Motion, space, knots, and higher dimensional algebra

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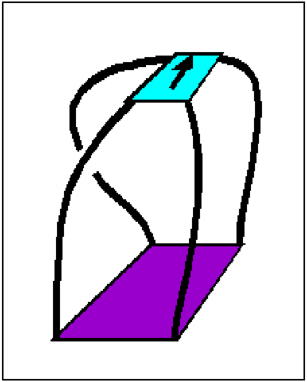
Space

The mathematical notion of space is the way data and change of data is encoded;
thus space encodes motion.

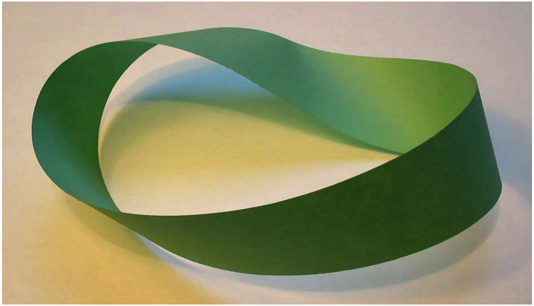
Dirac String Trick

We now show a strange feature of rotations in our 3-dimensional space.



Explanation

How can we explain this? To do this, we look at our modelling of the space of rotations, and in this, introduce our old friend, the Möbius Band.



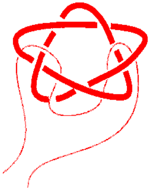
For those who have not seen it before, it is a one sided band, and has only one edge.
So in principle, you can sew a disc onto the Möbius Band!
But if you do try, you get yourself quite tangled!

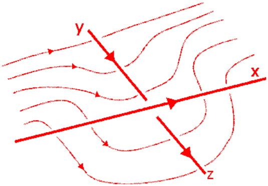
Pivoted lines and the Möbius Band

This is a video which was made in 1992 for my Royal Institution Friday Evening Discourse “Out of Line”.
Moral?
There may be many representations of a given situation, and one wants to find the simplest to make things clear.
The job of maths is to make difficult things easy.

How algebra can structure space

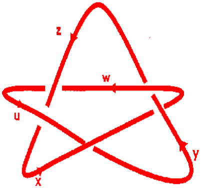
Moving in the space around a knot



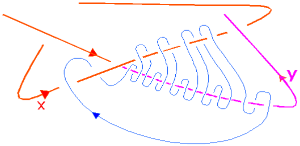


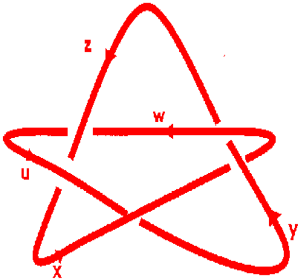
$y = x z x^{-1}$

Relation at a crossing



$x y x y x y^{-1} x^{-1} y^{-1} x^{-1} y^{-1} = 1$





$$x\,y\,x\,y\,x\,y^{-1}\,x^{-1}\,y^{-1}\,x^{-1}\,y^{-1} = 1$$

$$u = xyx^{-1},$$
$$x = ywy^{-1},$$
$$y = wzw^{-1},$$
$$w = zuz^{-1},$$
$$z = uxu^{-1}.$$

$$y = wzw^{-1} = zuz^{-1}.z.z^{-1}u^{-1}z^{-1} = zuzu^{-1}z^{-1} =$$
$$uxu^{-1}.u.uxu^{-1}.u^{-1}.ux^{-1}u^{-1} =$$
$$= uxuxu^{-1}x^{-1}u^{-1} =$$
$$xyx^{-1}.x.xyx^{-1}.x.xy^{-1}x^{-1}.x^{-1}.xy^{-1}x^{-1} =$$
$$xyxyxy^{-1}x^{-1}y^{-1}x^{-1}.$$

For the trefoil, you get the simpler relation

$$xyxy^{-1}x^{-1}y^{-1} = 1.$$

Local to global

The above emphasises an important class of problems in mathematics and science:

Local to global

For this we give another theme, relevant to my title, which is the notion of gluing.

Modern theme in mathematics: structure, rather than numbers; and indeed it is often difficult to describe structure completely in terms of numbers. You may be able to measure or count this or that, but that is unlikely to give a description of the structure.

The area of mathematics which has grown up since the 1950s to talk about varieties of structure, and to compare them, is that of category theory.

A category C has objects, arrows between objects, and a composition of arrows which is associative and has an identity 1_x for each object x. The composition fg of arrows is defined if and only if the endpoint of f is the initial point of g.

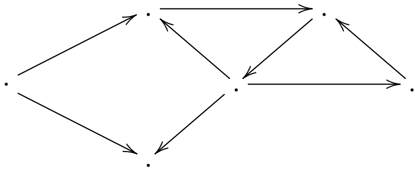
Aim: Describe constructions common to many mathematical situations.

Developed from a useful notation for a function: moving from $y = f(x)$ to $f : X \to Y$. The composition of functions then suggests the first step in the notion of a **category** C, which consists of a class Ob(C) of 'objects' and a set of 'arrows', or 'morphisms' $f : x \to y$ for any two objects x, y, and a composition $fg : x \to z$ if also $g : y \to z$. The only rules are associativity and the existence of identities 1_x at each object x.

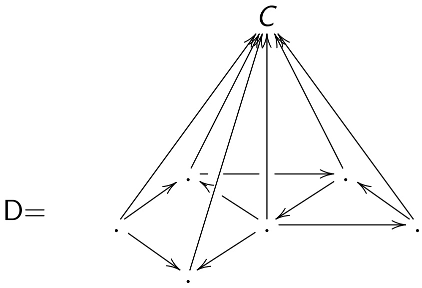
A colimit has 'input data', a 'cocone', and output from the 'best' cocone (when it exists).
Example: $X \cup Y$ has input data the two inclusions $X \cap Y \to X, X \cap Y \to Y$; the cocone is functions $f : X \to C, g : Y \to C$ which agree on $X \cap Y$. The output is a function $(f, g) : X \cup Y \to C$.

'Input data' for a colimit: a diagram D, that is a collection of some objects in a category C and some arrows between them, such as:

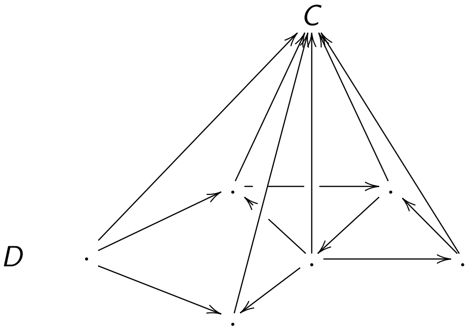
$D =$

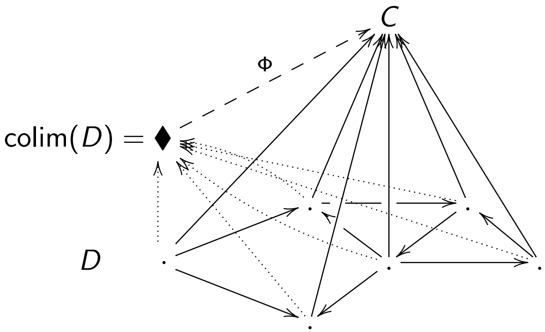


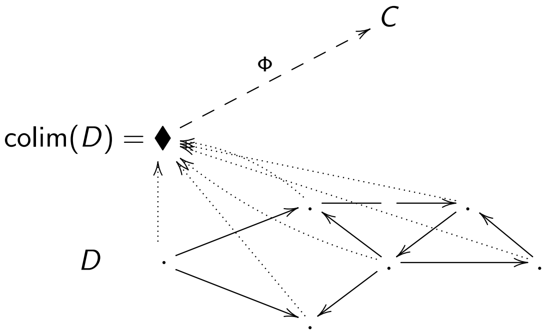
'Functional controls': cocone *with base D and vertex an object C.*



such that each of the triangular faces of this cocone is commutative.


C
D


C
Φ
$\mathrm{colim}(D) = \blacklozenge$
D


C
Φ
$\operatorname{colim}(D) = \blacklozenge$
D

Intuitions:

The object colim(D) is 'put together' from the constituent diagram D by means of the colimit cocone. From beyond (or above our diagrams) D, an object C 'sees' the diagram D 'mediated' through its colimit, i.e. if C tries to interact with the whole of D, it has to do so via colim(D). The colimit cocone is a kind of program: given any cocone on D with vertex C, the output will be a morphism

$$\Phi : \mathrm{colim}(D) \to C$$

constructed from the other data. How is this done?

Email analogy

You want to send an email Φ of a document D to a receiver C. The document D made up of lots of parts. The email programme
splits D up in some way into pieces,
labels each piece at the beginning and end, and
sends these labelled pieces separately to C which
combines them.
Also you want that the final received email is independent of all the choices that have been made.

Neuroscience

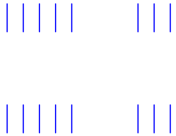
Does this give a model for the notion of structure in the brain and the way a structure communicates?
Compare: Ehresmann, A. and Vanbremeersch. *Memory Evolutive Systems: Hierarchy, Emergence, Cognition*, *Studies in Multidisciplinarity*, Volume 4. Elsevier, Amsterdam (2008).

Higher Dimensional Algebra

The idea is that we may need to get away from 'linear' thinking in order to express intuitions clearly.
Thus the equation

$$2 \times (5 + 3) = 2 \times 5 + 2 \times 3$$

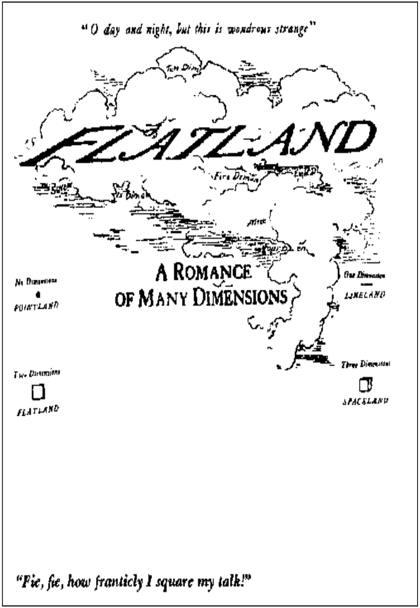
is more clearly shown by the figure



But we seem to need a linear formula to express the general law

$$a \times (b + c) = a \times b + a \times c.$$

Flatland

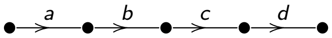


By
Rev. A. Abbott

Published in 1884, available on the internet.

The linelanders had limited interaction capabilities!

We often translate geometry into algebra. For example, a figure as follows:

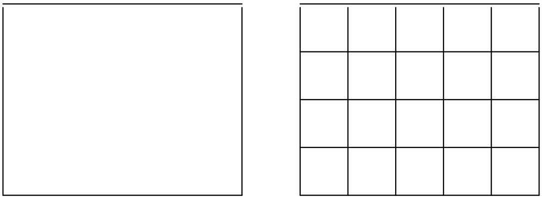


is easily translated into

$$abcd$$

and the language for expressing this is again that of category theory. It is useful to express this intuition as composition is an algebraic inverse to subdivision'. The labelled subdivided line gives the composite word, *abcd*.

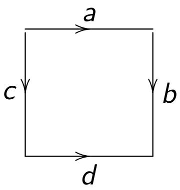
Consider the figures:



From left to right gives subdivision.
From right to left should give composition.
What we need for local-to-global problems is:
Algebraic inverses to subdivision also in dimension 2.
We know how to cut things up, but how to control algebraically putting them together again?

Double Categories

In dimension 1, we still need the 2-dimensional notion of commutative square:



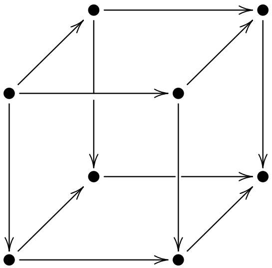
$ab = cd \quad a = cdb^{-1}$

Easy result: any composition of commutative squares is commutative.
In ordinary equations:

$$ab = cd, ef = bg \text{ implies } aef = abg = cdg.$$

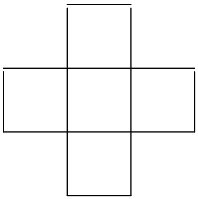
The commutative squares in a category form a double category!

What is a commutative cube?

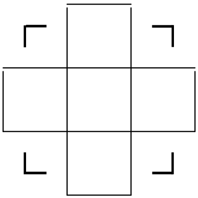


We want the faces to commute!

We might say the top face is the composite of the other faces: so fold them flat to give:



which makes no sense! Need fillers:



To resolve this, we need some special squares called thin:
First the easy ones:

$$\begin{pmatrix} 1 & \begin{matrix} 1 \\ 1 \end{matrix} & 1 \end{pmatrix} \qquad \begin{pmatrix} a & \begin{matrix} 1 \\ 1 \end{matrix} & a \end{pmatrix} \qquad \begin{pmatrix} 1 & \begin{matrix} b \\ b \end{matrix} & 1 \end{pmatrix}$$

$$\square \qquad \overline{\underline{\;\;}} \text{ or } \varepsilon_2 a \qquad | \; | \text{ or } \varepsilon_1 b$$

laws $\qquad [a \;\; \overline{\underline{\;\;}}] = a \qquad \begin{bmatrix} b \\ | \; | \end{bmatrix} = b$

Then we need some new ones:

$$\begin{pmatrix} a & \begin{matrix} a \\ 1 \end{matrix} & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & \begin{matrix} 1 \\ a \end{matrix} & a \end{pmatrix}$$

These are the connections

$$\lrcorner \qquad \ulcorner$$

What are the laws on connections?

$$[\ulcorner\ \lrcorner] = |\ | \qquad \begin{bmatrix}\ulcorner\\ \lrcorner\end{bmatrix} = \begin{matrix}-\\ -\end{matrix} \qquad \text{(cancellation)}$$

$$\begin{bmatrix}\ulcorner & \begin{matrix}-\\ -\end{matrix}\\ |\ | & \ulcorner\end{bmatrix} = \ulcorner \qquad \begin{bmatrix}\lrcorner & |\ |\\ \begin{matrix}-\\ -\end{matrix} & \lrcorner\end{bmatrix} = \lrcorner \qquad \text{(transport)}$$

These are equations on turning left or right, and so are a part of 2-dimensional algebra.
The term transport law and the term connections came from laws on path connections in differential geometry.
It is a good exercise to prove that any composition of commutative cubes is commutative.

Rotations in a double groupoid with connections

To show some 2-dimensional rewriting, we consider the notion of rotations σ, τ of an element u in a double groupoid with connections:

$$\sigma(u) = \begin{bmatrix} \| & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & \| \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \urcorner & \| \\ \ulcorner & u & \lrcorner \\ \| & \llcorner & = \end{bmatrix}.$$

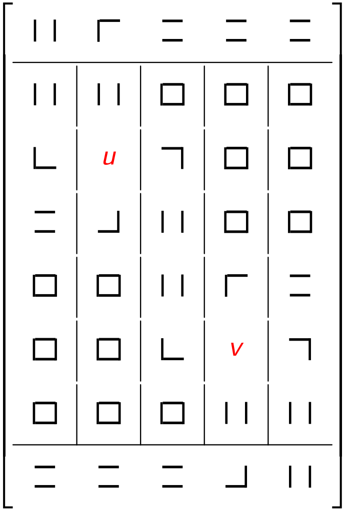
For any $u, v, w \in G_2$,

$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \text{and} \quad \sigma\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\sigma w, \sigma u]$$

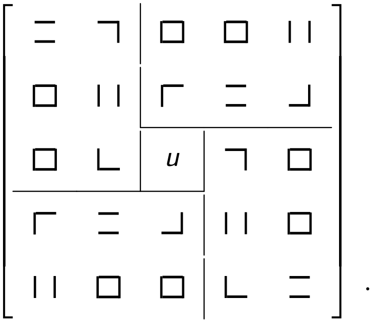
$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \text{and} \quad \tau\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\tau u, \tau w]$$

whenever the compositions are defined.
Further $\sigma^2\alpha = -_1 -_2 \alpha$, and $\tau\sigma = 1$.

To prove the first of these one has to rewrite $\sigma(u +_2 v)$ until one ends up with an array, shown on the next slide, which can be reduced in a different way to $\sigma u +_2 \sigma v$. Can you identify σu, σv in this array? This gives some of the flavour of this 2-dimensional algebra of double groupoids.
This has a homotopical interpretation.


u
v
.

In the lecture, the proof was given on the blackboard that $\tau\sigma(u) = u$, for which a middle step was the diagram



Can you see the final steps?

Conclusion

The progress of mathematics is measured not just in the solution of famous problems,
but also in the opening up of new worlds,
and the development of new structures,
with methods for relating them.
Mathematics develops languages for
description,
deduction,
verification,
calculation.
Some of these languages may be highly significant for the science and technology of the future.