Some intuitions of Higher Dimensional Algebra, and potential applications
Askloster

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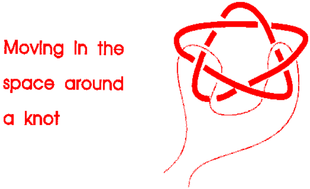
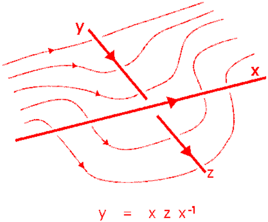
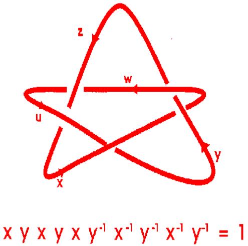
Consilience, John Robinson
www.popmath.org.uk
www.bangor.ac.uk/r.brown/hdaweb2.htm

This will be largely about work with Philip Higgins 1974-2001. ‘Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids’, R. Brown, P.J.Higgins and R. Sivera (to appear 2010) (downloadable as pdf: xx+496) Other large input in this area from C.B. Spencer (1971-73), Chris Wensley (1993-now) and research students Razak Salleh, Keith Dakin, Nick Ashley, David Jones, J.-L Loday, Graham Ellis, Ghaffar Mosa, Fahd Al-Agl, . . .

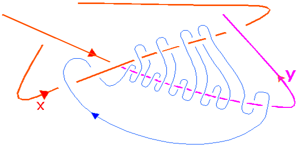
F.W. Lawvere: The notion of space is associated with representing motion.
How can algebra structure space?







Relation at a crossing

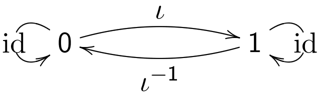


Local and global issue.
Use rewriting of relations.
Classify the ways of pulling the loop off the knot!

Now to groupoids in algebraic topology! Consider forming a pushout of spaces:

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \\ [0,1] & & \end{array} \qquad \begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ [0,1] & \longrightarrow & S^1 \end{array}$$

Let $\mathcal{I}$ be the indiscrete (transition) groupoid on $\{0,1\}$.



$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \\ \mathcal{I} & & \end{array} \qquad \begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \mathcal{I} & \longrightarrow & \mathbb{Z} \end{array}$$

pushout of groupoids

Advantages of $\pi_1(X, C)$ where C is a set of base points:

- C can be chosen according to the geometry of the situation;
- X may be the union of two open sets whose intersection has 150 path components;
- colimits (pushouts) rather than exact sequences;
- a nonabelian invariant, $\pi_1(X, x)$, is computed precisely from the larger structure, even though the given information is in two dimensions, 0 and 1;
- groupoids have structure in the two dimensions, 0 and 1,
- and they model weak homotopy 1-types;
- there are also appropriate accounts of:
  covering morphisms of groupoids
  and of orbit groupoids.

Some work on groupoids:
Brandt (1926): composition of quaternary quadratic forms, developing work of Gauss on the binary case
P.A. Smith (1951): Annals of Math.
Mackey: ergodic theory,
C. Ehresmann: bundles, foliations, differentiable groupoids, . . .,
Pradines: Lie groupoids and Lie algebroids, holonomy and monodromy
Ramsay, Connes, . . .: Non commutative geometry
Grothendieck: groupoids internal to a category, Teichmüller groupoid, . . .
Higgins (1964): applications to groups Grushko's theorem and graphs of groups (1976).

What is Higher Dimensional Algebra?
The idea is that we may need to get away from 'linear' thinking in order to express intuitions clearly.
Thus the equation

$$2 \times (5 + 3) = 2 \times 5 + 2 \times 3$$

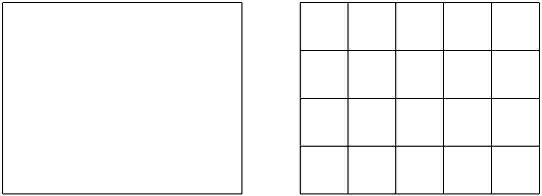
is more clearly shown by the figure

$$\begin{array}{cc} | \, | \, | \, | \, | & | \, | \, | \\ | \, | \, | \, | \, | & | \, | \, | \end{array}$$

But we seem to need a linear formula to express the general law

$$a \times (b + c) = a \times b + a \times c.$$

Note also the figures

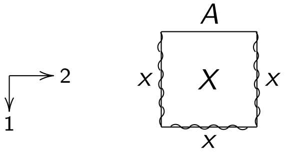


From left to right gives subdivision.
What we need for local-to-global problems is:
Algebraic inverses to subdivision.
i.e., we know how to cut things up, but how to control algebraically putting them together again?

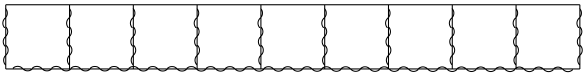
Higher homotopy theory?

groups ⊆ groupoids⊆ higher groupoids ?

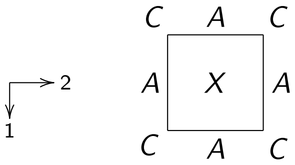
Consider second relative homotopy groups $\pi_2(X, A, x)$:



Definition involves choices, and is unsymmetrical w.r.t. directions.
Large compositions are 1-dimensional:

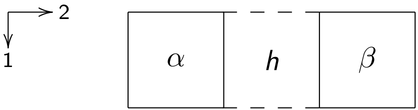


Brown-Higgins 1974 $\rho_2(X, A, C)$: homotopy classes rel vertices of maps $[0,1]^2 \to X$ with edges to A and vertices to C

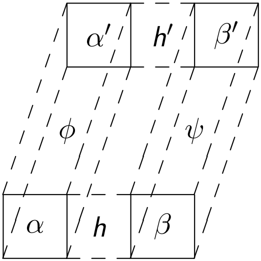


$$\rho_2(X, A, C) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \pi_1(A, C) \;\substack{\longrightarrow\\\longrightarrow}\; C$$

Horizontal composition in $\rho_2(X, A, C)$, where dashed lines show constant paths.

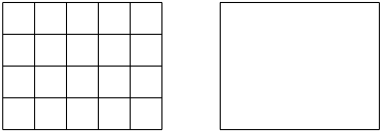


To show $+_2$ well defined, let $\phi : \alpha \equiv \alpha'$ and $\psi : \beta \equiv \beta'$, and let $\alpha' +_2 h' +_2 \beta'$ be defined. We get a picture



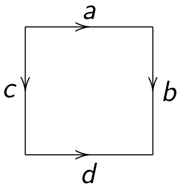
Thus $\rho(X, A, C)$ has in dimension 2 compositions in directions 1,2 satisfying the interchange law and is a double groupoid, containing as a substructure $\pi_2(X, A, x), x \in C$ and $\pi_1(A, C)$.

This structure of double groupoid enables multiple compositions.



Remarkable fact: given such a double composition in $\rho_2(X, A, C)$ then representatives can be chosen which actually fit together!
Aesthetic implies power!!

In dimension 1, we still need the 2-dimensional notion of commutative square:



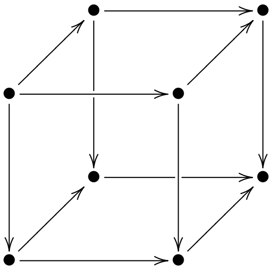
$ab = cd \quad a = cdb^{-1}$

Easy result: any composition of commutative squares is commutative.
In ordinary equations:

$$ab = cd, ef = bg \text{ implies } aef = abg = cdg.$$

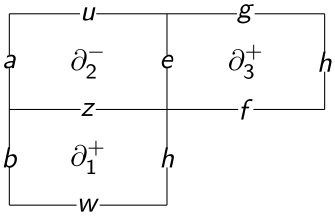
The commutative squares in a category form a double category!
Compare Stokes' theorem! Local Stokes implies global Stokes.

What is a commutative cube?

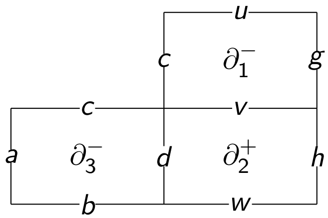


We want the faces to commute!

The problem is a cube has 6 faces divided into odd and even ones, which fit together as



even faces



odd faces

So the possible compositions do not make sense,
and the edges do not agree.
Need canonical ways of filling in the corners.
In 2-dimensional algebra, you need to be able to turn left or right.

To resolve this, we need some special squares called thin:
First the easy ones:

$$\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \qquad \begin{pmatrix} & 1 & \\ a & & a \\ & 1 & \end{pmatrix} \qquad \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix}$$

$$\square \qquad \overline{\underline{\quad}} \text{ or } \varepsilon_2 a \qquad |\,| \text{ or } \varepsilon_1 b$$

laws $\quad \begin{bmatrix} a & \overline{\underline{\quad}} \end{bmatrix} = a \quad \begin{bmatrix} b \\ |\,| \end{bmatrix} = b$

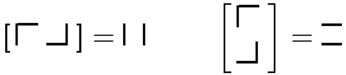
Then we need some new ones:

$$\begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \qquad \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}$$

These are the connections

$$\lrcorner \qquad \ulcorner$$

What are the laws on connections?



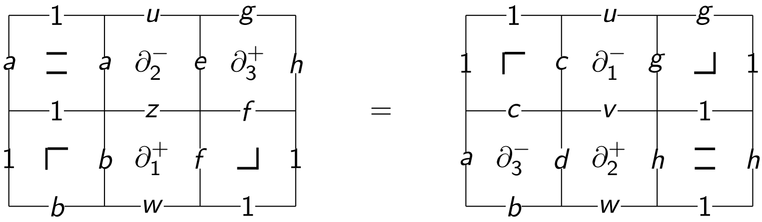
(cancellation)

(transport)

These are equations on turning left or right, and so are a part of 2-dimensional algebra.

The term transport law and the term connections came from laws on path connections in differential geometry.

Now you can use the thin elements to fill in the corners, and in fact you also need to expand out.



Thus 2-dimensional algebra needs some new basic constructions.
It is a good exercise to prove that the commutative cubes as defined here form a triple category!
You can see there might be problems in doing this in dimension n . Fortunately, these have been solved with the general notion of thin element.

Now add to the cubical singular complex of a space connections defined using the monoid structures

$$\max, \min : [0,1] \times [0,1] \to [0,1]$$

and get this structure on $\rho(X, A, C)$.
This enough to prove analogously to the 1-dimensional case a 2-dimensional van Kampen theorem for the functor ρ and so for the crossed module

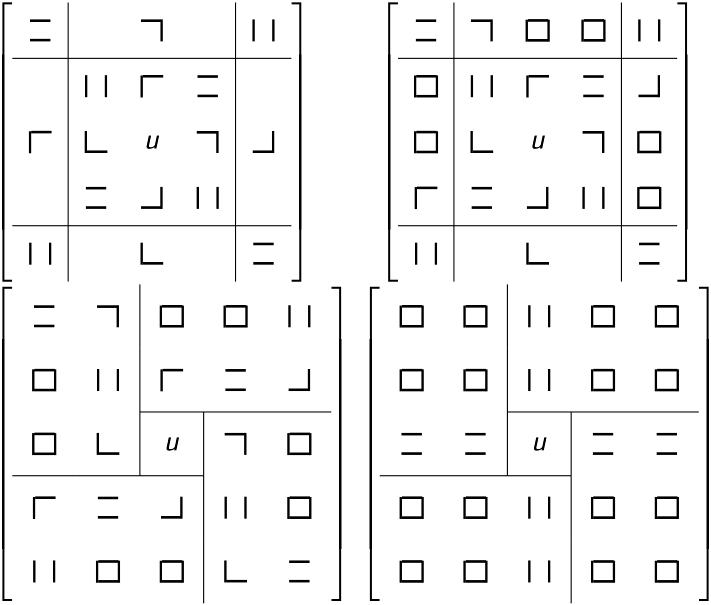
$$\pi_2(X, A, C) \to \pi_1(A, C) \rightrightarrows C$$

This 2d-vKT is difficult to prove directly. Published in 1978 in the teeth of opposition! A 1993 book on 2-dim homotopy theory and combinatorial group theory relates some consequences, but not the theorem!
Note that the crossed module above has structure in dimensions 0,1,2,
and models weak homotopy 2-types.

Rotations: clockwise and counterclockwise

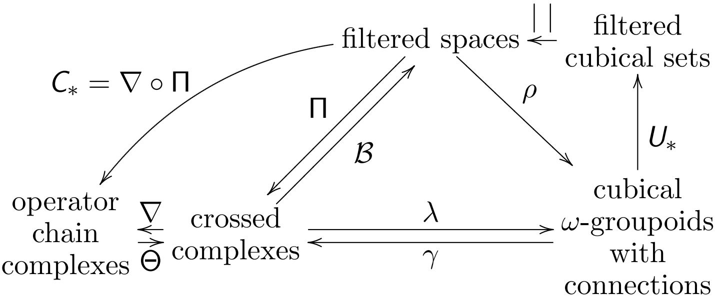
$$\sigma(u) = \begin{bmatrix} \vert\,\vert & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & \vert\,\vert \end{bmatrix} \qquad \tau(u) = \begin{bmatrix} = & \urcorner & \vert\,\vert \\ \ulcorner & u & \lrcorner \\ \vert\,\vert & \llcorner & = \end{bmatrix}$$

Now we prove $\tau\sigma(u) = u$ using 2-dimensional rewriting:


u
u
u
u

Further work shows $\sigma^2 u = -_1 -_2 u$, so $\sigma^4 u = u$. This algebra applied to $\rho(X, A, C)$ shows the existence of specific homotopies.
All this justifies the claim
higher dimensional nonabelian methods for local-to-global problems
and for the notion of
higher homotopy groupoids
46 years after Čech introduced higher homotopy groups.

Higher dimensions? Work of RB and Philip Higgins 1979-1991 gave the following diagram of an intricate situation:



No use of singular homology theory or simplicial approximation!
Π and ρ are homotopically defined.
Best since Poincaré???

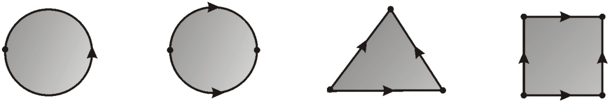
Here

- γ and λ are adjoint equivalences of monoidal closed categories;
- $\gamma\rho \simeq \Pi$, and both preserve certain colimits (HHvKT), and certain tensor products;
- $\Pi \circ \mathcal{B} \simeq 1$;
- ∇ is left adjoint to Θ, and preserves $\otimes$;
- if $B = U \circ \mathcal{B}$: (crossed complexes) $\to$ (spaces) then there is a homotopy classification theorem

$$[X, BC] \cong [\Pi X_*, C]$$

  for CW X and crossed complex C.

Cubical versus simplicial and globular?



Recent preprint of G. Maltsiniotis:
Cubical sets with connections form a strict test category, in sense of Grothendieck: i.e. as good as simplicial from a homotopy category viewpoint.
Notion of multiple composition is clear cubically but unclear simplicially or globularly.

Potential applications?

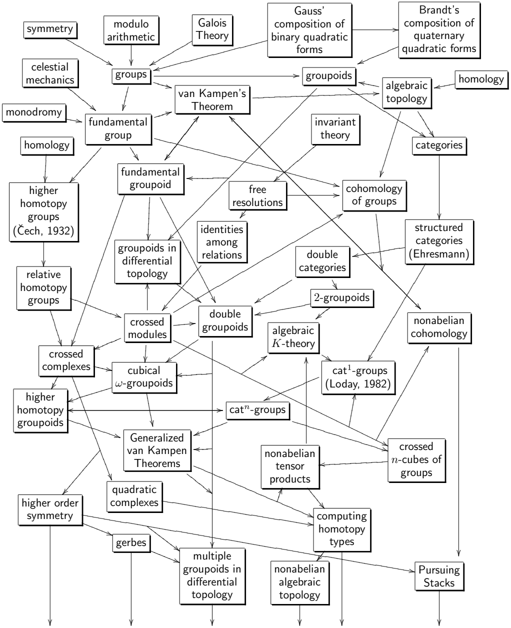
- n-fold groupoids in groups model weak pointed homotopy $(n+1)$-types, and there is a HHvKT for these, using n-cubes of spaces.
- That situation generates lots of algebra, including a nonabelian tensor product of groups which act on each other (bibliography of 100 items, including Lie algebras, ...).
- Use of these ideas, particularly colimit ideas, in general applications of homological/homotopical algebra?
- Higher dimensional algebra is concerned with partial algebraic structures the domains of whose operations are defined by geometric conditions.
- Many object Lie algebras? (I have a preprint on this!) Poincaré-Birkhoff-Witt for these?
- Free crossed resolutions for algebras?

Multiple groupoids are 'more noncommutative' than groupoids. How can they interact with noncommutative geometry and with physics?

Curious point: a groupoid has many objects. Its convolution algebra has only one.

Link with analysis and geometry may need algebras with many objects. Higher dimensional versions of these have been investigated by Ghaffar Mosa in his 1987 Bangor thesis (now scanned to internet).

Some Context for Higher Dimensional Group Theory



Ronnie Brown, April, 2004.