Exposition of part of Chapter 6

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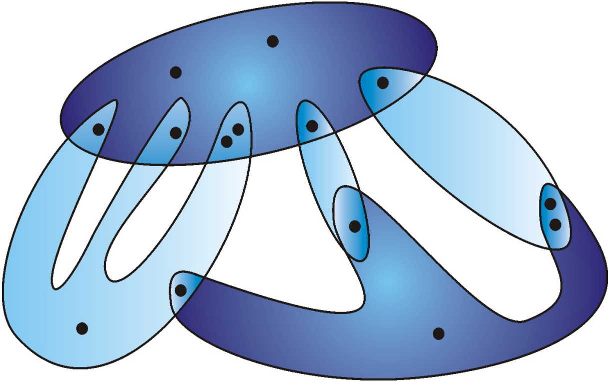
......people are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base-points chosen with care - or equivalently working in the algebraic context of groupoids, rather than groups. Choosing paths for connecting the base points natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids.



A connected union of two non connected spaces, with many base points.

The geometry is not captured by a choice of one base point.

Much more complicated situations than this occur in combinatorial group theory.

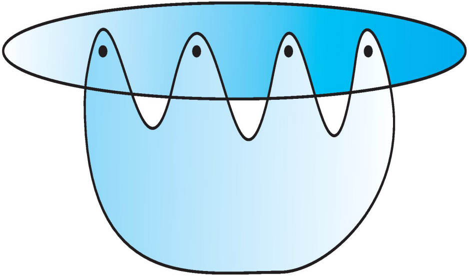
Amazing fact: there is one theorem that makes the transition from topology to algebra by giving the fundamental groupoid of the union on an appropriate set of base points. From that, calculation of the fundamental group of the union at some particular base point involves choices and needs more algebra of what is called ‘combinatorial group theory’; so in this section we go straight for the key theorem 6.7.2, and then proceed to do one so-called ‘retraction’ which is enough to compute the fundamental group of the circle, in the spirit of the discussion on p. xxi, where we gave an analogy between the following two diagrams (which are both pushouts!)

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ [0,1] & \longrightarrow & \mathbb{S}^1 \end{array} \qquad\qquad \begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \mathcal{I} & \longrightarrow & \mathbb{Z} \end{array}$$

spaces groupoids

The point to be made about Theorem 6.7.2 is that it goes via the case $A = X$, a case in which the proof is simple and elegant, and includes most of the topology. However for computation we want to get the set A as small as possible. For the case $X = \mathbb{S}^1$ we need at least two points in A, a fact which is relevant to the Phragmen-Brouwer Theorem discussed in Section 9.2.

An example which occurs in Section 8.4 is the following:



Here we would take A to consist of 4 points, one in each point of the intersection, and would like to reduce this to one point, and so compute the fundamental group at that point.

More generally, we choose a subset A' of $A \cap X_1$ which is representative in X_1, so that we can choose paths from each point of $A \cap X_1$ to some point of A'. This defines a retraction $r : \pi X_1 A \to \pi X_1 A'$, and hence also a retraction $r' : \pi XA \to \pi XA_1$ where A_1 is the union of the points of A' and those of $A \backslash X_1$. This gives us our diagram (*) of 6.7.4.
The fact that this diagram is a pushout allows the computation of πXA_1, and in particular in the case when A_1 is a single point a it computes the fundamental group $\pi(X, a)$. But notice the description of this group is in terms of groupoids. This is surely surprising from a traditional viewpoint.

The next result applies 6.7.4 to compute the fundamental group of the circle. It was the fact that the standard result in terms of groups alone did not compute this basic example seemed to Ronnie Brown in 1965 unaesthetic, an anomaly which needed correction. It later seemed that all of 1-dimensional homotopy theory could be better expressed in terms of groupoids rather than groups. This led naturally to the question of whether, or not, groupoids could be useful in higher homotopy theory.

Note that the circle $\mathbb{S}^1$ is the union of two open subsets $X_1 = \mathbb{S}^1 \backslash \{i\}, X_2 = \mathbb{S}^1 \backslash \{-i\}$ whose intersection is the union of two disjoint open arcs. Thus it is perhaps easier to draw the more general, and also interesting, case:

Theorem

Suppose the space X is the union of two open 1-connected sets X_1, X_2, whose intersection X_0 has two path components. Let $x \in X_0$. Then the fundamental group $\pi(X, x)$ is isomorphic to the group $\mathbb{Z}$ of integers.

Proof.

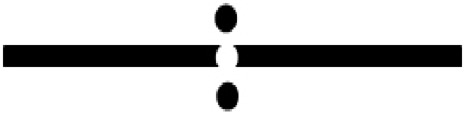
Let $A = \{x, y\}$ consist of one point in each path-component of X_0. Since X_1 is path-connected there is a path in X_1 joining y to x and this defines a retraction $r : \pi X_1\{x, y\} \to \pi(X_1, x)$. Since X_1 is 1-connected, the latter group is trivial. Since X_2 is 1-connected, $\pi X_2\{x, y\}$ is isomorphic to $\mathcal{I}$. So 6.7.4 yields the pushout diagram:

$$\begin{array}{ccc} \{0,1\} & \xrightarrow{r} & \{0\} \\ \downarrow & & \downarrow \\ \mathcal{I} & \xrightarrow[r']{} & \pi(X, x). \end{array}$$

The conclusion follows. □

Example

Consider the space X obtained from the union of two unit intervals $[-1,1] \times \{1,2\}$ by identifying $(t,1)$ with $(t,2)$ for all $t \in [-1,1]$ except for $t = 0$.
Let X_i be the image of $[-1,1] \times \{i\}$ for $i = 1,2$. Then each X_i is homeomorphic to $[-1,1]$. The space X is an analogue of Example 1 on p. 107. It is a non Hausdorff space which looks like the interval $[-1,1]$ but with two copies of 0.



The above theorem shows that the fundamental group of X at one of the copies of 0 is isomorphic to $\mathbb{Z}$.