \text{and} \quad \mathbb{S}(n-1) \cong \pi\mathbf{S}^{n-l}.$$

We now define a particular kind of morphism $j : A \to D$ which we call a *crossed complex morphism of relative free type*. Let A be any crossed complex. A sequence of morphisms $j_n : D^{n-l} \to D^n$ may be defined inductively as follows. Set $D^0 = A$. Supposing D^{n-l} given, choose any family of morphisms

$$f_n^\lambda : \mathbb{S}(m_\lambda - 1) \to D^{n-l}\,, \text{ for } \lambda \in \Lambda_n$$

and any m_λ, and to construct $j_n : D^{n-l} \to D^n$ form the pushout:

$$\begin{tikzcd} \bigsqcup_{\lambda\in\Lambda_n} \mathbb{S}(m_\lambda - 1) \arrow[r, "(f_n^\lambda)"] \arrow[d] & D^{n-1} \arrow[d] \\ \bigsqcup_{\lambda\in\Lambda_n} \mathbb{C}(m_\lambda) \arrow[r] & D^n \end{tikzcd}$$

Let $D = \operatorname{colim}_n D^n$, and let $j : A \to D$ be the canonical morphism. We call $j : A \to D$ a *crossed complex morphism of relative free type*. The images x^{m_λ} of the elements c_{m_λ} in D are called *basis elements* of D relative to A. We can conveniently write:

$$D = A \cup \{x^{m_\lambda}\}_{\lambda\in\Lambda_n, n\geqslant 0}$$

and may abbreviate this in some cases. For example we may write $D = A \cup x^n \cup x^m$, analogously to standard notations for CW-complexes.

We remark that for $A = \emptyset$ we get by this construction the crossed complexes of *free type* which were considered in [9] under the name "free crossed complexes" and in [11].

If $\{x_\lambda\}_{\lambda\in\Lambda}$ are all the cells of the crossed complex of free type C, then

$$\mathbb{C}(n) \otimes C = (\mathbf{S}(n-1) \otimes C) \cup \{c_n \otimes x_\lambda\}_{\lambda\in\Lambda}.$$

Hence the morphism $\mathbb{S}(n-1) \otimes C \to \mathbb{C}(n) \otimes C$ is also a morphism of relative free type. The functor $-\otimes C$ on Crs has a right adjoint, and so preserves pushouts. Let $f : A \to B$ be any morphism. If $j : A \to D$ is a morphism of relative free type, then so also is the pushout $\bar{j} : B \to Q$ of j and f. Therefore we get the following

Proposition 1.1 *Let C be a crossed complex of free type. If $A \to D$ is a morphism of relative free type then $A \otimes C \to D \otimes C$ is also of relative free type. In particular if D is a crossed complex of free type then the tensor product $D \otimes C$ is also of free type.* □

We now follow [10] in defining, for $n \geqslant 0$, the n-*fold left homotopies* $B \to C$ from a crossed complex B to a crossed complex C. These homotopies may also be taken to be the elements of $\mathsf{CRS}(B, C)$ in dimension n ([10], Proposition 3.3). A 0-*fold left homotopy* $B \to C$ is simply a morphism $B \to C$. For $n \geqslant 1$, an n-*fold left homotopy* $B \to C$ is to be a pair (H, f), where $f : B \to C$ is a morphism of crossed complexes (the *base morphism* of the homotopy) and H is a map of degree n from B to C (ie., $H : B_k \to C_{k+n}$ for each $k \geqslant 0$) satisfying

(i) $\beta H(b) = \beta f(b)$ for all $b \in B$;

(ii) if $b, b' \in B_1$ and $b + b'$ is defined, then
$$H(b + b') = H(b)^{f(b)} + H(b');$$

(iii) if $b, b' \in B_n (n \geqslant 2)$ and $b + b'$ is defined, then
$$H(b + b') = H(b) + H(b');$$

(iv) if $b \in B_n (n \geqslant 2), b_1 \in B_1$ and b^{b_1} is defined, then
$$H(b^{b_1}) = H(b)^{f(b_1)}.$$

Let $\mathcal{I}$ be the crossed complex which has vertices $0, 1$ and is freely generated by an element c_1 from 0 to 1 of dimension 1. Thus $\mathcal{I}$ may be regarded as a groupoid. Put $PC = \mathsf{CRS}(\mathcal{I}, C)$ for any crossed complex C. Then P is a functor on Crs and there are natural transformations
$$p^0, p^1 : P \to \mathrm{id}_{\mathsf{Crs}} \quad \text{and} \quad s : \mathrm{id}_{\mathsf{Crs}} \to P$$
such that
$$p^0 s = p^1 s = \mathrm{id}_{\mathsf{Crs}} .$$
Hence the quadruple $\mathbf{P} = (P : p^0, p^1, s)$ forms a cohomotopy in the sense of Kamps [18].

For crossed complexes B, C we have the induced cubical set $Q(B, C)$ such that $Q(B, C)_n = \mathsf{Crs}(B, P^n C)$, for $n \geqslant 0$. By [8], Corollary 9.6 and [10], Proposition 2.2
$$\mathsf{Crs}(B, P^n C) = \mathsf{Crs}(\otimes^n \mathcal{I}, \mathsf{CRS}(B, C)) = (\lambda \mathsf{CRS}(B, C))_n, n \geqslant 0,$$
where λ is the functor inducing an equivalence of the category of crossed complexes and ω-groupoids ([7], Theorem 6.2). Therefore $Q(C, D)$, being an ω-groupoid ([8], Corollary 9.6), satisfies the Kan extension condition for all dimensions ([7], Proposition 7.2).

According to [18], the cohomotopy system $\mathbf{P}$ defines in Crs a notion of homotopy between morphisms. In fact, this is essentially the notion of 1-fold left homotopy given above. This notion of homotopy also leads to the notions of homotopy equivalence, Hurewicz fibration and Hurewicz cofibration, as defined in [18]. Following Kamps [18] and using the analogue of the wellknown standard procedure for spaces we get

Lemma 1.2 *For any crossed complex morphism* $f : B \to C$ *there exists the following factorisation*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "j"'] & & C \\ & B \times_C PC \arrow[ur, "q"'] & \end{tikzcd}$$

where q *is a Hurewicz Fibration and* j *is a strong deformation retract morphism, hence a homotopy equivalence.* $\square$

Suppose C is a crossed complex and $p \in C_0$. Following Brown-Higgins [7], p.258 and Howie [16] we define $\pi_0(C)$ to be the set of components of the groupoid C_1. Define $\pi_1(C, p)$ to be the cokernel of $\delta_2 : C_2(P) \to C_1(p)$ and, for $n \geqslant 2$, define $\pi_n(C, p)$ to be the subquotient $\operatorname{Ker}\delta_n(p)/\operatorname{Im}\delta_{n+1}(p)$ of $C_n(p)$.

A morphism $f : B \to C$ in Crs is said to be a *weak equivalence* if the induced maps

$$\pi_0 B \to \pi_0 C \quad \text{and} \quad \pi_n(B, p) \to \pi_n(C, fp).$$

are isomorphisms, for all $n \geqslant 1$ and $p \in B_0$. It follows by standard arguments that any homotopy equivalence of crossed complexes is a weak equivalence.

2 CLOSED MODEL CATEGORY STRUCTURE ON Crs

Recall that a morphism $f : G \to H$ of groupoids is a fibration [3] if, whenever $p \in G_0$ and $y \in H$ with $\delta^0 y = fp$, there exists $z \in G_1$ such that $fz = y$ and $\delta^0 z = p$.

This notion was extended to morphisms in Crs by Howie in [16] in the following way. A morphism $f : E \to B$ in Crs is a fibration if each groupoid morphism $f_n : E_n \to B_n (n \geqslant 1)$ is a fibration of groupoids. Other equivalent descriptions of fibrations in Crs were given by Brown-Higgins [28] using the notion of what we call here "crossed complex of free type", which is the same notion as that of "free crossed complex" in [9]. A main fact we need is that if X_* is the skeletal filtration of a CW-complex, then the fundamental crossed complex πX_* of X_* is of free type.

Proposition 2.1 [28]. *Let $f : E \to B$ be a morphism of crossed complexes. Then the following conditions are equivalent:*

(i) f *is a fibration;*

(ii) *(Covering homotopy property) if C is a crossed complex of free type, $g : C \to E$ is amorphism, $n \geqslant 1$, and (H', fg) is an n-fold left homotopy $C \to B$, then there is an n-fold homotopy $(H, g) : C \to E$ such that $fH = H'$;'*

(iii) *the covering property holds for $n = 1$;*

(iv) *if C is a crossed complex of free type then the induced morphism $f_* : \mathsf{CRS}(C, E) \to \mathsf{CRS}(C, B)$ is a fibration.* □

Note that this Proposition implies that each Hurewicz fibration in Crs is a fibration.

A morphism which is both a fibration and a weak equivalence is said to be a *trivial fibration*.

We will say that a morphism $f : A \to D$ has the left lifting property (LLP) with respect to the class $\mathcal{F}$ of morphisms in Crs if the dotted arrow completion exists in any commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & E \arrow[d, "p"] \\ D \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where p is in the class $\mathcal{F}$. Similarly p has the *right lifting property* (RLP) with respect to $\mathcal{F}$ if the dotted arrow completion exists in any commutative square of the above form, where f is in $\mathcal{F}$.

Following Quillen [22] we define a *cofibration* in Crs to be a morphism which has the (LLP) with respect to trivial fibrations. *Trivial cofibrations* are morphisms which are cofibrations and weak equivalences. It is easily checked that in a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \bar{A} \arrow[d, "\bar{f}"] \\ D \arrow[r] & \bar{D} \end{tikzcd}$$

if f is a cofibration then so also is $\bar{f}$([22]. chap.II. §3).

Let $\emptyset$(resp.$*$) denote the initial (resp. final) object of Crs. An object C is called *cofibrant* if the unique morphism from $\emptyset$ to C is a cofibration. Not all crossed complexes are cofibrant. However for any C the unique morphism $C \to *$ is a fibration.

The next proposition, which is analogous to Proposition 2.1, is the key to our results on cofibrations. The proof uses explicitly the structure of $\mathsf{CRS}(B, C)$ defined in [10], and which is given above.

Proposition 2.2 *The following are equivalent for a morphism* $f : E \to B$ *in* Crs*:*

(i) f *is a trivial fibration:*

(ii) f_0 *is surjective; if* $p, q \in E_0$ *and* $b \in B_1(f_0p, f_0q)$*, then there is* $e \in E_1$ *such that* $f_1e = b$*; if* $n \geqslant 1$ *and* $d \in E_n$ *satisfies* $\delta^0 d = \delta_1 d$ *for* $n = 1, \delta d = 0$ *for* $n \geqslant 2$*, and* $b \in B_{n+1}$ *satisfies* $\delta b = f_n d$*, then there is*
$$e \in E_{n+1} \quad \textit{such that} \quad f_{n+1}e = b \quad \textit{and} \quad \delta e = d;$$

(iii) f *has the* RLP *with respect to* $\mathbb{S}(n-1) \to \mathbb{C}(n)$ *for all* $n \geqslant 0$*;*

(iv) *if* C *is a crossed complex of free type then* f *has the* RLP *with respect to* $\mathbb{S}(n-1) \otimes C \to \mathbb{C}(n) \otimes C$ *for all* $n \geqslant 0$*;*

(v) *if* C *is a crossed complex of free type then the induced morphism* $f_* : \mathsf{CRS}(C, E) \to \mathsf{CRS}(C, B)$ *is a trivial fibration.*

Proof. The equivalences (ii)$\Leftrightarrow$(iii) and (iv)$\Leftrightarrow$(v), and the implication (v)$\Rightarrow$ (i) are evident.

The implications (i)$\Rightarrow$ (ii) for $n \geqslant 2$ and (ii)$\Rightarrow$ (i) are straightforward and can be proved by standard procedures in homological algebra. So we prove only the non-Abelian case $n = 1$ of $(i) \Rightarrow (ii)$.

Let $p, q \in E_0$ and $b \in B_1(f_0p, f_0q)$. By the fibration property there is an element u in $E_1(p, q')$, say, such that $f_1u = b$. Hence $f_0q' = f_0q$ and the isomorphism $\pi_0(E) \to \pi_0(B)$ determines an element $v \in E_1(q', q)$. The isomorphism $\pi_1(E, q) \to \pi_1(B, f_0q)$ shows that there is an element $w \in \pi_1(E, q)$ such that $f_1w = f_1v$. Let $d = u + v + w$. Then $f_1d = b$.

To prove (i)$\Rightarrow$ (v) we assume (i) and show that the morphism
$$f_* : \mathsf{CRS}(C.E) \to \mathsf{CRS}(C, B)$$
satisfies the condition (ii), which can be represented diagrammatically by

$$\begin{tikzcd} \mathbf{S}(n-1) \arrow[r, "\hat{d}"] \arrow[d] & \mathsf{CRS}(C, E) \arrow[d, "f_*"] \\ \mathbb{C}(n) \arrow[r, "\hat{b}"] \arrow[ur, dashed, "\hat{e}"] & \mathsf{CRS}(C, B) \end{tikzcd}$$

for $n \geqslant 0$, where the morphisms $\hat{b}, \hat{d}$ and $\hat{e}$ are defined by their images b, d, e respectively. For $n = 0$, we write $\bar{H}$ for $\hat{b}(c_0)$. For $n = 1$, we write g^0, g for $\hat{d}(0), \hat{d}(1)$ respectively and $(\bar{H}, fh)$ for $\hat{b}(c_1)$, this last being a homotopy from fg_0 to fg. For $n \geqslant 2$, we write (K, g) for $\hat{d}(\delta C_n)$ and$(\bar{H}, fg)$ for $\hat{b}(c_n)$, Thus if $n = 2$,

$$\delta^0(K, g) = \delta^1(K, g) = g,$$

and for $n \geqslant 3, \delta(K, g) = 0_g$. Also for $n \geqslant 2$,

$$\delta(\bar{H}, fg) = f_*(K, g) = (fK, fg).$$

Recall that C is of free type. Let X_k be a basis for $C_k, k \geqslant 1$. We will construct by induction on $k \geqslant 0$ a family of maps $H_k : C_k \to E_{n+k}$.

If $n = 0$, then H. is to be a morphism $C \to E$. This H. is easily constructed on the basis X using the fact that $f : E \to B$ is a trivial fibration. Hence H. extends over C to give a morphism, also written $H. : C \to E$. For $n \geqslant 1$, we require the explicit formulae given in [10], Proposition 3.14 for the boundaries of n-fold left homotopies. These formulae (α_k^n) are as follows:

If (H, g) *is* 1*-fold left homotopy from* g_0 *to* g*, so that* $\delta^0(H, g) = g^0, \delta^1(H, g) = g$*, then*

$$
\begin{aligned}
&(\alpha_0) && g^0(c) = \delta^0 H.(c) && \text{if } c \in C,\\
&(\alpha_1^1) && g^0(c) = H.(\delta^0 c) + g(c) + \delta(H.c) - H.(\delta^1 C) && \text{if } c \in C_1,\\
&(\alpha_k^1) && g^0(c) = [g(c) + H.(\delta c) + \delta(H.c)]^{H.(\beta c)} && \text{if } c \in C_k (k \geqslant 2).
\end{aligned}
$$

If $n \geqslant 2$ *and* (H, g) *is an* n*-fold left homotopy, then* $\delta(H., g) = (K., g)$*, where*

$$
\begin{aligned}
&(\alpha_0^n) && K_0(c) = \delta H_0(c) && \text{if } c \in C_0,\\
&(\alpha_1^n) && K_1(c) = (-1)^{n+1} H_0(\delta_0 c)^{g(c)} + (-1)^n H_0(\delta^1 c) + \delta(H_1 C) && \text{if } c \in C_1,\\
&(\alpha_k^n) && K_k(c) = \delta H_k(c) + (-1)^{n+1} H_{k-1}(\delta c) && \text{if } c \in C_k (k \geqslant 2).,
\end{aligned}
$$

The above formulae will be used with H, g, K replaced by $\bar{H}, fg, \bar{K}$ in $\mathrm{CRS}(C, B)$, in order to construct an appropriate element (H, g) in $\mathrm{CRS}(C, E)$. Thus for all $n, k \geqslant 0$ we require also

$$(\beta_k^n) \qquad f_{k+n} H_k = \bar{H}_k\,.$$

Suppose now that H_i is defined for $0 \leqslant i \leqslant k-1$, so that (α_i^n) and (β_i^n) are satisfied for $0 \leqslant i \leqslant k-1$. Then H_k is defined using the fact that f is a trivial fibration and C is of free type. With the above information, the details are straightforward and are left to the reader. □

Corollary 2.3 *Let* C *be a crossed complex of free type. Then the morphism* $\mathbb{S}(n-1) \otimes C \to \mathbb{C}(n) \otimes C$ *is a cofibration for all* $n \geqslant 0$*. In particular,* C *is cofibrant.*

Corollary 2.4 *Let* $j : A \to D$ *be a morphism of relative free type. Then* j *is a cofibration.*

Proof By the definition of relative free type, we are given that D is a colimit $\mathrm{colim}_n D^n$, where $D^O = A$ and each $j_n : D^{n-l} \to D^n$ is a pushout of a coproduct of inclusions of the form $\mathbf{S}(m_\lambda - 1) \to \mathbb{C}(m_\lambda)$. By Proposition 2.2 (iii), such inclusions are cofibrations. Hence j_n is a cofibration. Hence $j : A \to D$ is a cofibration. □

To obtain a description of trivial cofibrations we need

Lemma 2.5 (i) *Let* C *be a crossed complex. Then the canonical morphisms* $p^0, p^1 : PC \to C$ *are trivial fibrations and the induced morphism* $(p^0, p^1) : PC \to C \times C$ *is a fibration;*

(ii) *for any fibration* $f : E \to B$ *the induced morphism* $(p^0, Pf) : PE \to E \times_B PB$ *is also a fibration.*

Proof (i) We prove only that the morphism $(p_0, p_1) : PC \to C \times C$ is a fibration. (Using the same methods one can also show that the canonical morphisms $p_0, p_1 : PC \to C$ are trivial fibrations.) For $n \geqslant 0$, the elements of $(PC)_n$ are the n-fold left homotopies $(H, f) : \mathcal{I} \to C$. The formulae for the boundary operators δ^0, δ^1 and δ of the crossed complex PC were given in the proof of Proposition 2.2. Let $n \geqslant 1$ and let

$$(x, x') \in C_n \times C_n \,, y \in (PC)_0 = C_1$$

be such that for $n = 1$

$$\delta^0(x, x') = (p^0 y, p^1 y)$$

and for $n \geqslant 2$

$$(x, x') \in C_n(p^0 y) \times C_n(p^1 y).$$

We define a morphism $f : \mathcal{I} \to C$ and for $k \geqslant 0$ a family of maps $H : \mathcal{I}_k \to C_{k+n}$ as follows, For $n = 1$ let $f(c_1) = -x + y + x'$ and for $n \geqslant 2$ let $f(c_1) = y$. We let $H : \mathcal{I}_k \to C_{k+n}$ be trivial for $k \geqslant 2$, and be given by $H(0) = x, H(1) = x'$ for $k = 1$. Then for $n \geqslant 1$,

$$(p^0, p^1)(H, f) = (x, x'),$$

and for $n = 1$,

$$\delta^0(H, f) = y.$$

(ii) The required property of the morphism

$$(p^0, Pf) : PE \to E \times_B PB$$

is to be that for $n \geqslant 1$ there is a completion H of the following commutative square

$$\begin{tikzcd}
\mathbb{C}(n) \vee \mathcal{I} \arrow[r] \arrow[d] & E \arrow[d, "f"] \\
\mathbb{C}(n) \otimes \mathcal{I} \arrow[r] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

where $\vee$ means the union in Crs with base points $0 \in \mathbb{C}(n), 0 \in \mathcal{I}$ identified. But this completion exists by the fibration property of the morphism $f : E \to B$. □

Now we can follow Quillen's proof ([22], p.3.4) to get:

Proposition 2.6 *The following are equivalent for a morphism* $j : A \to D$ *in* Crs

(i) j *is a trivial cofibration;*

(ii) j *has the* LLP *with respect to the fibrations;*

(iii) j *is a cofibration and a strong deformation retract morphism.*

It follows from Lemma 2.5 that each cofibration $j : A \to D$ is a Hurewicz cofibration, because j has the LLP with respect to the trivial fibration $p^0 : PC \to C$, for any crossed complex C. Hence j has the homotopy extension property. We do not expect the converse to hold, since for example in chain complexes Hurewicz cofibrations are not necessarily cofibrations in the Quillen sense.

Proposition 2.7 *Any morphism* $f : A \to B$ *in* Crs *may be factored* $f = p \to j$ *where* j *is of relative free type (and hence a cofibration) and* p *is a trivial fibration.*

Proof The essential fact needed to apply *Quillen's small object argument* ([22], chap. 11, p. 3.3 and 3.4) is the characterisation by Proposition 2.2 of trivial fibrations in Crs by the RLP with respect to the set of morphisms $\mathbb{S}(n-1) \to \mathbb{C}(n)\}_{n \geqslant 0}$ where each $\mathbb{S}(n-1)$ is "sequentially small" in the sense that $\mathsf{Crs}(\mathbb{S}(n-1), -)$ preserves sequential colimits.

For completeness and the convenience of the reader we give more details. We are given $f : A \to B$. We construct a diagram

$$\begin{tikzcd} A \arrow[r, "j_0"] \arrow[drr, "f"'] & E^0 \arrow[rr, "j_1"] \arrow[dr, "p_0"] & & E^1 \arrow[r, "j_2"] \arrow[dl, "p_1"'] & \cdots\cdots \\ & & B & & \end{tikzcd}$$

as follows. Let $E^{-1} = A$ and $p_{-1} = f$. Having obtained E^{n-l}, consider the set Λ of all commutative diagrams λ of the form

$$\begin{tikzcd} \mathbb{S}(m_\lambda - 1) \arrow[r, "(f_\lambda)"] \arrow[d] & E^{n-1} \arrow[d, "p_{n-1}"] \\ \mathbb{C}(m_\lambda) \arrow[r, "g_\lambda"] & B \end{tikzcd}$$

Define $j_n : E^{n-l} \to E^n$ by the pushout

$$\begin{tikzcd} \bigsqcup_{\lambda \in \Lambda_n} \mathbb{S}(m_\lambda - 1) \arrow[r, "f_n^\lambda"] \arrow[d] & E^{n-1} \arrow[d, "j_n"] \\ \bigsqcup_{\lambda \in \Lambda} \mathbb{C}(m_\lambda) \arrow[r, "i_n"'] & E^n \end{tikzcd}$$

Define $p_n : E^n \to B$ by

$$p_n j_n = p_{n-1} \quad \text{and} \quad p_n i_n = (g_\lambda)$$

Let $E = \operatorname{colim} E^n$, and let $j : A \to E$ and $p : E \to B$ be the canonical morphisms. By the above construction $j : A \to E$ is a morphism of relative free type. Proposition 2.2 implies that $p : E \to B$ is a trivial fibration. $\square$

If B is any crossed complex and $\emptyset \to B$ is the unique morphism from the initial object then from the above result we get:

Corollary 2.8 *Any crossed complex* B *is weakly equivalent to a crossed complex of free type.*

Corollary 2.9 *If* $f : A \to D$ *is a cofibration then it is a retract in the category of maps of* Crs *of a morphism of relative free type. In particular, each cofibrant object in* Crs *is a retract of a crossed complex of free type.*

Proof By Proposition 2.7 $f = pj$ where j is of relative free type and p is a trivial fibration. Hence by the LLP of f with respect to trivial fibrations, there exists a completion of the following commutative square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & E \arrow[d, "p"] \\ D \arrow[ur, dashed, "g"] \arrow[r, "\mathrm{id}_D"'] & B \end{tikzcd}$$

So the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "j"] & A \arrow[d, "f"] \\ D \arrow[r, "g"] & E \arrow[r, "p"] & D \end{tikzcd}$$

Hence $pg = \mathrm{id}_D$ and f is a retract of j. □

The next corollary generalises Corollary 2.6.

Corollary 2.10 *Let C be cofibrant. If $A \to D$ is a cofibration then $A \otimes C \to D \otimes C$ is also a cofibration. In particular, if D is cofibrant then $D \otimes C$ is also cofibrant.*

Corollary 2.11 *Any morphism $f : B \to C$ in Crs may be factored $f = pi$ where i is a trivial cofibration and p is a fibration.*

Proof By Lemma 1.1 $f = qj$ where q is a fibration and j is a homotopy equivalence. But by Proposition 2.7, $j = pi$ where p is a trivial fibration and i is a cofibration. Also j and p are weak equivalences, so i is a trivial cofibration. Finally,

$$f = qj = (qp)i$$

where qp is a fibration and i is a trivial cofibration. □

Theorem 2.12 *The category Crs of crossed complexes, together with the distinguished classes of weak equivalences, fibrations and cofibrations defined above, satisfies the following axioms:*

CM1: Crs *has all finite colimits and limits.*

CM2: *Suppose given a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "g"'] & & C \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

in Crs. If any two of f, g or h are weak equivalences, then so is the third.

CM3: *The classes of cofibrations, fibrations and weak equivalences are closed under retraction in the category of maps of Crs.*

CM4: *Suppose given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ D \arrow[r, "g"] & B \end{tikzcd}$$

in Crs*, where* p *is a fibration and* j *a cofibration. If either* j *or* p *is trivial, then there is a map* $h : D \to E$ *such that* $ph = g$ *and* $hj = f$.

CM5: *Any crossed complex morphism* f *may be factored as:*

a) $f = pi$*, where* p *is a fibration and* i *is a trivial cofibration and*

b) $f = qj$*, where* q *is a trivial fibration and* j *is a cofibration.*

CM1 – CM5 are the closed model axioms (cf. [22]); one says that the category of crossed complexes is a closed model category.

Proof. CM1 follows, because of the Brown-Higgins result from [8] that Crs is a complete and cocomplete category.

CM2 and CM3 are completely trivial.

CM4 follows from the definition of cofibrations and Proposition 2.5. The factorisation axiom CMS was proved by Proposition 2.7 and Corollary 2.11. □

3 WHITEHEAD THEOREM FOR CROSSED COMPLEXES.

By Proposition 2.5 a cofibration $j : A \to D$ in Crs is a trivial cofibration if and only if it is a strong deformation retract. Now we prove a dual fact for fibrations of cofibrant objects in Crs.

Proposition 3.1 *If* $p : E \to B$ *is a fibration of cofibrant objects in* Crs *then the following are equivalent:*

(i) p *is a trivial fibration;*

(ii) p *has the RLP with respect to cofibrations;*

(iii) p *is a fibration and a strong deformation coretract.*

Proof (i)$\Rightarrow$ (ii) follows from the definition of cofibration.

(ii)$\Rightarrow$ (i). In particular, p has the RLP with respect to $\mathbb{S}(n-1) \to \mathbb{C}(n)(n \geqslant 0)$, hence p is a trivial fibration by Proposition 2.2.

(iii)$\Rightarrow$ (i). This follows from the fact that a strong deformation coretract is a homotopy equivalence, and hence is a weak equivalence.

(i)$\Rightarrow$ (iii). Let $q : E \otimes \mathcal{I} \to E$ denote the canonical morphism. Thus q is the constant homotopy of the identity morphism on E. The coretract and strong deformation may be constructed by completing in

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B \arrow[r, "id_B"'] \arrow[ur, dashed, "s"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \sqcup E \arrow[r, "{sp, id_E}"] \arrow[d, "{(i^0, i^1)}"] & E \arrow[d, "p"] \\ E \otimes \mathcal{I} \arrow[r, "pq"'] \arrow[ur, dashed, "h"'] & B \end{tikzcd}$$

which is possible, since $\emptyset \to B$, by assumption, and

$$(i^0, i^1) : E \sqcup E \to E \otimes \mathcal{I}$$

by Corollary 2.3 for $n = 1$ and Corollary 2.10, are cofibrations. □

Theorem 3.2 (The Whitehead Theorem) *If a morphism $f : C \to D$ of cofibrant objects in Crs is a weak equivalence then f is also a homotopy equivalence.*

Proof By Theorem 2.12 we have the following factorisation of the morphism f:

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "i"'] & & D \\ & \bar{C} \arrow[ur, "p"'] & \end{tikzcd}$$

where p is a fibration and i is a trivial cofibration. But f is a weak equivalence and so p is a trivial fibration. Now C is a cofibrant object and $i : C \to \bar{C}$ is a trivial cofibration. So by lifting in the following diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[r] \arrow[d] & E \arrow[d, "q"] \\ C \arrow[r, "i"] & \bar{C} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where q is a trivial fibration, we obtain that C is also a cofibrant object. By Proposition 2.5, i is a strong deformation retract and by Proposition 3.1, p is a strong deformation coretract and so, finally, f is a homotopy equivalence. □

4 DOUBLE CATEGORY METHODS FOR CROSSED COMPLEXES.

Another view of abstract homotopy theory is given by Spencer and Spencer-Wong in [23,24]. Recall first that GabrielZisman [14] derive exact sequences in homotopy theory in the context of a 2-category in which all 2-morphisms are invertible. Spencer shows in [23] that 2-categories are equivalent to *special double categories with connection* or with *thin structure* [24], where the thin squares of the double category derive from the constant 2-morphisms of the 2-category.

Thus for crossed complexes one obtains a double category with thin structure from the 2-category of crossed complexes, morphisms of crossed complexes, and homotopies of morphisms. The general results of [23,24] now give the following applications to crossed complexes. Recall first that Vogt [25] has shown that for *spaces strong homotopy equivalence* is equivalent to homotopy equivalence. This result is placed in the abstract setting in [23], Proposition 3.1. So we obtain

Proposition 4.1 *A homotopy equivalence of crossed complexes is also a strong homotopy equivalence.*

The paper [23] has results on homotopy pullback and homotopy pushout squares - for example, a composite of homotopy pushouts is a homotopy pushout. The paper [24] has results on homotopy commutative cubes and homotopy pushouts and pullbacks. For example, Corollary 4.8 of [24] and its dual apply to give cogluing and gluing theorems for homotopy equivalences of crossed complexes. Roughly speaking, *homotopy pullbacks (pushouts) of homotopy equivalences are homotopy equivalences*.

(4.2) Open Problem. *Are homotopy pushouts of weak equivalences also weak equivalences?*

Note that the paper [2] obtains a type of model structure, there called a cofibration category, for the category of reduced crossed complexes of free type. In this category, all objects are cofibrant, weak equivalences are homotopy equivalences and standard arguments show that homotopy pushouts of homotopy equivalences are homotopy equivalence. However in many cases one wishes to deal with the non- free case, and it is in this context that (4.2) remains open.

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A Additional comments and references by R. Brown

A.1 Additional model properties

The Lausanne thesis *Orin Sauvageot, STABILISATION DES COMPLEXES CROISÉS* which is available from
http://hopf.math.purdue.edu/cgi-bin/generate?/Sauvageot/thesis
develops some more model properties of the category of crossed complexes. I find myself unable to follow section 4.2 which claims to prove that the model category of crossed complexes is left proper, i.e. that a pushout of a weak equivalence by a cofibration is also a weak equivalence. If this were true, it could be very useful for constructing free crossed resolutions. Part of the problem is that a pushout as in 4.2.1 does not correspond to a short exact sequence. Also, in general a cofibration need not be an inclusion, as shown by an example in [27], which gives the fundamental crossed complexes of the inclusions of CW-complexes $S^1 \vee S^2 \to E^2 \vee S^2$.

A.2 Crossed complexes and globular ∞-groupoids

The category of crossed complexes is equivalent to a category ∞-Gpds of ∞-groupoids, often called globular ω-groupoids, as shown in

[29] R. Brown and P.J. Higgins, "The equivalence of ∞-groupoids and crossed complexes", *Cah. Top. Géom. Diff.* 22 (1981) 371-386.

So we have a Quillen model structure on the category of ∞-groupoids. One of the difficulties with the latter category is to describe precisely the free object $\mathbb{G}(n)$ on one generator of dimension n, a concept which is clear for crossed complexes. It seems likely that the crossed complex corresponding to $\mathbb{G}(n)$ is exactly the fundamental crossed complex of the n-globe.

Another problem is to write down precisely the notion of homotopy for ∞-Gpds, and the monoidal closed structure.

[6] Subsumed in the published version of [11].

The well known category 2-Gpds of 2-groupoids arises as the 2-truncation of ∞-Gpds, and seems to be quite popular. A Quillen model category structure is established for this in

[30] I. Moerdijk and J. Svensson, 'Algebraic classification of equivariant homotopy 2-types', J. Pure Appl. Algebra, 89 (1993) 187-216.

This does not establish in full detail the functor from pairs of spaces to 2-groupoids, but the details are included in the more general

[31] R. Brown and G. Janelidze, 'A new homotopy double groupoid of a map of spaces', Applied Categorical Structures 12 (2004) 63-80.

I have found that the useful categories to work in for the groupoid case are crossed complexes or cubical ω-groupoids, as in [7,8,10]. With the latter one can formulate and prove theorems, while with the former one can calculate, as shown in

[32] R, Brown and A. Razak Salleh, 'Free crossed crossed resolutions of groups and presentations of modules of identities among relations', LMS J. Comp. and Math. 2 (1999) 28-61.

One also has a clear relation with traditional notions of chain complexes with operators [12]. This is useful for example to describe, as in [12], the tensor product of crossed complexes in dimensions > 2. The paper

[33] R. Brown and T. Porter, "On the Schreier theory of non-abelian extensions: generalisations and computations". *Proceedings Royal Irish Academy* 96A (1996) 213-227.

shows the utility of crossed complexes for extension theory. See also

[34] R. Brown and O. Mucuk, "Covering groups of non-connected topological groups revisited", *Math. Proc. Camb. Phil. Soc*, 115 (1994) 97-110.

It should be pointed out that only an idea for a proof of an Eilenberg-Zilber theorem for crossed complexes is sketched in [11]. A detailed description is given in

[35] A. Tonks, 'On the Eilenberg-Zilber theorem for crossed complexes', J. Pure Applied Algebra, 179 (2003) 199-230.

This result is used explicitly in

[36] R. Brown , M. Golasinski, T. Porter, and A. Tonks, "On function spaces of equivariant maps and the equivariant homotopy theory of crossed complexes", *Indag. Math.* 8 (1997) 157-172.

[37] R. Brown , M. Golasinski, T. Porter, and A. Tonks, "On function spaces of equivariant maps and the equivariant homotopy theory of crossed complexes II: the general topological group case", K-Theory 23 (2001)129-155.

A main aim of the papers with Philip Higgins was to establish a higher dimensional Seifert-van Kampen theorem which could enable the computation of higher homotopy information. This theorem established in [7,8] is a higher dimensional nonabelian local-to-global theorem. For a general survey see

[38] R. Brown, 'Groupoids and crossed objects in algebraic topology', Homology, homotopy and applications, 1 (1999) 1-78.