Triadic van Kampen and Hurewicz Theorems*

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Abstract

In [BH5] it is shown how the Relative Hurewicz Theorem follows from a Generalised Van Kampen Theorem (GVKT) for the fundamental crossed complex of a filtered space, and in [BL3] it is shown how a new multirelative Hurewicz Theorem follows from a GVKT for the fundamental cat^n-group of an n-cube of spaces. The purpose of this paper is to advertise and explain some implications and special cases of these GVKTs, and also to show how they came to be found.

1 Colimits of relative homotopy groups and the Relative Hurewicz Theorem[1]

Although the GVKT is stated in [BH2,BH5] for crossed complexes (over groupoids), it is an important point that the main content of the final result ([BH5] Theorem C) can be summarised as a theorem on relative homotopy groups considered as modules or crossed modules over the fundamental group.

Recall that if P is a group then a *crossed P-module* consists of a group M, an action of P on M on the left, say, written $(m, p) \mapsto {}^p m$, and a morphism of groups $\mu : M \to P$ satisfying the axioms:

CM1) $\mu({}^p m) = pmp^{-1}$;

CM2) $mnm^{-1} = {}^{\mu m} n$;

for all $m, n \in M, p \in P$. For background in examples and applications of crossed modules, see [BHu] and [B6].

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If (X, Y) is a pointed pair of spaces, we write $(\pi_n(X, Y), \pi_1(Y))$ for the relative homotopy group $\pi_n(X, Y)$ considered as a $\pi_1(Y)$-module if $n \geqslant 3$, and as a crossed $\pi_1(Y)$-module if $n = 2$. The notion of crossed module, and the description of this structure on $\pi_2(X, Y)$, are due to Whitehead [W2]. In the following theorem, the colimits are taken in the categories of modules or crossed modules according as $n > 2$ or $n = 2$. The computation of these colimits in terms of other presentations is discussed in [BH5].

Theorem 1.1 (Brown-Higgins) *Let (X, Y) be a pointed pair and suppose that X is the union of a family of open sets $U^\lambda, \lambda \in \Lambda$, which is closed under finite intersection, and such that each U^λ contains the base point. Suppose that for each $\lambda \in \Lambda$, U^λ and $U^\lambda \cap Y$ are path-connected and the pair $(U^\lambda, U^\lambda \cap Y)$ is $(n-1)$-connected. Then*

(C) *the pair (X, Y) is $(n-1)$-connected, and*

(I) *the natural morphism*

$$\operatorname{colim}_\lambda(\pi_n(U^\lambda, U^\lambda \cap Y), \pi_1(U^\lambda \cap Y)) \to (\pi_n(X, Y), \pi_1(Y))$$

is an isomorphism of modules over groups if $n > 2$, and of crossed modules over groups if $n = 2$.

Although Theorem 1.1 is stated in the context of modules and crossed modules, we do not know of a direct proof. Theorem 1.1 is a corollary of Theorem C and the discussion of colimits of crossed complexes in Section 6 of [BH6]. Conversely, the special case of Theorem C of [BH5] for crossed complexes over groups follows from Theorem 1.1. The proof of Theorem C of [BH5] is not short, and does involve a number of new techniques. For example, a key aspect is the use of ω-groupoids as defined in [BH1,4] in order to model the usual proof of the Van Kampen Theorem but replacing paths and squares by higher dimensional cubes. For an expression of the intuitive basis of the argument, see [B3]. We also make some remarks on this later.

Note however that the two papers [BH4,BH5] are together under 60 pages, and for all except the last section of [BH5] assume only basic facts on cubical sets and CW-complexes. So comparisons of length should be with expositions of homotopy theory which include the usual Van Kampen theorem and the relative Hurewicz theorem, where the usual form of the latter theorem also assumes knowledge of singular homology. Further, Whitehead's theorem on free crossed modules, which is discussed below and which is a deduction from Theorem 1.1, is sometimes stated but rarely proved in texts on homotopy theory or algebraic topology.

A useful Corollary of Theorem 1.1 requires the notion of *induced* module and crossed module. The H-module f_*M induced from a G-module M by a morphism $f : G \to H$ is given by $f_*M = M \otimes_{\mathbb{Z}G} \mathbb{Z}H$. Thus f_* is a left $\mathbb{Z}G$-adjoint to the pull-back functor

$$f^* : (H\text{-modules}) \to (G\text{-modules}).$$

In the case of crossed modules, the inducing functor

$$f_* : (\text{crossed } G\text{-modules}) \to (\text{crossed } H\text{-modules})$$

is defined in [BH-3] to be the left adjoint of the pull-back functor

$$f^* : (\text{crossed } H\text{-modules}) \to (\text{crossed } G\text{-modules}).$$

Corollary 1.2 (Homotopical Excision:[BH5] Theorem E) *Let X be the union of open subsets U and V such that U, V, and $W = U \cap V$ are path-connected. Let $n > 1$. Suppose also that the pair (V, W) is $(n-1)$-connected. Then the pair (X, U) is $(n-1)$-connected and the morphism induced by the excision inclusion*

$$\pi_n(V, W) \to \pi_n(X, U)$$

presents $\pi_n(X, U)$ as the $\pi_1(U)$-module induced from the $\pi_1(W)$-module $\pi_n(V, W)$ by the morphism of fundamental groups $\pi_1(W) \to \pi_1(U)$. A similar result holds for $n = 2$ but with module replaced by crossed module.

Proof The pair (X, U) is a pushout of pairs

$$\begin{array}{ccc} (W, W) & \longrightarrow & (U, U) \\ \downarrow & & \downarrow \\ (V, W) & \longrightarrow & (X, U). \end{array}$$

By Theorem 1.1 one obtains a pushout of modules or crossed modules two of which are of a degenerate kind. This pushout is easily shown to yield the universal property for the induced construction. □

C.T.C.Wall has shown me a proof of this Corollary for $n > 2$, using covering spaces and the relative Hurewicz theorem. Such a proof is not available for $n = 2$, essentially because of the non-abelian nature of crossed modules.

Corollary 1.3 (Relative Hurewicz Theorem as in [BH6]) *Let (V, W) be an $(n-1)$-connected pair of connected spaces. Then the space $V \cup CW$ is $(n-1)$-connected and the homotopy group $\pi_n(V \cup CW)$ is obtained from the relative homotopy group $\pi_n(V, W)$ by killing the action of $\pi_1(W)$.*

Proof Apply Corollary 1.2 with $U = CW$. □

Notice that this version of the Relative Hurewicz Theorem makes no mention of homology. However the traditional version may easily be deduced by using the Absolute Hurewicz Theorem applied to $V \cup CW$.

Other applications of Theorem 1.1 are given in [BH3,BH5] and in [B6][2].

Corollary 1.2 implies Whitehead's results [W1,W3] that if $X = Y \cup \{e^n_\lambda\}$ is obtained from Y by attaching n-cells, then $\pi_n 1(X, Y)$ is a free $\pi_1(Y)$-module on the n-cells if $n > 1$, and a free crossed $\pi_1(Y)$-module on the 2-cells if $n = 2$ [BH3,5].

A number of proofs of Whitehead's theorem for $n = 2$ are available ([B2] gives an exposition of the original proof; other proofs are given in [R],[GH], and [EP] includes a neat proof, which also appears in [H], of a key lemma in [R]), but no other proof of Corollary 1.2 has been given. In [BH3,5] it is pointed out that if A is $(n-2)$-connected and $f : A \to X$ is a map, then as a consequence of Corollary 1.2, $\pi_n(X \cup_f CA, X)$ is isomorphic to $\pi_{n-1}A \oplus \mathbb{Z}\pi_1 X$ if $n > 1$, and is isomorphic to the free crossed module on $f_* : \pi_1 A \to \pi_1 X$ if $n = 1$.

One suspects that there should be applications of Corollary 1.2 in geometric topology.

2 A Triadic Van Kampen Theorem and a Triadic Hurewicz Theorem

In order to deal with triads, one needs a generalisation of the notion of module or crossed module. The appropriate concept is that of crossed square due to Guin-Waléry and Loday [GWL], see also [L2].

A *crossed square* consists of a commutative square of morphisms of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

together with left actions of P on L, M, N , and a function : $M \times N \to L$. Let M and N act on M, N, and L via P. This structure must satisfy the following axioms for all $m, m' \in M, n, n' \in N, l \in L$ and $p \in P$:

CS1) the morphisms $\lambda, \lambda', \mu, \nu$ and $\kappa = \mu\lambda = \nu\lambda'$ are crossed modules and λ, λ' are equivariant;

CS2) h is a biderivation in the sense that

$$h(mm', n) = h({}^m m', {}^m n)h(m, n), h(m, nn') = h(m, n)h({}^n m, {}^n n');$$

CS3) $\lambda h(m, n) = m\, {}^n m^{-1}, \lambda' h(m, n) = {}^m n\, n^{-1}$;

CS4) $h(\lambda l, n) = l\, {}^n l^{-1}, h(m, \lambda' 1) = {}^m l l^{-1}$;

CS5) $h({}^p m, {}^p n) = {}^p h(m, n)$.

A *morphism* of crossed squares is a morphism of squares of groups commuting with all the structure. This gives a category (crossed squares).

The principal topological example arises as follows. Suppose given a square of pointed spaces

$$\mathsf{X} : \quad \begin{array}{ccc} W & \xrightarrow{a} & U \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[g]{} & X \end{array}$$

Let $F(\mathsf{X})$ be the homotopy fibre of the induced map $F(b) \to F(f)$, so that $F(\mathsf{X})$ is also homeomorphic to the homotopy fibre of the induced map $F(a) \to F(g)$. Then we have a commutative diagram of maps

$$\begin{array}{ccc} F(\mathsf{X}) & \longrightarrow & F(f) \\ \downarrow & & \downarrow \\ F(g) & \longrightarrow & W \end{array}$$

Proposition 2.1 (Loday [L2]) *The diagram of fundamental groups*

$$\begin{array}{ccc} \pi_1 F(\mathsf{X}) & \longrightarrow & \pi_1 F(f) \\ \downarrow & & \downarrow \\ \pi_1 F(g) & \longrightarrow & \pi_1 W \end{array}$$

may be given in a natural way the structure of crossed square.

The above crossed square is written $\Pi\mathsf{X}$. So we have a functor

$$\Pi : \text{(squares of spaces)} \to \text{(crossed squares)}.$$

If the maps a, b, f, g are inclusions and $W = U \cap V$, then the square X is said to be determined by the triad $(X; U, V)$. In this case the diagram of fundamental groups of Proposition 2.1 is isomorphic to the diagram of homotopy groups

$$\begin{array}{ccc} \pi_3(X; U, V) & \longrightarrow & \pi_2(U, W) \\ \downarrow & & \downarrow \\ \pi_2(V, W) & \longrightarrow & \pi_1(W), \end{array}$$

with the standard action of $\pi_1 W$ on the other groups, with morphisms the boundary maps, and h-map given by the generalised Whitehead product [A].

It is useful to generalise this crossed square to a higher dimensional version which we write

$$\Pi_{p+q+1;p+1:q+1}\mathsf{X} : \quad \begin{array}{ccc} \pi_{p+q+1}(X; U, V) & \longrightarrow & \pi_{p+1}(U, W) \\ \downarrow & & \downarrow \\ \pi_{q+1}(V, W) & \longrightarrow & \pi_1(W), \end{array}$$

where the morphisms are either the boundary map or are zero. Again the action of $\pi_1 W$ is the standard one, the other groups act via $\pi_1 W$, and the h-map is the generalised Whitehead product. The validity in this case of the rules for a crossed square is well known (except that we have changed conventions slightly to ensure that a crossed square is always a commutative square, whereas in topology texts it is common to have squares of boundary maps anti-commutative, as in [A]).

Let us say that the triad $\mathsf{X} = (X; U, V)$ is $(t; r\colon s)$-connected if all the spaces are path connected, the pair (U, W) is r-connected, the pair (V, W) is s-connected, and the triad $(X; U, V)$ is t-connected in the usual sense. Then the following is a consequence of results in [BL-3], though not stated in precisely this form.

Theorem 2.2 (GVKT for triads: [BL2,BL3]) *Suppose given the triad* $\mathsf{X} = (X; U, V)$ *and suppose that* X *is the union of a family* $\{U^\lambda\}_{\lambda\in\Lambda}$ *of open sets which is closed under finite intersection and is such that each* U^λ *contains the base point. Suppose that each triad*

$$\mathsf{U}^\lambda = (U^\lambda; U^\lambda \cap U, U^\lambda \cap V)$$

is $(p+q; p\colon q)$*-connected. Then*

(C) *the triad* X *itself is* $(p+q; p\colon q)$*-connected, and*

(I) *the natural morphism of crossed squares*

$$\operatorname{colim}_\lambda \Pi_{p+q+1;p+1\colon q+1}\mathsf{U}^\lambda \to \Pi_{p+q+1;p+1\colon q+1}\mathsf{X}$$

is an isomorphism.

This Theorem is a deduction from the GVKT for the fundamental cat^n-group of an n-cube of spaces [BL2] Theorem 5.4, using the relation between particular kinds of cat^n-groups and crossed squares given in Section 3 of [BL3][3].

Theorem 2.2 implies the Blakers-Massey triad connectivity theorem [BW], together with a determination of the critical group. For this last determination one needs the notion of *non-abelian tensor product of groups* defined in [BL2]. Let M and N be groups each of which acts on the other (on the left) and which act on themselves by conjugation. The *tensor product* $M \otimes N$ is the group with generators $m \otimes n, m \in M, n \in N$, and relations

$$mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n), \quad m \otimes nn' = (m \otimes n)({}^n m \otimes {}^n n')$$

for all $m, m' \in M, n, n' \in N$. It is known, assuming only a mild compatibility condition on the actions, that the group $M \otimes N$ is finite if M and N are finite [E3]. Calculations of $M \otimes M$ are given in [Ab,BJR,G1,J] for the conjugation action, and for $M = \mathbb{Z}$ and the non-trivial action in [GH].

Corollary 2.3 (Blakers-Massey Theorem; [BL3]) *Let* $(X; U, V)$ *be a triad such that* U *and* V *are open in* X *and cover* X*; that* U, V*, and* $W = U \cap V$ *are connected; and* (U, W) *is* p*-connected and* (V, W) *is* q*-connected. Then* $(X; U, V)$ *is* $(p+q)$*-connected and the morphism*

$$\pi_{p+1}(U, W) \otimes \pi_{q+1}(V, W) \to \pi_{p+q+1}(X; U, V)$$

given by the generalised Whitehead product is an isomorphism.

The Corollary is obtained from Theorem 2.2 by considering the open cover $\{U, V\}$ of X, so that we have a pushout of triads

$$\begin{array}{ccc} (W;W,W) & \longrightarrow & (U;U,W) \\ \downarrow & & \downarrow \\ (V;W,V) & \longrightarrow & (X;U,V). \end{array}$$

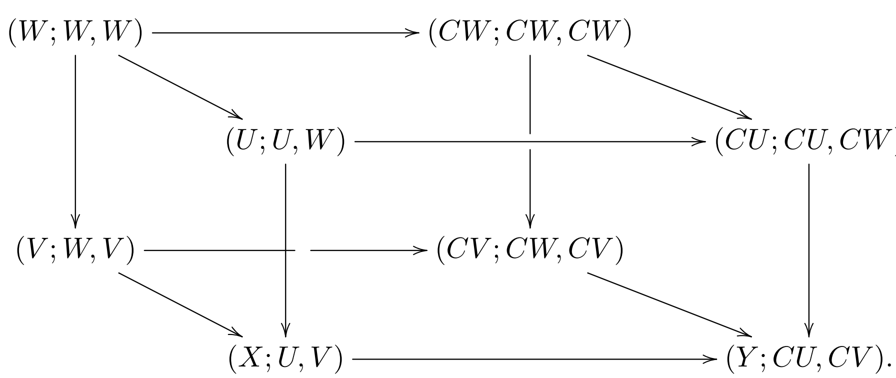
The assumption of Theorem 2.2 are clearly satisfied, and so the conclusion (C) of Theorem 2.2 implies the $(p+q)$-connectivity of $(X;U,V)$. To obtain the determination of the critical group one applies the functor $\Pi_{p+q+1;p+1:q+1}$ to the pushout of triads to obtain by Theorem 2.2 a pushout of crossed squares; this implies the tensor product formulation [BL6]. This determination of the critical group was not known previously if $\pi_1 W \neq 0$ or if p or q is 2.

Similar methods are used in [ES] to determine the critical homotopy group of an $(n+1)$-ad. They obtain a generalisation of the main result of [BW], and in particular obtain results in the non-simply connected case not previously available. Further, the formulae are obtained from an analysis of the universal properties which apply to a particular kind of colimit of crossed n-cubes of groups.

We now show that Theorem 2.2 also implies a triadic Hurewicz theorem. The proof uses a cover of a space by three open sets.

Theorem 2.4 (Triadic Hurewicz Theorem; Brown-Loday) *Suppose the triad $(X;U,V)$ is $(p+q;p:q)$-connected. Let $Y = X \cup C(U \cup V)$. Then Y is $(p+q)$-connected and $\pi_{p+q+1}Y$ is obtained from $\pi_{p+q+1}(X;U,V)$ by killing the action of $\pi_1 W$ and the generalised Whitehead product elements.*

Proof Consider the cover $\{CU, CV, X\}$ of Y. This is not an open cover, but its elements are easily enlarged up to homotopy to form an open cover. Also the cover is not closed under finite intersections. The finite intersections of the cover when intersected with the triad $(Y;CU,CV)$ form a cubical pushout of triads:



Our assumptions imply the $(p+q;p:q)$-connectivity of all the triads of this cube except possibly $(Y;CU,CV)$. By Theorem 2.2, the triad $(Y;CU,CV)$ is also $(p+q;p:q)$-connected, and this says simply that Y is $(p+q)$-connected. Further, the functor $\Pi_{p+q+1;p+1:q+1}$ applied to this cube of triads gives a pushout cube of crossed squares. An analysis of the universal property this implies yields the description of $\pi_{p+q+1}Y$. □

Theorem 2.2 also implies an analogue of Corollary 1.2 which involves the notion of induced crossed square. The corresponding notion for cat^n-groups is discussed in [BL3], which gives also an analogue in dimension 3 of Whitehead's theorem on free crossed modules.

The general Hurewicz theorem for n-cubes of spaces has two formulations for cat^n-groups in [BL3], but the clearest formulation probably uses the crossed n-cubes of groups of [ES]. An explanation of this result is given in [BE], where it is applied to give a formula for H_{n+1} of a group G in terms of an "n-fold presentation" of G, thus generalising the Hopf formula on H_2 of a group.

3 Historical background

The above results are clear and concrete. What is not so clear is why they should be expected to be true. In fact they were found by following through consistently a particular set of ideas, and seeing what came out. These ideas may have other implications, and so I hope it will be useful to explain the ideas which led to these results.

In 1965 I found a Van Kampen theorem for the fundamental groupoid $\pi_1(X,A)$ on a *set* A of base points rather than the usual fundamental group $\pi_1(X,a)$ [B1]. This theorem has the advantage that it enables the computation of the fundamental group $\pi_1(U \cup V, a)$ even in the case when $U \cap V$ is not connected. The important point is that you can choose the set A to consist of one element in each path component of the intersection $U \cap V$. From this computation, you can by making suitable choices determine completely the fundamental group of $U \cup V$ at a point a of A. In particular, this theorem gives the computation of the fundamental group of the circle directly, and without invoking the theory of covering spaces.

For a recent exposition, see [B2] (new edition), which also uses the general computation for steps in the proof of the Jordan Curve theorem.

The utility of the fundamental groupoid in this context and in the theory of covering spaces suggested the idea of using groupoids to consider problems for the higher homotopy groups. The following statement was made in the Introduction to [B1]: "The success of groupoids in this theory suggests the principle that one should consider seriously all the structure in a given situation......At the moment I can prove the n-dimensional version of the Van Kampen theorem, but the implications of this result need further consideration."

The rashness of this statement is shown by the fact that the first published theorems of this type are in 1978 [BH2,3]. However, as explained in [B4], the proofs do include the key elements expected in 1965.

One more point should be made about the peculiarity of this Van Kampen theorem for the fundamental groupoid. It is surprising to be able to make a complete computation of the fundamental group in the non connected case, since the usual theorems of algebraic topology by and large give rise to exact sequences which yield results only up to extension. Now in gluing spaces together, the kind of gluing in low dimensions can affect what happens homotopically in high dimensions. In particular, identifying vertices will introduce extra free summands in the fundamental group.

Groupoids can cope with this because they have structure in dimensions 0 and 1, and so they can model algebraically the geometrical identifications. Thus to model algebraically similar results for higher homotopy groups one should be looking for algebraic gadgets which have structure in a range of dimensions. Further, the way the proof goes in dimension one suggests that one should look for some kind of "higher homotopy groupoid".

It is well known that the higher homotopy groups were first defined by Čech. It appears that he submitted his paper on this to the 1932 International Congress of Mathematicians. However it was quickly proved by others that the higher homotopy groups were abelian, and on this ground Hopf persuaded Čech to withdraw his paper, so that only a brief paragraph appeared in the published proceedings [C]. Hopf remarked to Dyer in 1962 [D] on his later embarrassment at this, in view of the subsequent importance of the higher homotopy groups.

Hopf's decision should be understood in the light of the expectations of the time. There was already a considerable body of work on presentations of the fundamental group. It was difficult to make sense of the idea that the corresponding structure in dimension 2 was simpler, because it was abelian, than that in dimension 1. In particular, it was difficult to see how abelian higher homotopy groups could be handled by presentations in a manner similar to that for the fundamental group. This same attitude explains also the fascination of early workers in homotopy theory with the action of the fundamental group. This became less fashionable in the 1950's and 60's, but is now again at the forefront of aspects of homotopical research, particularly in low dimensional topology.

The early worry about the foundations of homotopy theory can now be resolved by working with higher homotopy groupoids. Such gadgets do model (truncated) homotopy types; they are non-Abelian structures; they do have structure in a range of dimensions; they do satisfy a Generalised Van Kampen Theorem (GVKT), so that one can compute to some extent with presentations of these gadgets, and in particular one can in some instances compute presentations of homotopy types. However the route to such structures and GVKTs was not straightforward.

It seemed to me initially that one of the virtues of groupoids was to get rid of the non-canonical base points. So the initial idea for using higher homotopy groupoids was to define a functor

$$\text{(spaces)} \to \text{(multiple groupoids)}$$

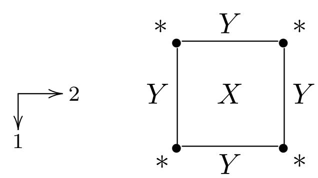
by starting with the singular cubical complex $S^{\square}X$ of a space, preferably defined using maps $[0, r_1] \times \cdots \times [0, r_n] \to X$, so that $S^{\square}X$ has n category structures in dimension n, making $(S^{\square}X)_n$ an n-fold category. The latter concept had already been defined by Ehresmann [Eh].

The proposal was then to factor $S^{\square}X$ by an appropriate equivalence relation so as to obtain in dimension n an n-fold groupoid rather than category. Unfortunately, it was difficult to decide whether or not an appropriate equivalence relation existed, or whether a Van Kampen theorem in this situation would have any computational value, even if, as seemed likely, it could be proved. So there was an idea for a proof, in search of a theorem.

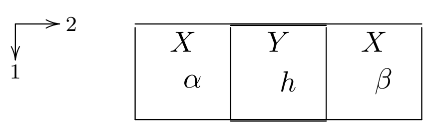
Progress was made on the algebraic side. Work with C.B.Spencer in 1971 [BS1,2] suggested an appropriate algebraic category of double groupoids with connection and one vertex; the special role of these double groupoids was shown by the equivalence of this category with the category of crossed modules. I should also mention that an important clue was given by G.Segal in 1971, who told me of a remark of Deligne that group objects in the 1category of groupoids are classified by a cohomology group H^3. This led to the rediscovery of a result of Verdier (1965) that group objects in the category of groupoids are equivalent to crossed modules [BS2].

A further clue came from Whitehead's theorem on free crossed modules referred to in section 1. This theorem showed that some universal properties were available in dimension 2. It also suggested that if a 2-dimensional Van Kampen Theorem was to be any good, then it should at least prove Whitehead's theorem. However Whitehead's theorem is about pairs of spaces, and relative homotopy groups, not about the absolute case. This suggested to Higgins and me in 1974 that we should look for homotopy double groupoids *defined on pairs of pointed spaces*. It seemed advisable to look at the simplest ideas, since time on our research contract was running out.

The simplest idea involving squares and pairs seemed to be start with a pointed pair (X, Y) and to look at maps of $f : I^2 \to X$ which take the edges to Y and the vertices to the base point $*$, as in the following picture:



Let the set of homotopy classes of such maps be written $\rho_2(X, Y)$. Then $\rho_2(X, Y)$ inherits composition structures in each of the horizontal and vertical directions, where $[\alpha]$ and $[\beta]$ are composable horizontally to give $[\alpha] +_2 [\beta]$ if the appropriate edges agree, i.e. have representatives which as loops in Y are homotopic in Y. To obtain the composite one chooses a homotopy in Y, say h, of these edges and then takes the class of $\alpha +_2 h +_2 \beta$ as in the following picture:



It is then not hard to prove that this composite is independent of the choices made [BH3], and that with this composition and the vertical one, together with the group $\pi_1 Y$ and the appropriate boundary maps, we obtain a "double groupoid" $\rho(X, Y)$. The pair (X, Y) is called *connected* if Y and X are path-connected and the pair (X, Y) is 1-connected in the usual sense (i.e. the homotopy fibre of the inclusion $Y \to X$ is path-connected). A generalisation of the argument for the usual Van Kampen theorem then proves that if X is the union of open sets $U^\lambda, \lambda \in \Lambda$, and each pair $(U^\lambda, U^\lambda \cap Y)$ is connected, then the natural morphism

$$\operatorname{colim}_\lambda \rho(U^\lambda, U^\lambda \cap Y) \to \rho(X, Y)$$

of double groupoids is an isomorphism [BH3]. A key element in the proof is the use of connections in double groupoids to express the homotopy addition lemma in dimension 2. It should be remarked that the word "connection" is chosen expressly because of an analogy with the term in differential geometry, see [BS1].

This theorem is then translated, using the equivalence between double groupoids of this type and crossed modules proved in [BS1], to obtain a corresponding result for the second relative homotopy group $\pi_2(X, Y)$ considered as a crossed $\pi_1(Y)$-module. The generalisation of these results to a set of base points required the later notion of a crossed module over a groupoid [BH4].

Once it was realised that sensible results could be obtained in dimension 2 by using pairs of spaces, it was not hard to conjecture that extensions would be obtained in higher dimensions using filtered spaces and crossed complexes, which are a generalisation of Whitehead's homotopy systems [W3]. The constructions and proofs needed a number of new ideas.

The equivalence in dimension 2 between crossed modules and double groupoids (with connections) is replaced by an equivalence between crossed complexes and ω-groupoids [BH4]. This equivalence is one of a family of non trivial equivalences between some equationally defined categories of algebraic objects [A, BH6,BH7, Jo], which in a sense give different algebraic views of some classical notions in homotopy theory. The construction of the fundamental ω-groupoid of a filtered space [BH5] is not so straightforward, and uses ideas of collapsing. Handling multiple compositions of homotopy addition lemma type formulae in higher dimensions requires the notion of *T-complex*, due in the simplicial context to K.Dakin [Da]. The use of cubical rather than simplicial techniques may seem disconcerting, but is required in order to handle multiple compositions. It is also convenient for higher homotopies [BH8,BH9].

The success of these methods suggests that the idea of obtaining absolute results was in fact misguided. Indeed, even for the fundamental groupoid, the sensible theory is for the fundamental groupoid on a *set* of base points. Thus in general one expects that a Generalised Van Kampen theorem will assert that a functor

$$\Pi : (\text{topological data}) \to (\text{algebraic data})$$

preserves certain colimits. In the situations considered to date there is further structure, namely a *classifying functor*

$$\mathsf{B} : (\text{algebraic data}) \to (\text{topological data})$$

such that $\Pi\mathsf{B}$ is naturally equivalent to the identity. Also there is a forgetful functor

$$U : (\text{topological data}) \to (\text{spaces}),$$

so that we have a classifying space functor

$$B = U \circ \mathsf{B} : (\text{algebraic data}) \to (\text{spaces}).$$

For example, this situation occurs when the algebraic data is *crossed complexes*; the topological data is *filtered spaces*; Π is the *fundamental crossed complex* of a filtered space, made from the fundamental groupoid $\pi_1(X_1, X_0)$ and the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, p)$, $n \geqslant 2, p \in X_0$, and the associated actions of π_1 and boundary maps; the classifying space BC of a crossed complex C is defined to be the geometric realisation of the underlying cubical complex of the ω-groupoid determined by C [BH5,9]; and the filtered space $\mathsf{B}C$ is induced on BC by the skeletal filtration of C. The space BC is the cubical analogue of a simplicial construction which goes back to Blakers [Bl]. A nice feature of crossed complexes is that if C is a crossed complex and X_* is the skeletal filtration of a CW-complex X , then there is a bijection of sets of homotopy classes [BH9]

$$[X, BC] \cong [\Pi X_*, C].$$

This generalises a classical homotopy classification theorem of Eilenberg-Mac Lane, including also the local coefficient case. The proof is obtained by rephrasing in terms of crossed complexes and ω-groupoids arguments well known for chain complexes and simplicial abelian groups.

Not every space is of the homotopy type of the classifying space BC for some crossed complex C. Nonetheless, crossed complexes form the first level in approximating homotopy types by non-abelian structures, as is made clear in the tower of homotopy theories of Baues [Ba], and, as usual, approximations have their uses. Also the category of crossed complexes has additional structures [BH9, BGo] which have not yet been developed for categories modeling a wider range of homotopy types such as cat^n-groups.

I talked about these results and methods to J.-L.Loday at Strasbourg in November, 1981. The reason for the visit was the common interest in generalisations of crossed modules. Crossed modules had arisen for Loday in algebraic K-theory for the following reason.

Quillen had reformulated Whitehead's description of the second relative homotopy group as a crossed module to say that if $F \to E \to B$ is a fibration sequence, then the map of fundamental groups $\pi_1 F \to \pi_1 E$ may be given the structure of crossed module (essentially obtained by conjugating a loop in F by a loop in E and deforming the result into F). Loday [L1] had applied this to the homotopy fibration sequence

$$F(\Lambda, I) \to BGL(\Lambda, I) \to BGL(\Lambda, I)^+$$

to obtain a crossed module $St(\Lambda, I) \to GL(\Lambda, I)$ with cokernel $K_1(\Lambda, I)$ and with kernel a version of $K_2(\Lambda, I)$ different from that given by Milnor [M].

Because of the importance of central extensions of groups in algebraic K-theory, it seemed reasonable to consider central extensions of crossed modules. The notion of a crossed module of crossed modules lead Loday and Dominique Guin-Waléry to the notion of crossed square [GWL], which they exploited in studying the obstruction to excision. The precise form of the axioms for a crossed square arises as follows.

Loday had rephrased the notion of group object in groupoids (or, equivalently, groupoid object in groups) to that of a cat^1-group. This can be defined as a group G with endomorphisms s and t of G such that $st = t, ts = s$, and $[\mathrm{Ker}\, s,\ Ker t] = 1$. This led to the notion of a cat^2-group as a group G with two commuting cat^1-group structures. The axioms for a crossed square are those which give an equivalence between crossed squares and cat^2-groups (theorem of Guin-Waléry and Loday [L2]).

Loday then defined the notion of cat^n-group as a group G with n commuting cat^1-group structures. Equivalently, this is a group with n commuting groupoid structures, and so is a special case of an $(n+1)$-fold groupoid.

In [L2] there is defined a functor, now written Π, from n-cubes of spaces to cat^n-groups (see also [BL2], [G2] for more details). This functor can thus be regarded as a higher homotopy groupoid. The construction of Π with its structure is sophisticated, and is given elegantly by means of n-simplicial techniques. Consider now the special case that the n-cube of spaces X derives from an $(n+1)$-ad $\mathsf{X} = (X; A_1, \ldots, A_n)$ by taking repeated intersections of the sets A_i. The cat^n-group $\Pi\mathsf{X}$ can be described simply. Let F be the function space of maps f of the n-cube I^n into X such that f maps the faces of I^n in direction i into A_i for $i = 1, \ldots, n$. Note that F has partial additions $+_i, i = 1, \ldots, n$, obtained by the usual gluing of cubes in direction i. The fundamental group $\pi_1(F, *)$ is certainly a group. The surprising fact is that the additions $+_i$ of elements of F are inherited by $\pi_1(F, *)$ so as to give this group the extra structure of n compatible groupoid structures, and so to make it a cat^n-group. The proof [G2] that this extra structure is well defined is by identifying $\pi_1(F, *)$ with Loday's $\Pi\mathsf{X}$, in which the groupoid structures are derived from the simplicial construction [L2].

There is a classifying space functor

$$B : (\mathrm{cat}^n\text{-groups}) \to (\mathrm{spaces})$$

[L2] which is the composite of the n-fold nerve functor

$$(\mathrm{cat}^n\text{-groups}) \to ((n)\text{-simplicial sets})$$

(derived from the n groupoid structures and the single group structure) and the usual realisation functor to spaces. It is shown in [L2] that if G is a cat^n-group, then $\pi_i BG$ is zero for $i > n+1$. Further, if X is a CW-complex with $\pi_i X = 0$ for $i > n+1$, then there is a cat^n-group G such that $X \simeq BG$ ([L2],[S]).

The study of the generalisation to all dimensions of the equivalence between cat^2-groups and crossed squares was initiated in [E1] and completed to an equivalence between cat^n-groups and *crossed n-cubes of groups* in [ES] (see [E6] for the case of algebras).

In 1981, Loday was interested in the following question. Suppose given the square X of spaces of section 2, and suppose that all the spaces in X and all the homotopy fibres $F(a), F(b), F(f), F(g)$ and $F(\mathsf{X})$ are connected spaces. Now instead of forming homotopy fibres form homotopy cofibres $C(a), C(b), C(f), C(g)$ and $Y = C(\mathsf{X}) = C(C(a) \to C(g))$. M. Zisman, in answer to a question of Loday, had proved that Y is 2-connected (in fact he had proved the n-dimensional version of this). The problem was the determination of $\pi_3 Y$.

We agreed that such a determination was a kind of triadic Hurewicz theorem, and that in view of the results of Brown-Higgins, it was natural to try and deduce such a theorem from a generalised Van Kampen theorem for 1cat -groups. This lead to the joint papers. In particular, the non-abelian tensor product arose from discussing the simplest possible non trivial computations of pushouts of crossed squares, and further work led to the deduction of the Hurewicz theorem for n-cubes of spaces.

Crossed n-cubes of groups and the GVKT for cat^n-groups have not yet been widely exploited to give presentations of homotopy types. We mention a result due to Brown-Loday that if $1 \to M \to P \to Q \to 1$ and $1 \to N \to P \to R \to 1$ are exact sequences of groups, then the homotopy pushout X of the maps $K(R,1) \leftarrow K(P,1) \to K(Q,1)$ has its 3-type given by the classifying space of the crossed square

$$\begin{array}{ccc} M \otimes N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array}$$

with h-map given by $(m,n) \mapsto m \otimes n$. It is difficult to see how one could obtain a more tidy description of this 3-type. From this crossed square one can deduce the homotopy groups $\pi_1 X$ for $i = 1,2,3$ (see [BL1]), together with a description of the Whitehead product $\pi_2 \times \pi_2 \to \pi_3$ as well as the composition with the Hopf map $h : S^3 \to S^2$. Thus the GVKT methods, when they apply, give rather complete results. However we have no information on π_4 of the above homotopy pushout X.

The proof of the GVKT for cat^n-groups in [BL2] is by induction, assuming the classical Van Kampen theorem, and uses the spectral sequence of a simplicial space and a result on simplicial groups G such that G_2 is generated by degenerate elements. Thus the results of Section 2 here, and their proofs, are more sophisticated than those of Section 1, but their formulation depended on the experience with the earlier results. It is curious that this GVKT is still a pointed result, unlike the GVKT for crossed complexes [BH5]. However the formulation and proof of a result of this kind with a set of base points has not yet been found.

There are connections of these results with nonabelian homology and cohomology which are beginning to be exploited [E7], [Gu1,2]. In view of the wide importance of groupoids [B7] it is difficult to see what might be the chief obstructions to higher dimensional groupoids and more generally "higher dimensional algebra" having in the long run a widespread influence in mathematics. In any case, these attempts to investigate new algebraic structures which in some sense behave well under subdivision (i.e. are involved in GVKTs) has lead to new results and methods in homotopy theory [BH5,9], [B5], [BL2,3], [ES], group theory [Ab], [BJR], [E3], [GH], homology of groups [BL2], [BE], the homology of Lie algebras [E2], [Gu2], homological algebra [E7], and algebraic K-theory [E4,5], [Gu1].

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Notes

[1]p. 39 The results of this Section are all contained in the additional reference [BHS].

[2]p. 41 See also Part I of [BHS].

[3]p. 44 In order to raise dimension and apply the GVKT we construct “degenerate” n-cubes of spaces. These derive from n-ads of the form $(X; U, \dots, U, V, \dots, V)$ with p U’s and q V’s. This leads to cat^n-groups of the form $(G; s, t, \dots, s, t, s', t', \dots, s', t')$ with p, resp. q, parts the same. One then identifies such cat^n-groups with crossed squares with some zero maps if p or q are > 1. The details are in [BL3].