The algebra of cubes[*][†]

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Introduction

This is the first of two papers whose main purpose is to prove a generalization of the Seifert-Van Kampen theorem on the fundamental group of a union of spaces. This generalisation (Theorem C of [8]) will give information in all dimensions and will include as special cases not only the above theorem (without the usual assumptions of path-connectedness) but also

- the Brouwer degree theorem ($\pi_n S^n = \mathbb{Z}$);
- the relative Hurewicz theorem;
- Whitehead's theorem that $\pi_n(X \cup \{e^2_\lambda\}, X)$ is a free crossed module, and
- earlier work [5] of the authors on the case of dimension 2.

The Seifert-Van Kampen theorem describes the fundamental group of a space X with base-point as, under certain circumstances, the colimit of the fundamental groups of subspaces whose interiors cover X. To generalise this to all dimensions we replace the space X by a filtered space

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

We replace the fundamental group $\pi_1(X, *)$ by the *homotopy crossed complex* πX_* which consists of the family of groups $C_n(p) = \pi_n(X_n, X_{n-1}, p)$ for $n \geqslant 2$ and all $p \in X_0$, together with the fundamental groupoid $C_1 = \pi_1(X_1, X_0)$ over $C_0 = X_0$, all with the standard boundary maps $C_n(p) \to C_{n-1}(p)$ and action of C_1.

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Our theorem describes this crossed complex πX_* as, under certain circumstances the colimit of the homotopy crossed complexes of certain filtered subspaces.

The path to this theorem is not direct. Its proof is based on techniques of subdivision, multiple composition, cancellation and the homotopy addition lemma, and the direction of our work has been largely determined by the need to develop these techniques. This explains, what may surprise many readers, the cubical rather than simplicial approach. Subdivisions and multiple compositions are very easily handled cubically; we have been unable to develop them simplicially. Thus, at least for the present, the cubical approach is essential, in spite of the fact that simplicial versions of a number of aspects of this work have been developed [1,12].

This first paper is primarily concerned with setting up the algebraic apparatus needed to imitate the proof of the Seifert-Van Kampen theorem using n-dimensional cubes instead of paths. We introduce a new algebraic category, that of ω-groupoids which models closely the composition and cancellation of cubes. To each ω-groupoid G is associated in canonical fashion a crossed complex γG and we show, by means of certain 'folding operations' on cubes, that this association defines an equivalence γ from the category $\mathcal{G}$ of ω-groupoids to the category $\mathcal{C}$ of crossed complexes. The folding operations are designed to facilitate the composition of the faces of a cube, and we use them also to obtain a purely algebraic form (Lemma 7.1) of the homotopy addition lemma. The equivalence $\gamma : \mathcal{G} \to \mathcal{C}$ is of interest to algebraists as giving a rare example of two equationally defined categories of (many-sorted) algebras which are nontrivially equivalent. The interest is increased by the fact that we know of three other quite different, but equally natural, equational descriptions of categories equivalent to $\mathcal{C}$ and $\mathcal{G}$ (see Section 7 below and [1,6,9]).

In the second paper [8] we define a new homotopy invariant ϱX_*, the *homotopy ω-groupoid of the filtered space* X_*. We generalise the proof of the Seifert-Van Kampen theorem to obtain a union theorem for ϱX_* (Theorem B of [8]). We then prove that $\gamma\varrho X_*$ is naturally isomorphic to πX_*. The result for πX_* follows immediately. (All these results were announced, with definitions but no proofs, in [6,7].)

A reader who wishes to see quickly the main direction of these two papers should read first Sections 1, 2 and 3 of this paper for the definitions of the categories $\mathcal{G}$ and $\mathcal{C}$ and the functor $\gamma : \mathcal{G} \to \mathcal{C}$. He should then turn to the second paper: Sections 1 and 2 for the construction of the ω-groupoid ϱX_*; Section 4 for the statement of the union theorem (Theorem B); and Sections 5 and 7 for the applications of this theorem to colimit theorems for relative homotopy groups. However, the proof of the union theorem does require detailed knowledge of properties of 'thin' elements in ϱX_* and in an arbitrary ω-groupoid, and the establishment of these properties requires most of the algebraic apparatus of this first paper.

The reader may also note that the proofs of our main results make no use of such classical methods of algebraic topology as simplicial approximation, covering spaces, fibrations or homology; indeed, we do not know how to obtain our colimit theorem for πX_* by these traditional methods. The two papers should therefore be read as an attempt to remodel certain aspects of elementary homotopy theory by using algebraic structures which mimic the geometry more closely than those most commonly used. This approach is justified by the progress made in extending the Seifert-Van Kampen theorem to higher dimensions and by the emergence of interesting structures that seem to warrant further investigation in their own right.

1 Connections and compositions in cubical complexes

1.1 Let K be a cubical complex, that is, a family of sets $\{K_n; n \geqslant 0\}$ with the face and degeneracy maps

$$\partial_i^\alpha : K_n \to K_{n-1}, \qquad \varepsilon_i : K_{n-1} \to K_n \ (i = 1, 2, \ldots, n; \alpha = 0, 1)$$

satisfying the usual cubical relations:

$$\text{(1.1)(i)} \qquad \partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \qquad (i < j),$$

$$\text{(1.1)(ii)} \qquad \varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \qquad (i \leqslant j),$$

$$\text{(1.1)(iii)} \qquad \partial_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1} \partial_i^\alpha & (i < j) \\ \varepsilon_j \partial_{i-1}^\alpha & (i > j) \\ \mathrm{id} & (i = j) \end{cases}$$

1.2 We say that K is a *cubical complex with connections* if it has additional structure maps

$$\Gamma_i : K_{n-1} \to K_n \ (i = 1, 2, \ldots, n-1)$$

(called *connections*) satisfying the relations:

$$\text{(1.2)(i)} \qquad \Gamma_i \Gamma_j = \Gamma_{j+1} \Gamma_i \qquad (i \leqslant j)$$

$$\text{(1.2)(ii)} \qquad \Gamma_i \varepsilon_j = \begin{cases} \varepsilon_{j-1} \Gamma_i & (i < j) \\ \varepsilon_j \Gamma_{i-1} & (i > j) \end{cases}$$

$$\Gamma_j \varepsilon_j = \varepsilon_j^2 = \varepsilon_{j+1} \varepsilon_j,$$

$$\partial_i^\alpha \Gamma_j = \begin{cases} \Gamma_{j-1} \partial_i^\alpha & (i < j) \\ \Gamma_j \partial_{i-1}^\alpha & (i > j+1), \end{cases}$$

$$\text{(1.2)(iii)} \qquad \partial_j^0 \Gamma_j = \partial_{j+1}^0 \Gamma_j = id,$$

$$\partial_j^1 \Gamma_j = \partial_{j+1}^1 \Gamma_j = \varepsilon_j \partial_j^1.$$

The connections are to be thought of as extra 'degeneracies'. (A degenerate cube of type $\varepsilon_j x$ has a pair of opposite faces equal and all other faces degenerate. A cube of type $\Gamma_i x$ has a pair of adjacent faces equal and all other faces of type $\Gamma_j y$ or $\varepsilon_j y$.) Cubical complexes with this, and other, structures have also been considered by Evrard [15].

The prime example of a cubical complex with connections is the singular cubical complex KX of a space X. Here K_n is the set of singular n-cubes in X (i.e. continuous maps $I^n \to X$) and the connection $\Gamma_i : K_{n-1} \to K_n$ is induced by the map $\gamma_i : I^n \to I^{n-1}$ defined by

$$\gamma_i(t_1, t_2, \ldots, t_n) = (t_1, t_2, \ldots, t_{i-1}, \max(t_i, t_{i+1}), t_{i+2}, \ldots, t_n).$$

The complex KX has some further structure which we wish to exploit, namely the composition of n-cubes, and their reversal, in the n different directions. Accordingly, we define a *cubical complex with connections and compositions* to be a cubical complex K with connections in which each K_n has n partial compositions $+_j$ and n unary operations $-_j$ $(j = 1, 2, \ldots, n)$ satisfying the following axioms.

1.3 If $a, b \in K_n$, then $a +_j b$ is defined if and only if $\partial_j^0 b = \partial_j^1 a$, and then

$$(1.3)(\mathrm{i}) \qquad \begin{cases} \partial_j^0(a +_j b) = \partial_j^0 a \\ \partial_j^1(a +_j b) = \partial_j^1 b \end{cases} \qquad \partial_i^\alpha(a +_j b) = \begin{cases} \partial_j^\alpha a +_{j-1} \partial_i^\alpha b & (i < j) \\ \partial_i^\alpha a +_j \partial_i^\alpha b) & (i > j), \end{cases}$$

If $a \in K_n$, then $-_j a$ is defined and

$$(1.3)(\mathrm{ii}) \qquad \begin{cases} \partial_j^0(-_j a) = \partial_j^1 a \\ \partial_j^1(-_j a) = \partial_j^0 a \end{cases} \qquad \partial_i^A(-_j a) = \begin{cases} -_{j-1}\partial_i^\alpha a & (i < j) \\ -_j \partial_i^\alpha a & (i > j) \end{cases}$$

$$(1.3)(\mathrm{iii}) \qquad -_j(a +_j b) = (-_j b) +_j (-_j a), \qquad -_j(-_j a) = a.$$

1.4 The *interchange laws.* If $i \neq j$ then

$$(1.4)(\mathrm{i}) \qquad (a +_i b) +_j (c +_i d) = (a +_j c) +_i (b +_j d)$$

whenever both sides are defined. (The diagram

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \begin{array}{l} \longrightarrow j \\ \downarrow \\ i \end{array}$$

will be used to indicate that both sides of the above equation are defined and also to denote the unique composite of the four elements.)

If $i \neq j$ then

$$(1.4)(\mathrm{ii}) \qquad -_i(a +_j b) = (-_i a) +_j (-_i b) \text{ and } -_i (-_j a) = -_j(-_i a).$$

1.5

$$(1.5)(\mathrm{i}) \qquad \varepsilon_i(a +_j b) = \begin{cases} \varepsilon_i a +_{j+1} \varepsilon_i b & (i \leqslant j) \\ \varepsilon_i a +_j \varepsilon_i b & (i > j) \end{cases}$$

$$(1.5)(\mathrm{ii}) \qquad \varepsilon_i(-_j b) = \begin{cases} -_{j+1}\varepsilon_i a & (i \leqslant j) \\ -_j \varepsilon_i a & (i > j) \end{cases}$$

1.6

$$(1.6)(\mathrm{i}) \qquad \Gamma_i(a +_j b) = \begin{cases} \Gamma_i a +_{j+1} \Gamma_i b & (i < j) \\ \Gamma_i a +_j \Gamma_i b & (i > j) \end{cases}$$

$$\Gamma_j(a +_j b) = \begin{bmatrix} \Gamma_j a & \varepsilon_j b \\ \varepsilon_{j+1} b & \Gamma_j b \end{bmatrix} \quad \begin{array}{l} \longrightarrow j+1 \\ \downarrow \\ j \end{array}$$

(This last equation is the *transport law.*)

$$(1.6)(\mathrm{ii}) \qquad \Gamma_i(-_j a) = \begin{cases} -_{j+1}\Gamma_i a & (i < j) \\ -_j \Gamma_i a & (i > j) \end{cases}$$

It is easily verified that the singular cubical complex KX of a space X satisfies these axioms if $+_j, -_j$ are defined by

$$(a +_j b)(t_1, t_2, \ldots, t_n) = \begin{cases} a(t_1, \ldots, t_{j-1}, 2t_j, t_{j+1}, \ldots, t_n) & (t_j \leqslant \frac{1}{2}) \\ b(t_1, \ldots, t_{j-1}, 2t_j - 1, t_{j+1}, \ldots, t_n) & (t_j \geqslant \frac{1}{2}) \end{cases}$$

whenever $\partial_j^0 b = \partial_j^1 a$; and

$$(-_j a)(t_1, t_2, \ldots, t_n) = a(t_1, \ldots, t_{j-1}, 1 - t_j, t_{j+1}, \ldots, t_n).$$

2 ω-groupoids

An ω-groupoid $G = \{G_n\}$ is a cubical complex with connections and compositions such that each $+_j$ is a groupoid structure on G_n with identity elements $\varepsilon_j y (y \in G_{n-1})$ and inverse $-_j$.

The example we have in mind is constructed as follows. We start with a filtered space

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_\infty = X.$$

Let I_*^n denote the n-cube with its skeletal filtration (using its standard cell-structure). Let $R_n X_*$ be the set of filtered maps $I_*^n \to X_*$, that is, maps $I^n \to X$ sending the r-skeleton I_r^n of I^n into X_r for $r \leqslant n$. Then $RX_* = \{R_n X_*\}$ is clearly, with the standard operations on cubes defined above, a cubical complex with connections and compositions.

Two filtered maps $I_*^n \to X_*$ are *filter-homotopic* if, as maps $I^n \to X$, they are homotopic by a map $H : I^n \times I \to X$ sending $I_r^n \times I$ into X_r for $r \leqslant n$. If $\varrho_n X_*$ denotes the set of filter-homotopy classes rel vertices of I^n in $R_n X_*$, then $\varrho X_* = \{\varrho_n X_*\}$ is clearly a cubical complex with connections. In [8] we prove the nontrivial fact that the compositions on RX_* induce compositions on ϱX_* making ϱX_* an ω-groupoid. This geometric example is one of the foundations of our work.

In defining ω-groupoids some of the previous laws are redundant. Thus if one assumes that each $+_j$ is a groupoid structure on G_n with identities $\varepsilon_j y$ $(y \in G_{n-1})$ and inverse $-_j$, then one may omit parts (ii) of all the laws (1.3), (1.4), (1.5), (1.6) since they follow from parts (i) and the groupoid laws. One may also rewrite the transport law (1.6)(i) in the form

(2.1)(i) $$\Gamma_j(a +_j b) = (\Gamma_j a +_{j+1} \varepsilon_j b) +_j \Gamma_j b = (\Gamma_j a +_j \varepsilon_{j+1} b) +_{j+1} \Gamma_j b$$

and deduce that

(2.1)(ii) $$\Gamma_j(-_j a) = (-_j \Gamma_j a) -_{j+1} \varepsilon_j a = (-_{j+1} \Gamma_j a) -_j \varepsilon_{j+1} a.$$

Further, in any ω-groupoid G , the interchange law (1.4)(i) and associativity imply that a rectangular array of n-cubes $x_{pq} \in G_n$ $(1 \leqslant p \leqslant P, 1 \leqslant q \leqslant Q)$ satisfying for some $i \neq j$ the relations

$$\partial_i^1 x_{pq} = \partial_i^0 x_{p+1,q} \ (1 \leqslant p < P, 1 \leqslant q \leqslant Q)$$
$$\partial_j^1 x_{pq} = \partial_j^0 x_{p,q+1} \ (1 \leqslant p \leqslant P, 1 \leqslant q < Q)$$

has a unique composite $x \in G_n$ obtained by applying the operations $+_i, +_j$ in any well-formed fashion; for example

$$x = (x_{11} +_i x_{21} +_i \cdots +_i x_{P1}) +_j \cdots +_j (x_{1Q} +_i x_{2Q} +_i \cdots +_i x_{PQ}).$$

Such a composite is written

$$[x_{pq}] \; \begin{smallmatrix} \longrightarrow j \\ \downarrow \\ i \end{smallmatrix}$$

or simply $[x_{pq}]$ if no confusion will arise. The same is true for multi-dimensional arrays, and the most general situation can be described as follows. Let $(m) = (m_1, m_2, \dots, m_n)$ be a sequence of positive integers. A *composable array* in G_n of type (m) is a family of cubes $x_{(p)} \in G_n$, where $(p) = (p_1, p_2, \dots, p_n), 1 \leqslant p_i \leqslant m_i$, satisfying the relations

$$\partial_i^1 x_{(p)} = \partial_i^0 x_{(p)'_i} \text{ for all } i$$

where $(p)'_i = (p_1, p_2, \dots, p_{i-1}, p_i + 1, p_{i+1}, \dots, p_n)$. We denote the unique composite in G_n of such an array by $[x_{(p)}]$. The previous case is obtained by taking $m_k = 1$ for $k \neq i, j$. We shall also sometimes write $[x_1, x_2, \dots, x_r]_j$ for the linear composite $x_1 +_j x_2 +_j \cdots +_j x_r$, and an unlabelled $-x$ in such a composite will always mean $-_j x$.

An *ω-subgroupoid* of G is a cubical subcomplex closed under all the connections and all the operations $+_j, -_j$. Any set S of elements of G generates an ω-subgroupoid, namely, the intersection of all ω-subgroupoids containing S. This ω-subgroupoid can be built from S by repeated applications of all the structure maps and operations. First, it can be verified that the elements of the form $\varepsilon \dots \varepsilon \Gamma \dots \Gamma \partial \dots \partial x$ $(x \in S)$ make up the subcomplex-with-connections K generated by S. (Here ∂ stands for various ∂_i^α, etc.) The ω-subgroupoid generated by S then consists, as again can be verified, of all composites of arrays of cubes of the form $-_i -_j \dots -_l \, y$ $(y \in K)$.

A *morphism* of ω-groupoids is a morphism of cubical complexes preserving all the connections and all the groupoid operations. We denote the resulting category of ω-groupoids by $\mathcal{G}$. This category is complete and cocomplete, as follows from general theorems of Freyd [16], Bastiani-Ehresmann [2] and Coates [11]. It is in fact tripleable over the category of cubical complexes. We will prove below that $\mathcal{G}$ is equivalent to a category $\mathcal{C}$ of crossed complexes which Howie [17] has shown to be Cartesian-closed; it follows that $\mathcal{G}$ is Cartesian-closed.

We also use finite-dimensional versions of the above definitions. An m-tuple groupoid is an m-truncated cubical complex $G = (G_m, G_{m-1}, \dots, G_0)$ with connections, having n groupoid structures in dimension n $(n \leqslant m)$, and satisfying all the laws for an ω-groupoid in so far as they make sense. We denote by $\mathcal{G}_m$ the category of m-tuple groupoids. (The category $\mathcal{G}_2$ of double groupoids, the prototype for $\mathcal{G}$, was introduced in [10]).

3 Crossed complexes

The relative homotopy groups $C_n = \pi_n(X_n, X_{n-1}, *)$ of a filtered space X_* with base-point form (with respect to the usual boundary maps) a sequence of groups

$$\cdots \longrightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1$$

on which $C_1 = \pi_1(X_1, *)$ acts. The formal properties of such sequences have been studied by Blakers [3], Whitehead [21] and Huebschmann [18,19] under the names 'group systems', 'homotopy systems' and 'crossed resolutions', respectively. If the base-point is allowed to vary in a subspace X_0, one obtains such a sequence over each point of X_0, and the fundamental groupoid $\pi_1(X_1, X_0)$ acts on the collection of sequences (in a manner which we axiomatise below) forming a 'crossed complex' πX_*. This complex is closely related to the ω-groupoid ϱX_* described in Section 2, and we shall show that this relationship exists at the algebraic level by exhibiting between the categories of crossed complexes and ω-groupoids an equivalence in which πX_* and ϱX_* correspond.

A *crossed complex* C (over a groupoid) is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_0$$

satisfying the following axioms;

3.1 C_1 is a groupoid with C_0 as its set of vertices and δ^0, δ^1 as its initial and final maps.

We write $C_1(p,q)$ for the set of arrows from p to q $(p, q \in C_0)$ and $C_1(p)$ for the group $C_1(p,p)$.

3.2 For $n \geqslant 2$, C_n is a family of groups $\{C_n(p)\}_{p \in C_0}$ and for $n \geqslant 3$, the groups $C_n(p)$ are Abelian.

3.3 The groupoid C_1 operates on the right on each $C_n\,(n \geqslant 2)$ by an action denoted $(x, a) \mapsto x^a$. Here, if $x \in C_n(p)$ and $a \in C_1(p,q)$ then $x^a \in C_n(q)$. (Thus $C_n(p) \cong C_n(q)$ if p and q lie in the same component of the groupoid C_1.)

We use additive notation for all groups $C_n(p)$ and for the groupoid C_1, and we use the same symbol 0 for all their identity elements.

3.4 For $n \geqslant 2$, $\delta : C_n \to C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1, where C_1 acts on the groups $C_1(p)$ by conjugation: $x^a = -a + x + a$.

3.5 $\delta\delta = 0 : C_n \to C_{n-2}$ for $n \geqslant 3$ (and $\delta^0\delta = \delta^1\delta : C_2 \to C_0$, as follows from 3.4).

3.6 If $c \in C_2$, then δc operates trivially on C_n for $n \geqslant 3$ and operates on C_2 as conjugation by c, that is, $x^{\delta c} = -c + x + c\,(x, c \in C_2(p))$.

We observe that these laws make $C_2(p)$ a crossed module over $C_1(p)$, and this is the reason for the name 'crossed complex'. In the case when C_0 is a single point we call C a *crossed complex over a group*, or a *reduced* crossed complex.

A *morphism* of crossed complexes $f : C \to D$ is a family of morphisms of groupoids $f_n : C_n \to D_n\,(n \geqslant 1)$ all inducing the same map of vertices $f_0 : C_0 \to D_0$, and compatible with the boundary maps $\delta : C_n \to C_{n-1}, D_n \to D_{n-1}$ and the actions of C_1, D_1 on C_n, D_n. We denote by $\mathcal{C}$ the resulting category of crossed complexes. Again we use a finite-dimensional version of these definitions. An *m-truncated crossed complex* is a finite sequence

$$C_m \xrightarrow{\delta} C_{m-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_0$$

satisfying all the axioms 3.1 - 3.6 in so far as they make sense. (Alternatively, it could be described as a crossed complex with $C_n = 0$ for $n > m$.) Thus a 1-truncated crossed complex is simply a groupoid, and a 2-truncated crossed complex is what we will term a crossed module over a groupoid; in the case of one vertex, it is a crossed module in the usual sense. We denote by $\mathcal{C}_m$ the category of m-truncated crossed modules.

In [10] it was shown that the category $\mathcal{G}_2^*$ of double groupoids with one vertex is equivalent to the category $\mathcal{C}_2^*$ of crossed modules over groups. The construction given there extends easily to an equivalence $\mathcal{G}_2 \simeq \mathcal{C}_2$ between double groupoids and crossed modules over groupoids. We prove below that it extends further to equivalences $\mathcal{G}_m \simeq \mathcal{C}_m$ and $\mathcal{G} \simeq \mathcal{C}$.

For any ω-groupoid G, we construct the crossed complex $C = \gamma G$ associated with G as follows. Let $C_0 = G_0, C_1 = G_1$ and $\delta^\alpha = \partial_1^\alpha : G_1 \to G_0 (\alpha = 0, 1)$. For $n \geqslant 2$ and $p \in C_0 = G_0$, let $C_n(p) = \{x \in G_n : \partial_i^\alpha x = (\varepsilon_1)^{n-1}p \text{ for all } (\alpha, i) \neq (0, 1)\}$, the set of n-cubes x all of whose faces except $\partial_1^0 x$ are *concentrated* at p. We observe that a concentrated r-cube $(\varepsilon)^r p$ is an identity for all compositions $+_k$ of G since $(\varepsilon)^r p = \varepsilon_k (\varepsilon)^{r-1} p$ for $1 \leqslant k \leqslant r$; accordingly, we will write 0 (sometimes 0_p) for such a cube $(\varepsilon)^r p \, (p \in C_0)$. With this convention, we have the rules $\partial_i^\alpha 0 = 0, \varepsilon_i 0 = 0, \Gamma_i 0 = 0$.

Lemma 3.7 *Let $n \geqslant 2$ and $p \in C_0$. Then each composition $+_j$ of G_n, for $2 \leqslant j \leqslant n$, induces a group structure on $C_n(p)$. For $n \geqslant 3$ this group structure is independent of j and is Abelian.*

Proof The first part is easy to verify, while the last part is proved by applying the interchange law to the composites

$$\begin{bmatrix} x & 0_p \\ 0_p & y \end{bmatrix} \qquad \begin{bmatrix} 0_p & x \\ y & 0_p \end{bmatrix} \quad \begin{array}{l} \xrightarrow{\;k\;} \\ \downarrow j \end{array}$$

for $x, y \in C_n(p)$ and $2 \leqslant j < k \leqslant n$. □

We write $x + y$ for $x +_j y$ if $x, y \in C_n(p)$ and $2 \leqslant j \leqslant n$, and the zero element for this addition is 0_p. If $n = 1$ we also write $+$ for the groupoid operation $+_1$ on $C_1 = G_1$.

It is easily verified that the face map $\partial_1^0 : G_n \to G_{n-1}$ restricts to a group homomorphism $\delta : C_n(p) \to C_{n-1}(p)$, and that $\delta^2 = 0$ (where $C_1(p)$ is defined to be $C_1(p, p)$).

Lemma 3.8 *Let $n \geqslant 1$ and $x \in C_n(p)$, $a \in C_1(p, q)$. Then the element*

$$x = [-\varepsilon_1^{n-1} a, x, \varepsilon_1^{n-1} a]_n$$

lies in $C_n(q)$, and the rule $(x, a) \mapsto x^a$ defines an action of the groupoid C_1 on the family of groups $C_n(p)$, $p \in C_0$. This action is preserved by the map $\delta : C_n(p) \to C_{n-1}(p)$ for $n \geqslant 2$. Further, if $x \in C_n(p), y \in C_2(p)$ and $a = \delta y$, then $x^a = x$ for $n \geqslant 3$ and $x^a = -y + x + y$ for $n = 2$.

Proof. Note that, for $1 \leqslant i < n$, $\partial_i^\alpha (x^a) = [-\varepsilon_1^{n-2} a, \partial_i^\alpha x, \varepsilon_1^{n-2} a]_{n-1}$, while $\partial_n^\alpha (x^a) = \varepsilon^{n-1} \partial_1^1 a = 0_q$. From this it follows that $x^a \in C_n(q)$ and $\delta(x^a) = (\delta x)^a$. That $x^{a+b} = (x^a)^b$ follows from the equation

$$\varepsilon_1^{n-1}(a + b) = [\varepsilon_1^{n-1} a, \varepsilon_1^{n-1} b]_n.$$

Now, if $a = \delta y$ and $n \geqslant 2$, the two ways of composing

$$\begin{bmatrix} -_n \varepsilon_1^{n-1} a & x & \varepsilon_1^{n-1} a \\ -_n \varepsilon_1^{n-2} y & 0_p & \varepsilon_1^{n-2} y \end{bmatrix} \quad \begin{array}{l} \xrightarrow{\;n\;} \\ \downarrow n-1 \end{array}$$

give $x^a = [-_n\varepsilon_1^{n-2}y, x, \varepsilon_1^{n-2}y_n]_n$, which is the result we require when $n = 2$. For $n \geqslant 3$ we may also compose

$$\begin{bmatrix} -_n -_{n-1}\varepsilon_1^{n-2}y & 0_p & -_{n-1}\varepsilon_1^{n-2}y \\ -_n\varepsilon_1^{n-2}y & x & \varepsilon_1^{n-2}y \end{bmatrix} \quad \begin{array}{l} \longrightarrow n \\ \downarrow n-1 \end{array}$$

in two ways to obtain, by what we have just proved, $x^a = x$. □

From the above results we obtain easily:

Proposition 3.9 *γG is a crossed complex, and γ defines a functor from the category $\mathcal{G}$ of ω -groupoids to the category $\mathcal{C}$ of crossed complexes. Also γ induces a functor from the category $\mathcal{G}_m$ of m-tuple groupoids to the category $\mathcal{C}_m$ of m-truncated crossed complexes.*

We shall show in Section 6 that the ω-groupoid G can be reconstructed from its crossed complex γG and hence that $\gamma : \mathcal{G} \to \mathcal{C}$ is an equivalence of categories.

For later use in proving Lemma 4.9, we record the following lemma.

Lemma 3.10 *The action of C_1 on C_n defined in Lemma 3.8 is also given by*

$$x^a = [-\varepsilon_1^{j-1}\varepsilon_2^{n-j}a, x, \varepsilon_1^{j-1}\varepsilon_2^{n-j}a]_j$$

for $x \in C_n(p)$, $a \in C_1(p,q)$ and any j with $2 \leqslant j \leqslant n$.

Proof. Let $2 \leqslant j \leqslant n$, and write $b_j = \varepsilon_1^{j-1}\varepsilon_2^{n-j}a = \varepsilon_n\varepsilon_{n-1}\dots\widehat{\jmath}\dots\varepsilon_1 a \in G_n$. Then b_j is an identity for all the compositions of G_n except $+_j$. Also $\partial_j^1(-_j b_j) = \partial_j^0(b_j) = 0$ and

$$\partial_{j+1}^\alpha(b_j) = \partial_j^\alpha(b_{j+1}) = \varepsilon_{n-1}\varepsilon_{n-2}\dots\widehat{\jmath}\dots\varepsilon_1 a = c,$$

say. Thus, if $j \geqslant 2$, we may form the composite

$$y = \begin{bmatrix} -_j -_{j+1}\Gamma_j c & -_j b_j & -_j\Gamma_j c \\ -_{j+1}b_{j+1} & x & b_{j+1} \\ -_{j+1}\Gamma_j c & b_j & \Gamma_j c \end{bmatrix} \quad \begin{array}{l} \longrightarrow j+1 \\ \downarrow j \end{array}$$

Since b_{j+1} is an identity for $+_j$, the composite of the last column is $\varepsilon_j\partial_j^1\Gamma_j c = 0_p$, and similarly the composites of the first column and of the first and last rows are 0_p. Hence, computing y by rows and by columns, we have

$$[-b_{j+1}, x, b_{j+1}]_{j+1} = [-b_j, x, b_j]_j \ (j \geqslant 2).$$

It follows that, for $j \geqslant 2$, $[-b_j, x, b_j]_j = [-b_n, x, b_n]_n$, which is the definition of x^a. □

4 Folding operations

In this section we introduce an operation Φ on cubes in an ω-groupoid G (or in an m-tuple groupoid) which has the effect of folding all faces of $x \varepsilon G_n$ onto the face $\partial_1^0 x$ so that they can be composed to form the 'sum of the faces of x'. This operation Φ transforms x into an element of the associated crossed complex γG.

First, in any m-tuple groupoid G, we define operations $\Phi_j : G_n \to G_n (1 \leqslant j < n \leqslant m)$ by the formula

$$\Phi_j x = [-\varepsilon_j \partial_j^1 x, -\Gamma_j \partial_{j+1}^0 x, x, \Gamma_j \partial_{j+1}^1 x]_{j+1} \tag{4.1}$$

It is easy to check that this composite is defined. Writing a, b, c, d for the relevant faces of x,

$$\begin{array}{ccc} & c & \\ \cline{2-2} b & \multicolumn{1}{|c|}{x} & d \\ \cline{2-2} & a & \end{array} \qquad \begin{array}{l} \rightarrow j+1 \\ \downarrow \\ j \end{array}$$

the effect of Φ_j can be seen from the diagram

$$\Phi_j x = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{-_j a} & \multicolumn{1}{c}{-_j b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline -_{j+1}\varepsilon_j a & -_{j+1}\Gamma_j b & b \quad x \quad d & \Gamma_j d \\ \hline \multicolumn{1}{c}{-_j a} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{} \end{array} \qquad \begin{array}{l} \rightarrow j+1 \\ \downarrow \\ j \end{array}$$

in which unlabelled faces are appropriate degenerate cubes.

The laws of Section 2 imply various laws for the operations Φ_j and their composites. To simplify the notation we write $\eta_j^\alpha x$ for $\varepsilon_j \partial_j^\alpha x$, the left ($\alpha = 0$) or right ($\alpha = 1$) identity for x with respect to $+_j$.

4.2

(4.2)(i) $$\partial_i^\alpha \Phi_j = \begin{cases} \Phi_{j-1}\partial_i^\alpha & (i < j), \\ \Phi_j \partial_i^\alpha & (i > j+1), \end{cases}$$

(4.2)(ii) $$\partial_j^0 \Phi_j x = [-\partial_j^1 x, -\partial_{j+1}^0 x, \partial_j^0 x, \partial_{j+1}^1 x]_j.$$

(4.2)(iii) $$\partial_{j+1}^\alpha \Phi_j = \partial_j^1 \Phi_j = \eta_j^1 \partial_j^1 = \eta_j^1 \partial_{j+1}^1.$$

(4.2)(iv) $$\begin{aligned} \partial_{j+1}^\alpha \Phi_j \Phi_{j+1} \cdots \Phi_{n-1} &= \partial_j^1 \Phi_j \Phi_{j+1} \cdots \Phi_{n-1} \\ &= \eta_j^1 \eta_{j+1}^1 \cdots \eta_{n-1}^1 \partial_n^1. \end{aligned}$$

Proof. (i) If $i < j$ then by (1.1), (1.2) and (1.3)

$$\begin{aligned} \partial_i^\alpha \Phi_j x &= [-\partial_i^\alpha \eta_j^1 x, -\partial_i^\alpha \Gamma_j \partial_{j+1}^0 x, \partial_i^\alpha x, \partial_i^\alpha \Gamma_j \partial_{j+1}^1 x]_j \\ &= [-\eta_{j-1}^1 \partial_i^\alpha x, -\Gamma_{j-1} \partial_j^0 \partial_i^\alpha x, \partial_i^\alpha x, \Gamma_{j-1} \partial_j^1 \partial_i^\alpha x]_j \\ &= \Phi_{j-1} \partial_i^\alpha x. \end{aligned}$$

The case $i > j+1$ is similar.

(ii) This is proved by a routine argument of the same kind and we will omit all such routine proofs from now on.

(iii) By (1.1), (1.2) and (1.3),

$$\begin{aligned} \partial_j^1 \Phi_j x &= [-\partial_j^1 \eta_j^1 x, -\partial_j^1 \Gamma_j \partial_{j+1}^0 x, \partial_j^1 x, \partial_j^1 \Gamma_j \partial_{j+1}^1 x]_j \\ &= [-\partial_j^1 x, \eta_j^1 \partial_{j+1}^0 x, \partial_j^1 x, \eta_j^1 \partial_{j+1}^1 x]_j. \end{aligned}$$

But $\eta_j^1\partial_{j+1}^0 x$ and $\eta_j^1\partial_{j+1}^1 x$ are identities for $+_j$, so

$$\partial_j^1\Phi_j x = [-\partial_j^1 x, \partial_j^1 x]_j = \eta_j^1\partial_j^1 x.$$

The other cases are easily verified.

(iv) This follows from (iii). □

4.3

(4.3)(i) $$\begin{cases} \Phi_j\varepsilon_i = \varepsilon_i\Phi_{j-1} \text{ and } \Phi_j\eta_i^\alpha = \eta_i^\alpha\Phi_j & \text{if } i < j; \\ \Phi_j\varepsilon_i = \varepsilon_i\Phi_j \text{ and } \Phi_j\eta_i^\alpha = \eta_i^\alpha\Phi_j & \text{if } i > j+1. \end{cases}$$

(4.3)(ii) $\Phi_j\varepsilon_j = \eta_{j+1}^1\varepsilon_j = \eta_j^1\varepsilon_{j+1}$ and $\Phi_j\eta_j^\alpha = \eta_{j+1}^1\eta_j^\alpha$.

(4.3)(iii) $\Phi_j\varepsilon_{j+1} = \eta_{j+1}^1\varepsilon_j = \eta_j^1\varepsilon_{j+1}$ and $\Phi_j\eta_{j+1}^\alpha = \eta_j^1\eta_{j+1}^\alpha$.

(4.3)(iv) $\Phi_1\Phi_2\cdots\Phi_{j-2}\eta_{j-1}^1 = \eta_1^1\eta_2^1\cdots\eta_{j-1}^1$.

(4.3)(v) $\Phi_1\Phi_2\cdots\Phi_{j-1}\varepsilon_j = \eta_1^1\eta_2^1\cdots\eta_{j-1}^1\varepsilon_j$.

Proof. (i) and (ii) are routine; the parts involving $\Phi_j\eta_j^\alpha$ use 4.2 as well as (1.1), (1.2) and (1.3).

(iii)

$$\begin{aligned} \Phi_j\varepsilon_{j+1}x &= [-\eta_j^1\varepsilon_{j+1}x, -\Gamma_j x, \varepsilon_{j+1}x, \Gamma_j x]_{j+1} \\ &= [-\eta_j^1\varepsilon_{j+1}x]_{j+1} = [-\eta_{j+1}^1\varepsilon_j x]_{j+1} \\ &= \eta_{j+1}^1\varepsilon_j x. \end{aligned}$$

The other equations follow easily.

(iv) and (v) follow from (iii). □

4.4

(4.4)(i) $$\Phi_j\Gamma_i = \begin{cases} \Gamma_i\Phi_{j-1} & (i < j), \\ \Gamma_i\Phi_j & (i > j+1). \end{cases}$$

(4.4)(ii) $\Phi_j\Gamma_j = \varepsilon_j\eta_j^1 = \varepsilon_{j+1}\eta_j^1$.

(4.4)(iii) $\Phi_j\Gamma_{j+1}x = [-\Gamma_{j+1}\eta_j^1 x, -\Gamma_j x, \Gamma_{j+1}x, \Gamma_j\eta_{j+1}^1 x]_{j+1}$.

Proof. (i) and (iii) are routine. For (ii),

$$\begin{aligned} \Phi_j\Gamma_j x &= [-\eta_j^1\Gamma_j x, -\Gamma_j\partial_{j+1}^0\Gamma_j x, \Gamma_j x, \Gamma_j\partial_{j+1}^1\Gamma_j x]_{j+1} \\ &= [-\varepsilon_j\eta_j^1 x, -\Gamma_j x, \Gamma_j x, \Gamma_j\eta_j^1 x]_{j+1} && \text{by (1.2)} \\ &= [-\varepsilon_{j+1}\eta_j^1 x, \varepsilon_{j+1}\eta_j^1 x]_{j+1} && \text{by (1.1) and (1.2)} \\ &= \varepsilon_{j+1}\eta_j^1 x = \varepsilon_j\eta_j^1 x. \end{aligned}$$

□

We now define the *folding operation* Φ by

$$\Phi x = \Phi_1 \Phi_2 \cdots \Phi_{n-1} x \qquad (x \in G_n, n \geqslant 2).$$

On G_0 and G_1 we shall interpret Φ as the identity map. For $x \in G_n$, we call $(\partial_1^1)^n x$ the *base-point* of x and denote it by βx.

Proposition 4.5 *If $(\alpha, j) \neq (0,1)$ then*

$$\partial_j^\alpha \Phi = \varepsilon_1^{n-1} \beta : G_n \to G_{n-1}.$$

Hence, for any $x \in G, \Phi x$ lies in the associated crossed complex γG.

Proof. If $2 \leqslant j \leqslant n$ then

$$\begin{aligned}
\partial_j^\alpha \Phi &= \Phi_1 \Phi_2 \cdots \Phi_{j-2} \partial_j^\alpha \Phi_{j-1} \cdots \Phi_{n-1} && \text{by (4.2)(i)} \\
&= \Phi_1 \Phi_2 \cdots \Phi_{j-2} \eta_{j-1}^1 \cdots \eta_{n-1}^1 \partial_n^1 && \text{by (4.2)(iv)} \\
&= \eta_1^1 \eta_2^1 \cdots \eta_{n-1}^1 \partial_n^1 && \text{by (4.3)(iv)} \\
&= \varepsilon_1^{n-1} (\partial_1^1)^n && \text{by (1.1).}
\end{aligned}$$

If $j = 1$ and $n \geqslant 2$, then $\alpha = 1$ and the equation follows from (4.2)(iv) and (1.1). The case $n = 1$ is trivial. Thus, for $x \in G_n$, we have $\partial_j^\alpha \Phi x = 0_p$ for $(\alpha, j) \neq (0,1)$, where $p = \beta x$. This shows that $\Phi x \in C_n(p)$, where $C = \gamma G$. $\square$

It is clear that if $x \in C_n(p)$, then (4.1) becomes $\Phi_j x = x$. This implies $\Phi x = x$, so we have:

Corollary 4.6 *$\Phi x = x$ if and only if x is in γG. In particular $\Phi^2 y = \Phi y$ for all y in G.* $\square$

Proposition 4.7 *If $n \geqslant 2$, then on G_{n-1},*

$$\Phi \varepsilon_j = \varepsilon_1^n \beta \text{ and } \Phi \Gamma_j = \varepsilon_1^n \beta.$$

Proof.

$$\begin{aligned}
\Phi_1 \Phi_2 \cdots \Phi_{n-1} \varepsilon_j &= \Phi_1 \Phi_2 \cdots \Phi_j \varepsilon_j \Phi_j \Phi_{j+1} \cdots \Phi_{n-2} && \text{by (4.3)(i)} \\
&= \Phi_1 \Phi_2 \cdots \Phi_{j-1} \eta_{j+1}^1 \varepsilon_j \Phi_j \cdots \Phi_{n-2} && \text{by (4.3)(ii)} \\
&= \Phi_1 \Phi_2 \cdots \Phi_{j-1} \varepsilon_j \varepsilon_j \partial_j^1 \Phi_j \cdots \Phi_{n-2} && \text{by (1.1)} \\
&= \eta_1^1 \eta_2^1 \cdots \eta_{j-1}^1 \varepsilon_j \varepsilon_j \eta_j^1 \cdots \eta_{n-2}^1 \partial_{n-1}^1 && \text{by (4.3)(v),(4.2)(iv)} \\
&= \varepsilon_1^n \beta && \text{by (1.1).} \\
\Phi_1 \Phi_2 \cdots \Phi_{n-1} \Gamma_j &= \Phi_1 \Phi_2 \cdots \Phi_j \Gamma_j \Phi_j \Phi_{j+1} \cdots \Phi_{n-2} && \text{by (4.4)(i)} \\
&= \Phi_1 \Phi_2 \cdots \Phi_{j-1} \varepsilon_j \eta_j^1 \Phi_j \cdots \Phi_{n-2} && \text{by (4.4)(ii)} \\
&= \varepsilon_1^n \beta && \text{as above.}
\end{aligned}$$

$\square$

The rules for folding sums of cubes are easy to state (see Proposition 4.9) but their proof involves more complicated rules for the partial folding Φ_j.

4.8

(4.8)(i) $$\left.\begin{array}{ll}\Phi_j(x +_i y) & = \Phi_j x +_i \Phi_j y \\ \Phi_j(-_i x) & = -_i \Phi_j x\end{array}\right\} \qquad \textit{if } i \neq j, j+1.$$

(4.8)(ii) $$\Phi_j(x +_j y) = [\Phi_j y, -\varepsilon_j \partial^1_{j+1} y, \Phi_j x, \varepsilon_j \partial^1_{j+1} y]_{j+1}.$$

(4.8)(iii) $$\Phi_j(x +_{j+1} y) = [-\eta^1_j y, \Phi_j x, \eta^1_j y, \Phi_j y]_{j+1}.$$

Proof. (i) This is routine, using the interchange law (1.4) for the directions i and $j+1$.

(ii) Let the relevant faces of x and y be given by

$$\begin{array}{ccc} & u & \\ a & \boxed{x} & b \\ & v & \end{array} \qquad \begin{array}{ccc} & v & \\ c & \boxed{y} & d \\ & w & \end{array} \quad \begin{array}{l} \rightarrow j{+}1 \\ \downarrow j \end{array}$$

Then

$$\Phi_j(x +_j y) = [-\varepsilon_j w, -\Gamma_j(a +_j c), (x +_j y), \Gamma_j(b +_j d)]_{j+1}.$$

Using the transport law (1.6), this can be written as the composite

$$A = \begin{bmatrix} -\varepsilon_j w & -\varepsilon_j c & -\Gamma_j a & x & \Gamma_j b & \varepsilon_j d \\ -\varepsilon_j w & -\Gamma_j c & -\varepsilon_{j+1} c & y & \varepsilon_{j+1} d & \Gamma_j d \end{bmatrix} \quad \begin{array}{l} \rightarrow j{+}1 \\ \downarrow j \end{array}$$

where $-$ stands for $-_{j+1}$. Consider the composite

$$B = \begin{bmatrix} -\varepsilon_j w & -\varepsilon_j c & \varepsilon_j \nu & \varepsilon_j d & -\varepsilon_j d & -\varepsilon_j v & -\Gamma_j a & x & \Gamma_j b & \varepsilon_j d \\ -\varepsilon_j w & -\Gamma_j c & y & \Gamma_j d & -\varepsilon_j d & -\varepsilon_j v & -\varepsilon_j \eta^1_j a & \varepsilon_j v & \varepsilon_j \eta^1_j b & \varepsilon_j d \end{bmatrix} \quad \begin{array}{l} \rightarrow j{+}1 \\ \downarrow j \end{array}$$

By composing the columns first, we see that B is equal to the right hand side of (4.8)(ii). However, the composites of the rows of B are the same as the composites of the rows of A, since $\varepsilon_j \eta^1_j b = \varepsilon_{j+1} \eta^1_j b$ is an identity of the horizontal composition as well as the vertical one. Hence $A = B$.

(iii) This is routine. $\square$

For $x \in G_n$, the edges of x terminating at the basepoint $\beta x = (\partial^1_1)^n x$ will have special importance and we denote them by

$$u_i x = \partial^1_1 \partial^1_2 \cdots \widehat{\imath} \cdots \partial^1_n x \qquad (1 \leqslant i \leqslant n).$$

Proposition 4.9 *Let $n \geqslant 2$ and $x, y, z \in G_n$ with $\partial^1_i x = \partial^0_i y$. Then, in C_n :*

(4.9)(i) $$\Phi(x +_i y) = \begin{cases} \Phi y + (\Phi x)^{u_1 y} & \textit{if } n = 2 \textit{and } i = 1, \\ (\Phi x)^{u_i y} + \Phi y & \textit{otherwise;} \end{cases}$$

(4.9)(ii) $$\Phi(-_i z) = -(\Phi z)^{-u_i z}.$$

Proof. (i) First consider the case $i = n \geqslant 2$. We have, by 4.8,

$$\begin{aligned} \Phi(x +_n y) &= \Phi_1 \Phi_2 \cdots \Phi_{n-2}[-\eta^1_{n-1} y, \Phi_{n-1} x, \eta^1_{n-1} y, \Phi_{n-1} y]_n \\ &= [-u, \Phi x, u, \Phi y]_n \end{aligned}$$

where

$$
\begin{aligned}
u &= \Phi_1\Phi_2\cdots\Phi_{n-2}\eta^1_{n-1}y \\
&= \eta^1_1\eta^1_2\cdots\eta^1_{n-1}y && \text{by (4.3)(iv)} \\
&= \varepsilon_1^{n-1}u_n y && \text{by (1.1).}
\end{aligned}
$$

Hence $\Phi(x +_n y) = (\Phi x)^{u_n y} + \Phi y$ in this case.

In the remaining cases we have $1 \leqslant i < n$, so we may put

$$
\begin{aligned}
X &= \Phi_{i+1}\Phi_{i+2}\cdots\Phi_{n-1}x, \\
Y &= \Phi_{i+1}\Phi_{i+2}\cdots\Phi_{n-1}y,
\end{aligned}
$$

and then

$$
\begin{aligned}
\Phi(x +_i y) &= \Phi_1\Phi_2\cdots\Phi_i(X +_i Y) && \text{by (4.8)(i)} \\
&= \Phi_1\cdots\Phi_{i-1}[\Phi_i Y, -\varepsilon_i\partial^1_{i+1}Y, \Phi_i X, \varepsilon_i\partial^1_{i+1}Y]_{i+1} && \text{by (4.8)(ii)} \\
&= [\Phi y, -V, \Phi x, V]_{i+1} && \text{by (4.8)(i) ,}
\end{aligned}
$$

where

$$
\begin{aligned}
V &= \Phi_1\cdots\Phi_{i-1}\varepsilon_i\partial^1_{i+1}\Phi_{i+1}\cdots\Phi_{n-1}y \\
&= \eta^1_1\eta^1_2\cdots\eta^1_{i-1}\varepsilon_i\eta^1_{i+1}\cdots\eta^1_{n-1}\partial^1_n y && \text{by (4.2)(iv), (4.3)(v)} \\
&= (\varepsilon_1)^i(\varepsilon_2)^{n-i-1}u_i y && \text{by (1.1).}
\end{aligned}
$$

Hence, by Lemma 3.10, $\Phi(x +_i y) = \Phi y + (\Phi x)^{u_i y}$ in this case. (Note that $i+1 \geqslant 2$, so addition in direction $i+1$ is addition in C_n.) If $n = 2$ and $i = 1$, this is the required formula. Otherwise, we have $n \geqslant 3$, so C_n is commutative and the formula can be rewritten in the required form.

(ii) Put $x = -_i x, y = z$ in (i) and note that, by (4.7), $\Phi((-_i z) +_i z) = \Phi\varepsilon_i\partial^1_i z = \varepsilon_1^n\beta z = 0$ in C_n. □

4.10

$$\Phi\Phi_j = \Phi : G_n \to G_n (1 \leqslant j \leqslant n-1).$$

Proof. By definition, for $x \in G_n$,

$$
\begin{aligned}
\Phi_j x &= [-\varepsilon_j\partial^1_j x, -\Gamma_j\partial^0_{j+1}x, x, \Gamma_j\partial^1_{j+1}x]_{j+1} \\
&= [a, b, x, c]_{j+1}, \text{say.}
\end{aligned}
$$

By Proposition 4.7 and (4.9)(ii), Φa, Φb and Φc are all zero in C_n, so Proposition 4.9 gives $\Phi\Phi_j x = (\Phi x)^u$, where

$$
\begin{aligned}
u = u_{j+1}c &= \partial^1_1\cdots\partial^1_j\partial^1_{j+2}\cdots\partial^1_n\Gamma_j\partial^1_{j+1}x \\
&= \varepsilon_1\partial^1_1\partial^1_2\cdots\partial^1_n x \qquad \text{by (1.1) and (1.2).}
\end{aligned}
$$

Thus $\Phi\Phi_j x = (\Phi x)^{\varepsilon_1\beta x} = \Phi x$. □

Definition 4.11 An element $x \in G_n (n \geqslant 1)$ is *thin* if it can be written as a composite $x = [x_{(r)}]$, where each entry is either of the form $\varepsilon_i y$ or of the form $-_i -_j \cdots -_l \Gamma_m y$.

The collection of all thin elements of G is clearly closed under all the ω-groupoid operations except the face operations. It is useful to think of the thin elements as the most general kind of degenerate cubes. They are important in the topological applications and we establish their main properties in Section 7. For the present we prove only the following characterisation.

Proposition 4.12 *Let $x \in G_n (n \geqslant 1)$. Then x is thin if and only if $\Phi x = 0$.*

Proof. We have shown that $\Phi \varepsilon_j y = 0, \Phi \Gamma_j y = 0$ for all $y \in G_{n-1}$ (see Proposition 4.7). It follows from Proposition 4.9 that $\Phi x = 0$ whenever x is thin. To see the converse, we recall the definition

$$\Phi_j x = [-\varepsilon_j \partial_j^1 x, -\Gamma_j \partial_{j+1}^0 x, x, \Gamma_j \partial_{j+1}^1 x]_{j+1}$$

which can be rewritten as

$$x = [\Gamma_j \partial_{j+1}^0 x, \varepsilon_j \partial_j^1 x, \Phi_j x, -\Gamma_j \partial_{j+1}^1 x]_{j+1}.$$

These two equations show that $\Phi_j x$ is thin if and only if x is thin. Hence Φx is thin and only if x is thin. In particular, if $\Phi x = 0$ (i.e. $\Phi x = \varepsilon_1^n \beta x$) then Φx is thin, so x is also thin. □

5 Skeleton and coskeleton

If one ignores the elements of dimension higher than n in an ω-groupoid one obtains an n-tuple groupoid. The truncation functor $tr^n : \mathcal{G} \to \mathcal{G}_n$ thus defined has (as we shall show) both a left adjoint $sk^n : \mathcal{G}_n \to \mathcal{G}$, the *n-skeleton functor*, and a right adjoint $cosk^n : \mathcal{G}_n \to \mathcal{G}$, the *n-coskeleton functor.* (Here we follow the notation and terminology of Duskin [14].) The coskeleton is easily described in terms of 'shells' as follows.

In any cubical complex K, an *r-shell* is a family $\mathbf{x} = (x_i^\alpha)$ of r-cubes $(i = 1, 2, \cdots, r+1; \alpha = 0, 1)$ satisfying

$$\partial_j^\beta x_i^\alpha = \partial_{i-1}^\alpha x_j^\beta \qquad \text{for} \qquad 1 \leqslant j < i \leqslant r+1 \qquad \text{and} \qquad \alpha, \beta = 0, 1.$$

In particular the faces $\partial_j^\alpha y$ of any $(r+1)$-cube form an r-shell $\boldsymbol{\partial} y$. We denote by $\square K_r$ the set of all r-shells of K (cf. Duskins's "simplicial kernel").

Let $K = (K_n, K_{n-1}, \cdots, K_0)$ be an n-truncated cubical complex. Then $K' = (\square K_n, K_n, K_{n-1}, \cdots, K_0)$ is an $(n+1)$-truncated cubical complex in which, for any $\mathbf{x} \in \square K_n, \partial_i^\alpha \mathbf{x}$ is defined to be x_i^α and, for any $y \in K_n$, $\boldsymbol{\varepsilon}_j y$ is defined to be the n-shell $\mathbf{z}$, where (cf. (1.1), (iii))

$$z_i^\alpha = \begin{cases} \varepsilon_{j-1} \partial_i^\alpha y & (i < j), \\ \varepsilon_j \partial_{i-1}^\alpha y & (i > j), \\ y & (i = j). \end{cases} \tag{5.1}$$

If K has connections, we can also define $\boldsymbol{\Gamma}_j \mathbf{y} = \mathbf{w}$, where (cf. (1.2)(iii))

$$w_i^\alpha = \begin{cases} \Gamma_{j-1} \partial_i^\alpha y \quad (i < j), & w_j^0 = w_{j+1}^0 = y, \\ \\ \Gamma_j \partial_{i-1}^\alpha y \quad (i > j+1); & w_j^1 = w_{j+1}^1 = \eta_j^1 y. \end{cases} \tag{5.2}$$

In this way K' becomes an $(n+1)$-truncated cubical complex with connections. If K has compositions, we can also define compositions in $\square K_n$ as follows. Let $\mathbf{x}, \mathbf{y} \in \square K_n$ with $y^0_j = x^1_j$. Define $\mathbf{x} +_j \mathbf{y} = \mathbf{t}$ and $-_j\mathbf{x} = \mathbf{s}$, where (cf. (1.3))

(5.3)
$$\begin{cases} t^0_j = x^0_j, \\ t^1_j = y^1_j, \end{cases} \qquad t^\alpha_i = \begin{cases} x^\alpha_i +_{j-1} y^\alpha_i & (i < j), \\ x^\alpha_i +_j y^\alpha_i & (i > j), \end{cases}$$
$$\begin{cases} s^0_j = x^1_j, \\ s^1_j = x^0_j, \end{cases} \qquad s^\alpha_i = \begin{cases} -_{j-1}x^\alpha_i & (i < j), \\ -_j x^\alpha_i & (i > j). \end{cases}$$

Then K' becomes an $(n+1)$-truncated cubical complex with connections and compositions. If K is an n-tuple groupoid, then K' is an $(n+1)$-tuple groupoid. The verification of these facts is a tedious but entirely routine computation.

The coskeleton can now be obtained by iteration of this construction.

Proposition 5.4 *If $G = (G_n, G_{n-1}, \ldots, G_0)$ is an n-tuple groupoid, then the ω-groupoid $\overline{G}$ with*

$$\overline{G}_m = \begin{cases} G_m & \text{for } m \leqslant n, \\ \square^{m-n} G_n & \text{for } m > n \end{cases}$$

and operations defined as above, is the n-coskeleton of G. Its elements in dimension $n+2$ and higher are all thin.

Proof. If H is any ω-groupoid and $\theta_k : H_k \to G_k$ are defined for $k = 0, 1, 2, \cdots, n$ so as to form a morphism of n-tuple groupoids from $tr^n H$ to G, then there is a unique extension to a morphism of ω-groupoids $\theta : H \to \overline{G}$ defined inductively by

$$\theta_m y = \mathbf{z}, \quad \text{where} \quad z^\alpha_i = \theta_{m-1}\partial^\alpha_i y \qquad (m > n).$$

This shows that $\overline{G} \cong cosk^n G$.

To prove the last assertion, it is enough to show that, for any ω-groupoid G, elements of $\square^2 G_r$ are always thin. Let $\mathbf{z} \in \square^2 G_r$; then $\mathbf{w} = \Phi\mathbf{z} \in \square^2 G_r$ and all its $(r+1)$-dimensional faces $\partial^\alpha_i \mathbf{w}$ except $\partial^0_1 \mathbf{w}$ are 0_p, where $p = \beta\mathbf{z}$. This implies that *all* the r-dimensional faces of $\mathbf{w}$ are 0_p. Hence $\partial^0_1\mathbf{w}$ is an r-shell all of whose faces are 0_p. By definition, therefore $\partial^0_1 \mathbf{w} = 0_p$. Hence $\mathbf{w}$ itself is an $(r+1)$-shell all of whose faces are 0_p and therefore $\mathbf{w} = 0_p$. By Proposition 4.12, $\mathbf{z}$ is thin. □

We can now apply the folding operations Φ_i and Φ in the ω-groupoid $cosk^n G$, where $G = (G_n G_{n-1}, \cdots, G_0)$. Given an n-shell $\mathbf{y} = (y^\alpha_i) \in \square G_n$, we obtain n-shells $\Phi_i \mathbf{y}$ and $\Phi\mathbf{y} = \Phi_1\Phi_2 \cdots \Phi_{n-1}\mathbf{y}$. By Proposition 4.5, all faces of $\Phi\mathbf{y}$ except $\partial^0_1 \Phi\mathbf{y}$ are 0_p, where $p = \beta\mathbf{y} = (\partial^1_1)^n y^1_1$. If H is a given ω-groupoid, then adjointness gives a canonical morphism $\theta : H \to Cosk^n H = cosk^n(tr^n H)$, with $\theta_{n+1} x = \boldsymbol{\partial} x$ for $x \in H_{n+1}$. Since θ preserves the folding operations we have

(5.5)
$$\Phi\boldsymbol{\partial} x = \boldsymbol{\partial}\Phi x$$

for any element x of dimension at least two in an ω-groupoid. (Note that by 4.2 the faces of $\Phi_j x$ depend only on the faces of x, and this gives a recipe for $\Phi_j \boldsymbol{\partial} x$.) The same is true in any m-tuple groupoid.

Proposition 5.6 *Let G be an ω-groupoid, and let $C = \gamma G$ be its associated crossed complex. Let $\mathbf{x} \in \Box G_{n-1}$ and $\xi \in C_n(p)$, where $p = \beta \mathbf{x}$ and $n \geqslant 2$. Then a necessary and sufficient condition for the existence of $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$ is that $\delta\xi = \delta\Phi\mathbf{x}$. Furthermore, if x exists, it is unique.*

Proof. If $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$, then $\boldsymbol{\partial}\Phi x = \Phi\boldsymbol{\partial} x = \Phi\mathbf{x}$, by (5.5), so $\delta\Phi\mathbf{x} = (\Phi\mathbf{x}^0)_1 = \partial_1^0 \Phi x = \delta\xi$. Suppose, conversely, that we are given $\mathbf{x}$ and ξ with $\delta\xi = \partial\Phi\mathbf{x}$, i.e. $\partial_1^0 \xi = (\Phi\mathbf{x})_1^0$. Then, since all other faces of ξ and $\Phi\mathbf{x}$ are concentrated at p, we have $\boldsymbol{\partial}\xi = \Phi\mathbf{x} = \Phi_1\Phi_2 \cdots \Phi_{n-1}\mathbf{x}$, an equation in $\Box G_{n-1}$. We have to show that there is a unique $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$. By induction, it is enough to show that if $y \in G_n$ and $\boldsymbol{\partial} y = \Phi_i \mathbf{z}$ for some $1 \leqslant i \leqslant n-1$ and $\mathbf{z} \in \Box G_{n-1}$, then there is a unique $z \in G_n$ with $\boldsymbol{\partial} z = \mathbf{z}$ and $\Phi_i z = y$. But this is clear since the equation

$$[-\varepsilon_i \partial_i^1 z, -\Gamma_i \partial_{i+1}^0 z, z, \Gamma_i \partial_{i+1}^1 z]_{i+1} = y$$

becomes

$$[-\varepsilon_i z_i^1, -\Gamma_i z_{i+1}^0, z, \Gamma_i z_{i+1}^1]_{i+1} = y$$

under the stated conditions, and therefore has a unique solution for z in terms of y and $\mathbf{z}$. It is easy to check that this z has boundary $\mathbf{z}$. $\Box$

Corollary 5.7 *A thin element on an ω-groupoid is determined by its faces. Given a shell $\mathbf{x}$, there is a thin element t with $\boldsymbol{\partial} t = \mathbf{x}$ if and only if $\delta\Phi\mathbf{x} = 0$.*

Proof. Put $\xi = 0$ in Proposition 5.6 and use the fact that t is thin if and only if $\Phi t = 0$ (Proposition 4.12). $\Box$

Corollary 5.8 *Let G be an ω-groupoid. Then the associated crossed complex γG generates G as ω-groupoid.*

Proof. Let H be any ω-subgroupoid of G containing $C = \gamma G$. Then $\gamma H = C$ by definition. We show inductively that $H_n = G_n$. This is true for $n = 0, 1$ since $C_0 = G_0, C_1 = G_1$. Suppose $x \in G_n (n \geqslant 2)$. Then $\Phi x \in C_n$ and, by induction hypothesis, $\boldsymbol{\partial} x \in \Box H_{n-1}$. By Proposition 5.6, there is a unique $y \in H_n$ with $\boldsymbol{\partial} y = \boldsymbol{\partial} x$ and $\Phi y = \Phi x$. But x is the unique element of G_n with this property, so $H_n = G_n$. $\Box$

We can now describe the n-skeleton construction. A shell $\mathbf{x}$ will be called a *commuting shell* if 'the sum of its faces is 0', that is, if $\delta\Phi\mathbf{x} = 0$.

Proposition 5.9 *Given an n-tuple groupoid $G = (G_n, G_{n-1}, \cdots, G_0)$, the n-skeleton S of G is the ω-subgroupoid of $\overline{G} = cosk^n G$ generated by G. For $m \leqslant n, S_m = G_m$, while for $m > n$, S_m consists entirely of thin elements, namely, the commuting shells in $\Box S_{m-1}$. For $m \geqslant n+2$, all shells in $\Box S_{m-1}$ are commuting shells.*

Proof. Let S_m be defined by

$$S_m = \begin{cases} G_m & \text{if} \quad m \leqslant n, \\ \{\mathbf{x} \in \Box S_{m-1}; \delta\Phi\mathbf{x} = 0\} & \text{if} \quad m > n. \end{cases}$$

Then $G \subset S \subset cosk^n G$. By Corollary 5.7 applied to the ω-groupoid $\overline{G} = cosk^n G, S_m$ contains only thin elements for $m > n$. Clearly, S is closed under face maps, degeneracy maps and connections (since $\varepsilon_j y$ and $\Gamma_j y$ are always thin). Also, by induction on m, S_m is closed under $+_i, -_i (1 \leqslant i \leqslant m)$; for if $\mathbf{x}, \mathbf{y} \in S_m (m > n)$ and $\mathbf{x} +_i \mathbf{y}$ is defined, then $\mathbf{x} +_i \mathbf{y}$ has faces in S_{m-1} (by induction hypothesis) and $\delta\Phi(\mathbf{x} +_i \mathbf{y}) = 0$ because composites of thin elements in $\overline{G}$ are thin. Thus $\mathbf{x} +_i \mathbf{y} \in S_m$, and similarly $-_i \mathbf{x} \in S_m$. Hence S is an ω-subgroupoid of $\overline{G}$. Also, by Corollary 5.7, any ω-subgroupoid of $\overline{G}$ containing $S_{m-1} (m > n)$ must contain S_m, so S is generated by G.

If H is any ω-groupoid and $\psi : G \to tr^n H$ is a morphism of n-tuple groupoids, then ψ extends uniquely to a morphism of ω-groupoids $\psi : S \to H$ by the inductive rule that, for any commuting shell $\mathbf{x} \in \Box S_{m-1} (m > n), \psi_m(\mathbf{x})$ is the unique thin element t of H_m such that $\partial_i^\alpha t = \psi_{m-1} x_i^\alpha$ for $1 \leqslant i \leqslant m$ and $\alpha = 0, 1$. The element t exists by Corollary 5.7 since the elements $\psi_{m-1} x_i^\alpha$ form a commuting shell in H. This shows that $S = sk^n G$. If $m \geqslant n + 2$, all shells in $\overline{G}_m = \Box^{m-n} G_k$ are thin by Proposition 5.4 and are therefore commuting shells by Corollary 5.7. $\Box$

Given an ω-groupoid G, we define $Sk^n G = sk^n(tr^n G)$ and call this, by abuse of language, the n-skeleton of G. There is a unique morphism $\sigma : Sk^n G \to G$ of ω-groupoids (the adjunction) which is the identity in dimensions $0, 1, 2, \cdots, n$.

Proposition 5.10 *The adjunction $\sigma : Sk^n G \to G$ is an injection and identifies $Sk^n G$ with the ω-groupoid of G generated by G_n.*

Proof. For $m = 0, 1, 2, \cdots, n$, $\sigma_m : G_m \to G_m$ is the identity map. Let $S_m = (Sk^n G)_m$. Then, for $m > n$, S_m is the set of commuting shells in $\Box S_{m-1}$, by Proposition 5.9. Suppose that, for some $m > n$, $\sigma_{m-1} : S_{m-1} \to G_{m-1}$ is an injection. For any $\mathbf{x} \in S_m$, the elements $\sigma_{m-1} x_i^\alpha$ form a commuting shell $\mathbf{y}$ in $\Box G_{m-1}$ and $\sigma_m \mathbf{x}$ is the unique thin element t of G_m with $\boldsymbol{\partial} t = \mathbf{y}$. Thus $x_i^\alpha = \sigma_{m-1}^{-1} y_i^\alpha = \sigma_{m-1}^{-1} \partial_i^\alpha t$ is uniquely determined by t for all (i, α) and therefore σ_m is an injection. This shows, inductively, that σ is an injection. Now G_n generates $tr^n G$ as n-tuple groupoid (even as n-truncated complex) and therefore generates $Sk^n G$ as ω-groupoid, by Proposition 5.9. It follows that G_n generates the image of $Sk^n G$ in G. $\Box$

6 The equivalence of ω-groupoids and crossed complexes

We now show how to construct, from any crossed complex C, and ω-groupoid $G = \lambda C$ with $\gamma G \cong C$. The basis of the construction is Proposition 5.6 which shows (inductively) that the elements of any ω-groupoid G are uniquely determined by γG. Given an ω-groupoid G with associated crossed complex $C = \gamma G$, and given $\mathbf{x} \in \Box G_{n-1}, \xi \in C_n$ with $\delta\xi = \delta\Phi\mathbf{x}$, we write $\langle \mathbf{x}, \xi \rangle$ for the unique element $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$. Our next proposition shows that the compositions in G are also determined by γG.

Proposition 6.1 *If $x = \langle \mathbf{x}, \xi \rangle, y = \langle \mathbf{y}, \eta \rangle$ in G_n, and $x_i^1 = y_i^0$, then*

$$x +_i y = \begin{cases} \langle \mathbf{x} +_1 \mathbf{y}, \eta + \xi^{u_1 \mathbf{y}} \rangle & \text{if } n = 2 \text{ and } i = 1, \\ \langle \mathbf{x} +_i \mathbf{y}, \xi^{u_1 \mathbf{y}} + \eta \rangle & \text{otherwise,} \end{cases}$$

and

$$-_i x = \langle -_i \mathbf{x}, -\xi^{-u_i \mathbf{x}} \rangle.$$

Proof. This follows immediately from Proposition 4.9 and the rule $\boldsymbol{\partial}(x +_i y) = \boldsymbol{\partial} x +_i \boldsymbol{\partial} y$. □

Theorem 6.2 *There is a functor λ from the category $\mathcal{C}$ of crossed complexes to the category $\mathcal{G}$ of ω-groupoids such that $\lambda : \mathcal{C} \to \mathcal{G}$ and $\gamma : \mathcal{G} \to \mathcal{C}$ are inverse equivalences.*

Proof. Let C be any crossed complex. We construct an ω-groupoid $G = \lambda C$ and an isomorphism of crossed complexes $\sigma : C \to \gamma G$ by induction on dimension. We start with $G_0 = C_0, C_1 = C_1$, so that (G_1, G_0) is a groupoid. We write γG_n (in any cubical complex) for the set of n-cubes x with all faces except $\partial_1^0 x$ concentrated at a point. Then $\gamma G_0 = C_0, \gamma G_1 = C_1$, and we take $\sigma_0 : C_0 \to \gamma G_0$ and $\sigma_1 : C_1 \to \gamma C_1$ to be the identity maps. Suppose, inductively, that we have defined G_r and $\sigma_r : C_r \to \gamma G_r$ for $0 \leqslant r < n$ (where $n \geqslant 2$) so that $(G_{n-1}, G_{n-1}, \cdots, G_0)$ is an $(n-1)$-tuple groupoid and $(\sigma_{n-1}, \sigma_{n-2}, \cdots, \sigma_0)$ is an isomorphism of $(n-1)$-truncated crossed complexes. Then $(\square G_{n-1}, G_{n-1}, \cdots, G_0)$ is an n-tuple groupoid and we define

$$G_n = \{(\mathbf{x}, \xi); \mathbf{x} \in \square G_{n-1}, \xi \in C_n, \delta \Phi \mathbf{x} = \sigma_{n-1} \delta \xi\}.$$

For $y \in G_{n-1}$, let $\varepsilon_j y = (\boldsymbol{\varepsilon}_j y, 0)$, where $\boldsymbol{\varepsilon}_j$ is defined by (5.1). Then $\varepsilon_j y \in G_n$, since $\Phi \boldsymbol{\varepsilon}_j y = 0$ by Proposition 4.7. The maps $\varepsilon_j : G_{n-1} \to G_n$, together with the obvious face maps $\partial_i^\alpha : G_n \to G_{n-1}$ defined by $\partial_i^\alpha(\mathbf{x}, \xi) = x_i^\alpha$, give $(G_n, G_{n-1}, \cdots, G_0)$ the structure of an n-truncated cubical complex. Similarly one can define connections $\Gamma_j : G_{n-1} \to G_n$ by $\Gamma_j y = (\boldsymbol{\Gamma}_j y, 0)$, where $\boldsymbol{\Gamma}_j$ is defined by (5.2), and the laws (1.2) are clearly satisfied, since they are satisfied by $\boldsymbol{\Gamma}_j$.

With Proposition 6.1 in mind, we now define operations $+_i, -_i$ as follows. For $(\mathbf{x}, \xi), (\mathbf{y}, \eta) \in G_n$ with $x_i^1 = y_i^0$, let

$$(\mathbf{x}, \xi) +_i (\mathbf{y}, \eta) = \begin{cases} (\mathbf{x} +_1 \mathbf{y}, \eta + \xi^{u_1 \mathbf{y}}) & \text{if } n = 2 \text{ and } i = 1, \\ (x +_i \mathbf{y}, \xi^{u_1 \mathbf{y}} + \eta) & \text{otherwise,} \end{cases}$$

and

$$-_i(\mathbf{x}, \xi) = (-_i \mathbf{x}, -\xi^{u_i \mathbf{x}}).$$

By Proposition 4.9, in the general case,

$$\begin{aligned}
\delta\Phi(\mathbf{x} +_i \mathbf{y}) &= \delta((\Phi\mathbf{x})^{u_i\mathbf{y}} + \Phi\mathbf{y}) \\
&= (\sigma_{n-1}\delta\xi)^{u_i\mathbf{y}} + \sigma_{n-1}\delta\eta \\
&= \sigma_{n-1}\delta(\xi^{u_i\mathbf{y}} + \eta),
\end{aligned}$$

so G_n is closed under $+_i$. The case $n = 2, i = 1$ is similar. Also $\delta\Phi(-_i\mathbf{x}) = \delta(-\Phi\mathbf{x})^{-u_i\mathbf{x}} = \sigma_{n-1}\delta(-\xi^{u_i\mathbf{x}})$, and therefore $-_i\mathbf{x} \in G_n$.

We claim that $(G_n, G_{n-1}, \ldots, G_0)$ is now an n-tuple groupoid. Firstly, it is clear that, for $t \in G_{n-1}$, $\varepsilon_i t$ acts as an identity for $+_i$, and that $-_i$ is an inverse operation for $+_i$. The associative law is verified as for semi-direct products of groups. Secondly, the laws (1.3), (1.5) and (1.6) are true for $\Box G_{n-1}$. It remains, therefore, to prove the interchange law (1.4)(i) (from which (1.4)(ii) follows, using the groupoid laws).

Let $1 \leqslant i < j \leqslant n$ and let $x = (\mathbf{x}, \xi), y = (\mathbf{y}, \eta), z = (\mathbf{z}, \zeta), t = (\mathbf{t}, \tau)$ be elements of G_n such that the composite shell

$$\mathbf{w} = \begin{bmatrix} \mathbf{x} & \mathbf{y} \\ \mathbf{z} & \mathbf{t} \end{bmatrix} \quad \begin{tikzcd}[sep=small] \cdot \arrow[r, "j"] \arrow[d, "i"'] & {} \\ {} & \end{tikzcd}$$

is defined. Let $g = \partial_1^1\partial_2^1 \cdots \widehat{\imath} \cdots \widehat{\jmath} \cdots \partial_n \mathbf{t} \in G_2$ have boundary

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "g"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "a"'] & \cdot \end{tikzcd} \quad \begin{tikzcd}[sep=small] \cdot \arrow[r, "2"] \arrow[d, "1"'] & {} \\ {} & \end{tikzcd}$$

Then

$$(x +_i z) +_j (y +_i t) = (\mathbf{w}, \omega), \qquad (x +_j y) +_i (z +_j t) = (\mathbf{w}, \omega'),$$

say, and we have to show that $\omega = \omega'$ in C_n. If $n = 2$ then $i = 1$ and $j = 2$ and we find that

$$\omega = (\zeta + \xi^b)^a + (\tau + \eta^d), \omega' = (\zeta^a + \tau) + (\xi^c + \eta)^d.$$

To show that these are equal, it is enough to show that $\xi^{b+a} + \tau = \tau + \xi^{c+d}$. But this follows from the crossed module laws since

$$\delta\tau = \sigma_1\delta\tau = \delta\Phi\mathbf{t} = \delta\Phi g = -a - b + c + d$$

and therefore

$$-\tau + \xi^{b+a} + \tau = (\xi^{b+a})^{\delta\tau} = \xi^{c+d}.$$

If $n > 2$, we find that

$$\omega = (\xi^b + \zeta)^a + \eta^d + \tau, \qquad \omega' = (\xi^c + \eta^d) + \zeta^a + \tau,$$

and since addition is now commutative, the equation $\omega = \omega'$ reduces to $\xi^{a+b} = \xi^{c+d}$, that is, $\xi^{\delta\Phi g} = \xi$. But, by induction hypothesis, we have an isomorphism $\sigma_2 : C_2 \to \gamma G_2$ preserving the crossed module structure, and if $\theta \in C_2$ is the element with $\sigma_2(\theta) = \Phi g$, then $\xi^{\delta\Phi g} = \xi^{\delta\theta} = \xi$ by the crossed complex laws. This completes the proof of the interchange law.

We now have an n-tuple groupoid $(G_n, G_{n-1}, \cdots, G_0)$, and we must identify γG_n. For any $\xi \in C_n(p)$, let $\mathbf{d}\xi$ denote the shell $\mathbf{x} \in \Box G_{n-1}$ with $x_1^0 = \sigma_{n-1}\delta\xi$ and all other x_i^α concentrated at p. Define

$$\sigma_n\xi = (\mathbf{d}\xi, \xi).$$

Clearly $\sigma_n\xi \in \gamma G_n$ and every element of γG_n is of this form. The bijection $\sigma_n : C_n \to \gamma G_n$ is compatible with the boundary maps since $\delta\sigma_n\xi = \partial_1^0\sigma_n\xi = \sigma_{n-1}\delta\xi$. It preserves addition because, for $\xi, \eta \in C_n(p)$,

$$\begin{aligned}(\mathbf{d}\xi, \xi) + (\mathbf{d}\eta, \eta) &= (\mathbf{d}\xi +_n \mathbf{d}\eta, \xi^{u_n \mathbf{d}\eta} + \eta) \\ &= (\mathbf{d}(\xi + \eta), \xi + \eta).\end{aligned}$$

Furthermore, if $\xi \in C_n(p)$ and $a \in C_1(p, q) = G_1(p, q)$, then

$$\begin{aligned}(\sigma_n\xi)^a &= -_n\varepsilon_1^{n-1}a +_n \sigma_n\xi +_n \varepsilon_1^{n-1}a \\ &= (-_n\boldsymbol{\varepsilon}_1\varepsilon_1^{n-2}a, 0) +_n (\mathbf{d}\xi, \xi) +_n (\boldsymbol{\varepsilon}_1\varepsilon_1^{n-2}a, 0) \\ &= (\mathbf{y}, \xi^a),\end{aligned}$$

in all cases. Since $(\sigma_n\xi)^a \in \gamma G_n$, it follows that $\mathbf{y} = \mathbf{d}(\xi^a)$, making σ_n an isomorphism of crossed complexes up to dimension n.

This completes the inductive step in our construction, and we therefore obtain an ω-groupoid $G = \lambda C$ and an isomorphism $\sigma : C \to \gamma G$ of crossed complexes. This ω-groupoid has the following universal property: If G' is any ω-groupoid and $\sigma' : C \to \gamma G'$ any morphism of crossed complexes then there is a unique morphism $\theta : G \to G'$ of ω-groupoids making the diagram

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & \gamma G \arrow[r, hook] \arrow[d, "\gamma\theta"] & G = \lambda C \arrow[d, "\theta"] \\ & \gamma G' \arrow[r, hook] & G' \end{tikzcd}$$

commute. We define θ inductively, starting with $\theta_0 = \sigma_0', \theta_1 = \sigma_1'$. For $n \geqslant 2$, each $x' \in G_n'$ is uniquely of the form $\langle \mathbf{x}', \xi' \rangle$ where $\mathbf{x}' \in \Box G_{n-1}', \xi' \in \gamma G_n$ and $\delta\Phi\mathbf{x}' = \delta\xi'$. We define $\theta_n : G_n \to G_n'$ by $(\mathbf{x}, \xi) \mapsto \langle \mathbf{x}', \xi' \rangle$, where $(x')_i^\alpha = \theta_{n-1}x_i^\alpha$ and $\xi' = \sigma_n'\xi$. This definition is forced, and it clearly gives a morphism of ω-groupoids. From this universal property, it follows that λ is a functor from $\mathcal{C}$ to $\mathcal{G}$ and is left adjoint to $\gamma : \mathcal{G} \to \mathcal{C}$. The adjunction $\sigma_c : C \to \gamma\lambda C$ is an isomorphism for all C, so $1_{\mathcal{C}} \simeq \gamma\lambda$. Also, the adjunction $\lambda\gamma G' \to G'$ is obtained by putting $G = \gamma G', \sigma' =$ identity, in which case θ is an isomorphism $\lambda\gamma G' \to G'$, as is clear from its definition. Hence $\lambda\gamma \simeq 1_{\mathcal{G}}$ and we have inverse equivalences λ and γ between $\mathcal{C}$ and $\mathcal{G}$. $\Box$

7 Properties of thin elements

In any cubical complex K we define an *n-box* to be an n-shell with one missing face. More precisely (since we do not postulate the existence of an extra face completing the shell) it is a collection of

n-cubes $\mathbf{x} = (x_i^\alpha)$, where $1 \leqslant i \leqslant n+1, \alpha = 0,1$ and $(\alpha, i) \neq (\gamma, k)$ for some fixed γ, k, satisfying the incidence relations

$$\partial_j^\beta x_i^\alpha = \partial_{i-1}^\alpha x_j^\beta \qquad \text{for} \qquad 1 \leqslant j < i \leqslant n+1, (\alpha, i) \neq (\gamma, k), (\beta, j) \neq (\gamma, k).$$

An $(n+1)$-cube z is a *filler* for the box $\mathbf{x}$ if $\partial_i^\alpha z = x_i^\alpha$ for $(\alpha, i) \neq (\gamma, k)$. Similarly z is a filler for the shell $\mathbf{y}$ if $\partial_i^\alpha z = y_i^\alpha$ for *all* α, i. We recall that K is a (cubical) *Kan complex* if every box has a filler, and we now show that ω-groupoids are not only Kan complexes, but are provided with a set of *canonical* fillers, namely, the thin elements defined in Definition 4.11. First we prove:

Lemma 7.1 *(the Homotopy Addition Lemma). Let G be an ω-groupoid (or an m-tuple groupoid with $m \geqslant n$). Let $\mathbf{x} \in \square G_n$ and define $\Sigma \mathbf{x} \in C_n = (\gamma G)_n$ by*

$$\Sigma \mathbf{x} = \begin{cases} -x_1^1 - x_2^0 + x_1^0 + x_2^1 = -\Phi x_1^1 - \Phi x_2^0 + \Phi_1^0 + \Phi x_2^1 & \text{if } n = 1, \\ -\Phi x_3^1 - (\Phi x_2^0)^{u_2 \mathbf{x}} - \Phi x_1^1 + (\Phi x_3^0)^{u_3 \mathbf{x}} + \Phi x_2^1 + (\Phi x_1^0)^{u_1 \mathbf{x}} & \text{if } n = 2, \\ \sum_{i=1}^{n+1} (-)^i \{\Phi x_i^1 - (\Phi x_i^0)^{u_i \mathbf{x}}\} & \text{if } n \geqslant 3 \end{cases}$$

(where $u_i = \partial_1^1 \partial_2^1 \cdots \widehat{\imath} \cdots \partial_{n+1}^1$). Then $\delta \Phi \mathbf{x} = \Sigma \mathbf{x}$ in all cases. Hence, if t is a thin element of G, then $\Sigma \boldsymbol{\partial} t = 0$.

Proof. The case $n = 1$ is trivial, so we assume $n \geqslant 2$. It is enough to show that $\Sigma \Phi_j \mathbf{x} = \Sigma \mathbf{x}$ for $j = 1, 2, \cdots, n$; for this implies that $\Sigma \Phi \mathbf{x} = \Sigma \mathbf{x}$, and since all faces of $\Phi \mathbf{x}$ except one are 0, we have

$$\begin{aligned} \Sigma \Phi \mathbf{x} &= (\Phi \partial_1^0 \Phi \mathbf{x})^{u_1 \Phi \mathbf{x}} \\ &= \Phi \delta \Phi \mathbf{x} \qquad & \text{(because } u_1 \Phi \mathbf{x} = \varepsilon_1 \beta \mathbf{x}) \\ &= \delta \Phi \mathbf{x} \qquad & \text{(because } \delta \Phi \mathbf{x} \in C_n). \end{aligned}$$

To prove that $\Sigma \Phi_j \mathbf{x} = \Sigma \mathbf{x}$, put $\mathbf{y} = \Phi_j \mathbf{x}$ (for fixed j). By 4.2, we have

$$\begin{aligned} y_i^\alpha &= \begin{cases} \Phi_{j-1} x_i^\alpha & (i < j), \\ \Phi_j x_i^\alpha & (i > j+1); \end{cases} \\ y_{j+1}^\alpha &= y_j^1 = \eta_j^1 x_j^1; \\ y_j^0 &= [-x_j^1, -x_{j+1}^0, x_j^0, x_{j+1}^1]_j. \end{aligned}$$

Hence, by 4.10 and Proposition 4.12,

$$(*) \qquad \begin{aligned} \Phi y_i^\alpha &= \Phi x_i^\alpha (i \neq j, j+1), \\ \Phi y_{j+1}^\alpha &= \Phi y_j^1 = 0, \\ \Phi y_j^0 &= \Phi[-x_j^1, -x_{j+1}^0, x_j^0, x_{j+1}^1]_j. \end{aligned}$$

We write $a_j = [-x_j^1, -x_{j+1}^0, x_j^0, x_{j+1}^1]_j$ and use Proposition 4.9 to compute Φa_j.

First suppose that we are not in the case $n = 2, j = 1$. Then

$$\Phi_j = -(\Phi x_j^1)^{p_j} - (\Phi x_{j+1}^0)^{q_j} + (\Phi x_j^0)^{r_j} + \Phi x_{j+1}^1,$$

where $p_j = u_j a_j, q_j = u_j[x^1_j, a_j]_j, r_j = u_j x^1_{j+1}$. By (1.3), u_j is a morphism of groupoids from $(G_n +_j)$ to $(G_1, +)$ so $p_j = -u_j x^1_j - u_j x^0_{j+1} + u_j x^0_j + u_j x^1_{j+1}$ in G_1, and $q_j = u_j x^1_j + p_j$. The four terms of p_j are the edges of the square $s_j = \partial^1_1 \partial^1_2 \cdots \widehat{j} \widehat{j+1} \cdots \partial^1_n \mathbf{x}$; hence $p_j = \Sigma \boldsymbol{\partial} s_j = \delta \Phi s_j$. Also $u_j x^1_j = u_{j+1}\mathbf{x}$ and $u_j x^1_{j+1} = u_j \mathbf{x}$, so

$$(**) \qquad \Phi y^0_j = \Phi a_j = -(\Phi x^1_j)^{\delta \Phi s_j} - (\Phi x^0_{j+1})^{u_{j+1}\mathbf{x} + \delta \Phi s_j} + (\Phi x^0_j)^{u_j \mathbf{x}} + \Phi x^1_{j+1}.$$

If $n \geqslant 3$ then $\delta \Phi s_j$ acts trivially on C_n, by Proposition 3.9 and 3.6, and addition is commutative. Hence by (*),

$$\begin{aligned} \Sigma \mathbf{y} &= \sum_{i=1}^{n} (-)^i \{\Phi y^1_i - (\Phi y^0_i)^{u_i \mathbf{y}}\} \\ &= \sum_{i \neq j, j+1} (-)^i \{\Phi x^1_i - (\Phi x^0_i)^{u_i \Phi_j \mathbf{x}}\} + (-)^{j+1} (\Phi y^0_j)^{u_j \Phi_j \mathbf{x}}. \end{aligned}$$

But $u_i \Phi_j \mathbf{x} = u_i \mathbf{x}$ if $i \neq j, i \neq j+1$; and $u_j \Phi_j \mathbf{x} = 0$; so substituting from (**) we find $\Sigma \mathbf{y} = \Sigma \mathbf{x}$.

If $n = 2$ and $j = 2$ then $s_2 = \partial^1_1 \mathbf{x} = x^1_1$, and $\delta \Phi s_2 = \delta \Phi x^1_1$ acts on C_2 by $a^{\delta \Phi s_2} = -\Phi x^1_1 + a + \Phi x^1_1$. Hence (**) becomes

$$\Phi y^0_2 = -\Phi x^1_1 - \Phi x^1_2 - (\Phi x^0_3)^{u_3 x} + \Phi x^1_1 + (\Phi x^0_2)^{u_2 \mathbf{x}} + \Phi x^1_3$$

which, together with (*), gives

$$\begin{aligned} \Sigma \mathbf{y} &= -\Phi y^1_3 - (\Phi y^0_2)^{u_2 \Phi_2 \mathbf{x}} - \Phi y^1_1 + (\Phi y^0_3)^{u_3 \Phi_2 \mathbf{x}} + \Phi y^1_2 + (\Phi y^0_1)^{u_1 \Phi_2 \mathbf{x}} \\ &= 0 - \Phi y^0_2 - \Phi x^1_1 + 0 + 0 + (\Phi x^0_1)^{u_1 \mathbf{x}} \\ &= \Sigma \mathbf{x}. \end{aligned}$$

Finally, in the case $n = 2, j = 1$, we have

$$\begin{aligned} \Phi y^0_1 &= \Phi[-x^1_1, -x^0_2, x^0_1, x^1_2]_1 \\ &= \Phi x^1_2 + (\Phi x^0_1)^{r_1} - (\Phi x^0_2)^{q_1} - (\Phi x^1_1)^{p_1} \end{aligned}$$

by Proposition 4.9, where p_1, q_1, r_1 are as defined above. As in the previous cases, this gives

$$\Phi y^0_1 = \Phi x^1_2 + (\Phi x^0_1)^{u_1 \mathbf{x}} - \Phi x^1_3 - (\Phi x^0_2)^{u_2 \mathbf{x}} - \Phi x^1_1 + \Phi x^1_3$$

and hence

$$\Sigma \mathbf{y} = -\Phi x^1_3 + (\Phi x^0_3)^{u_3 \mathbf{x}} + \Phi x^1_2 + (\Phi x^0_1)^{u_1 \mathbf{x}} - \Phi x^1_3 - \Phi (x^0_2)^{u_2 \mathbf{x}} - \Phi x^1_1 + \Phi x^1_3.$$

Writing $b = (\Phi x^0_3)^{u_3 \mathbf{x}} + \Phi x^1_2 + (\Phi x^0_1)^{u_1 \mathbf{x}} - \Phi x^1_3$ and $c = -(\Phi x^0_2)^{u_2 \mathbf{x}} - \Phi x^1_1$, it can be verified that $\delta b = -\delta c$, and hence, by the crossed module laws, $b + c = c + b^{\delta c} = c + b^{-\delta b} = c + b$. It follows easily that $\Sigma \mathbf{y} = \Sigma \mathbf{x}$, as required.

If t is thin, then $\Sigma\boldsymbol{\partial} t = \delta\Phi\boldsymbol{\partial} t = \partial_1^0\Phi t = 0$ by (5.5) and Proposition 4.12. $\Box$

Note. The element $\Sigma\mathbf{x}$ in the case $n = 2$ is in the centre of $C_2(\beta\mathbf{x})$, because conjugation by $\Sigma\mathbf{x} = \delta\Phi x$ is the same as action by $\delta\delta\Phi\mathbf{x} = 0$. Hence $\Sigma\mathbf{x}$ can be rewritten, for example, by permuting its terms cyclically.

Proposition 7.2 *Let G be an ω-groupoid. Then each box in G has a unique thin filler.*

Proof. Let $\mathbf{y}$ be an n-box with missing $(\gamma, k)-$ face. The result is trivial if $n = 0$, so we assume $n \geqslant 1$. By Corollary 5.7, it is enough to prove that there is a unique n-cube y_k^γ which closes the box $\mathbf{y}$ to form an n-shell $\overline{\mathbf{y}}$ with $\delta\Phi\overline{\mathbf{y}} = \Sigma\overline{\mathbf{y}} = 0$.

If $n \geqslant 2$, the edges of the given box $\mathbf{y}$ form the complete 1-skeleton of an $(n+1)$-cube; in particular, $\mathbf{y}$ determines the $n+1$ edges $w_i = u_i\mathbf{y}$ terminating at $\beta\mathbf{y}$. We write $F(s_i^\alpha)$ for the word in the indeterminates $s_i^\alpha (i = 1, 2, \cdots, n+1; \alpha = 0, 1)$ obtained from the formula for $\Sigma\mathbf{x}$ in Lemma 7.1 by substituting s_i^α for Φx_i^α and the given edges $w_i = u_i\mathbf{y}$ for $u_i\mathbf{x}$. If $n = 1$, then

$$F(s_i^\alpha) = -s_1^1 - s_2^0 - s_1^0 + s_2^1$$

does not involve the w_i.

If we put $\mathbf{z}_i^\alpha = \boldsymbol{\partial} y_i^\alpha$ for $(\alpha, i) \neq (\gamma, k)$, then the $\mathbf{z}_i^\alpha$ form a box of $(n-1)$-shells, and there is a unique $(n-1)$-shell $\mathbf{z}_k^\gamma$ which closes this box to form an n-shell $\overline{\mathbf{z}} \in \Box^2 G_{n-1}$. Since δ preserves addition and the action of the edges w_i, we find

$$(*) \qquad F(\delta\Phi\mathbf{z}_i^\alpha) = \delta F(\Phi\mathbf{z}_i^\alpha) = \delta\Sigma\overline{\mathbf{z}} = \delta^2\Phi\overline{\mathbf{z}} = 0.$$

Next, put $\zeta_i^\alpha = \Phi y_i^\alpha$ for $(\alpha, i) \neq (\gamma, k)$ and let $\zeta_k^\gamma \in C_n$ be the unique element determined by the equation $F(\zeta_i^\alpha) = 0$. Then

$$\delta\zeta_i^\alpha = \delta\Phi y_i^\alpha = \delta\Phi\mathbf{z}_i^\alpha \text{ for } (\alpha, i) \neq (\gamma, k),$$

while

$$F(\delta\zeta_i^\alpha) = 0.$$

From these equations and (*) we deduce that $\delta\zeta_k^\gamma = \delta\Phi\mathbf{z}_k^\gamma$ also. Hence, by Proposition 5.6, there is a unique $y_k^\gamma \in G_n$ such that $\boldsymbol{\partial} y_k^\gamma = \mathbf{z}_k^\gamma$ and $\Phi y_k^\gamma = \zeta_k^\gamma$; this y_k^γ completes the box $\mathbf{y}$ to form a shell $\overline{\mathbf{y}}$ with $\Sigma\overline{\mathbf{y}} = F(\zeta_i^\alpha) = 0$, as required. $\Box$

Proposition 7.3 *Let t be a thin element in an ω-groupoid. If all faces except one of t are thin, then the last face is also thin.*

Proof. Let the faces of t be $t_i^\alpha (i = 1, 2, \cdots, n; \alpha = 0, 1)$. By Proposition 4.12, $\Phi t_i^\alpha = 0$ for $(\alpha, i) \neq (\gamma, k)$ say, so $\Sigma\boldsymbol{\partial} t = \pm(\Phi t_k^\gamma)^w$ for some edge w of t. But t is thin so, by the Homotopy Addition Lemma, $\Sigma\boldsymbol{\partial} t = 0$. Hence $\Phi t_k^\gamma = 0$ and t_k^γ is thin. $\Box$

Remark 1. The properties of Propositions 7.2 and 7.3 of thin elements, together with the fact that degenerate cubes are thin, can be taken as axioms for 'cubical T-complexes' or 'cubical complexes

with thin elements'. (The definition was first given by Dakin [12] in the simplicial case.) Precisely, a (*cubical*) *T-complex* is a cubical complex with a distinguished set of elements called 'thin', satisfying:
(i) all degenerate cubes are thin;
(ii) every box has a unique thin filler;
(iii) if a thin cube has all faces except one thin then the last face is also thin.

We have shown that every ω-groupoid is a T-complex, and it is a remarkable fact (see [9]) that the converse is also true: all the ω-groupoid structure can be recovered from the set of thin elements using these three assumptions. Thus the category of cubical T-complexes is equivalent (in fact isomorphic) to the category of ω-groupoids; it is therefore, by Theorem 6.2, equivalent to the category of crossed complexes. Ashley has shown [1] that the category of simplicial T-complexes is also equivalent to the category of crossed complexes. He has also shown that this result generalises the theorem of Dold and Kan [13, 20] which gives an equivalence between the category of simplicial groups and the category of chain complexes; the T-complex structure on a simplicial abelian group is obtained by defining the thin elements to be sums of degenerate elements.

Remark 2. If G is any ω-groupoid, we may define the fundamental groupoid $\pi_1 G$ and the homotopy groups $\pi_n(G,p)(p \in G_0, n \geqslant 2)$ as follows. For $a, b \in G_1(p,q)$, define $a \sim b$ if there exists $c \in G_2$ such that $\partial_1^0 c = a,\ \partial_1^1 c = b,\ \partial_2^0 c = \varepsilon_1 p,\ \partial_2^1 c = \varepsilon_1 q$. Then $\sim$ is a congruence relation on G_1 and we define $\pi_1 G = G_1/\sim$. For $n \geqslant 2$ and $p \in G_0$, let $Z_n(G,p) = \{x \in G_n; \partial_1^\alpha x = \varepsilon_1^{n-1} p \text{ for all } (\alpha, i)\}$. Then the $+_i (i = 1, 2, \cdots, n)$ induce on $Z_n(G,p)$ the same Abelian group structure. Two elements x, y of $Z_n(G,p)$ are *homotopic*, $x \sim y$, if there exists $h \in G_{n+1}$ such that $\partial_{n+1}^0 h = x, \partial_{n+1}^1 h = y$ and $\partial_i^\alpha h = \varepsilon_1^n p$ for $i \neq n+1$. This is a congruence relation on $Z_n(G,p)$ and we define $\pi_n(G,p)$ to be the quotient group $Z_n(G,p)/\sim$.

Now G is a Kan complex, by Proposition 7.2, so there is a standard procedure for defining $\pi_1 G$ and $\pi_n(G,p)$, without using the compositions $+_i$. As sets they coincide with the definitions above, but their groupoid and group structures are defined by a procedure using only the properties of Kan fillers. It is not hard to see that the special properties of thin fillers in G ensure that the groupoid and group structures obtained in this way coincide with those induced by the compositions $+_i$.

It is also clear that if $C = \gamma G$ then the groupoid $\pi_1 G$ and the groups $\pi_n(G,p)$ coincide with the *fundamental groupoid* $\pi_1 C$ and the *homology groups* $H_n(C,p)$, which are defined as follows. For any crossed complex C, $\pi_1 C$ is the quotient of the groupoid C_1 by the normal, totally disconnected subgroupoid $\delta(C_2)$; and $H_n(C,p)$, for $n \geqslant 2$, is the group $Z_n(C,p)/B_n(C,p)$, where

$$\begin{aligned} Z_n(C,p) &= Ker(\delta : C_n(p) \to C_{n-1}(p)), \\ B_n(C,p) &= Im(\delta : C_{n+1}(p) \to C_n(p)). \end{aligned}$$

Remark 3. It is well known that the crossed complex πI^n has one generator for each cell of I^n, with defining relations given by the Homotopy Addition Lemma. The corresponding statement for the ω-groupoid ϱI^n is that it is the free ω-groupoid on a single generator in dimension n; this fact is proved in [8].

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References

[1] N. Ashley, T-complexes and crossed complexes, Ph.D. thesis, University of Wales (1978).

[2] A. Bastiani and Ch. Ehresmann, Categories of sketched structures, Cahiers Topologie Géom. Différentielle 13 (1972) 105-214.

[3] A. L. Blakers, Some relations between homology and homotopy groups, Ann. of Math. 49 (1948) 428-461.

[4] R. Brown, Groupoids and Van Kampen's theorem, Proc. London Math. Soc. (3) 17 (1967) 385-401.

[5] R. Brown and P. J. Higgins, On the connection between the second relative homotopy groups of some related spaces, Proc. London Math. Soc. (3) (1978) 193-212.

[6] R. Brown and P. J. Higgins, Sur les complexes croisés, ω-groupoïds et T-complexes, C.R. Acad. Sci. Paris, Série A 285 (1977) 997-999.

[7] R. Brown and P. J. Higgins, Sur les complexes croisés d'homotopie associés à quelques escapes filtrés, C.R. Acad. Sci. Paris, Série A, 286 (1978) 91-93.

[8] R. Brown and P. J. Higgins, Colimit theorems for relative homotopy groups, J. Pure Appl. Algebra 22 (1981) 11-41.

[9] R. Brown and P. J. Higgins, (a) The equivalence of ω-groupoids and T-complexes (b) The equivalence of ∞-groupoids and crossed complexes, Cahiers Top. Géom. Diff., 22 (1981) 349-386.

[10] R. Brown and C. B. Spencer, Double groupoids and crossed modules, Cahiers Topologie Géom. Différentielle 17 (1976) 343-362.

[11] R. B. Coates, Semantics of generalised algebraic structures, Ph.D. thesis, University of London (1974).

[12] M. K. Dakin, Kan complexes and multiple groupoid structures, Ph.D. thesis, University of Wales (1977).

[13] A. Dold, Homology of symmetric products and other functors of complexes, Ann. of Math. 68 (1958) 54-80.

[14] J. Duskin, Simplicial methods and the interpretation of triple cohomology, Mem. Amer. Math. Soc. No. 163 (1975).

[15] M. Évrard, Homotopie des complexes simpliciaux et cubiques, Preprint.

[16] P. Freyd, Aspects of topoi, Bull. Austral. Math. Soc. 7 (1972) 1-76.

[17] J. Howie, Pullback functors and crossed complexes, Cahiers Topologie Géom. Differentielle, 20 (1979) 281-295.

[18] J. Huebschmann, Verschränkte n-fache Erweiterungen von Gruppen und Cohomologie, Diss. Eidg. Techn. Hochschule, Zurich (1977).

[19] J. Huebschmann, Crossed n-fold extensions of groups and cohomology, Comment. Math. Helvetii 55 (1980) 302-314.

[20] D. M. Kan, Functors involving c.s.s. complexes, Trans. Amer. Math. Soc. 87 (1958) 330-346.

[21] J. H. C. Whitehead, Combinatorial homotopy II, Bull. Amer. Math. Soc. 55 (1949) 453-96.