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Homotopies and Automorphisms of Crossed Modules of Groupoids

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Abstract. We give a detailed description of the structure of the actor 2-crossed module related to the automorphisms of a crossed module of groupoids. This generalises work of Brown and Gilbert for the case of crossed modules of groups, and part of this is needed for work on 2-dimensional holonomy to be developed elsewhere.

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Introduction

This is the first of several planned papers whose aim is to develop contexts for 2-dimensional versions of holonomy [11].

The notion of holonomy that we will use is that in Aof and Brown [1], which exposes ideas of Pradines justifying [24, Théorème 1]. This starts with a *locally Lie groupoid* (G, W) where G is a groupoid and W is a topological space such that as a set $Ob(G) \subseteq W \subseteq G$ and certain other conditions hold so that the groupoid operations on G are 'as smooth as possible' on W. The locally Lie groupoid is *extendible* if there is a topology on G making G a Lie groupoid in which W is an open subspace. In general (G, W) is not extendible but under extra conditions on the existence of certain local continuous coadmissible sections, there is a Lie groupoid $Hol(G, W)$ again containing W as an open subspace and with a morphism $Hol(G, W) \to G$ which is the identity on W. Then $Hol(G, W)$ is the 'smallest' overgroupoid of G with this property.

Here a *coadmissible section* is a section $s : X \to G$ of the target map β such that αs is a bijection on X. We are investigating 2-dimensional versions of these ideas, and so we need 2-dimensional versions of groupoids and of coadmissible sections.

There are various useful 2-dimensional notions of groupoid: double groupoid, 2-groupoid, crossed module of groupoids. In this paper we will consider homotopies in the context of crossed modules of groupoids, which are equivalent to 2-groupoids, and to edge symmetric double groupoids with the extra structure of connection. In Section 5 we briefly summarise the application to 2-groupoids. It is not clear at this stage whether a direct proof in that context would or would not be simpler than that given here. In any case, theory has been developed and applied for crossed modules of groupoids and crossed complexes which has not been developed for the corresponding 2-groupoids and globular ∞-groupoids, particularly the theory and applications of free objects (see, for example, [4]).

The aim is to replace the notion of coadmissible section by *coadmissible homotopy*. Our main results develop this relation between coadmissible homotopies and automorphisms of crossed modules of groupoids, and start by showing that a coadmissible homotopy is of the form $f \simeq 1$ where f is an automorphism.

The study of automorphisms of a crossed module $\mathcal{M} = (\mu : M \to P)$ of a group P was initiated by Whitehead in [25], where he showed the relation with derivations $s : P \to M$. These derivations occurred in his work on homotopies of his ‘homotopy systems’ in [26]. Work was continued by Lue [20] and Norrie [23], leading to the notion of the *actor crossed square*

$$\begin{array}{ccc} M & \longrightarrow & Der^*(P, M) \\ {\scriptstyle\mu}\downarrow & & \downarrow \\ P & \longrightarrow & \mathsf{Aut}(\mathcal{M}) \end{array} \qquad (1)$$

which should be thought of as the direct generalisation of the *actor crossed module of a group* $M \to \mathsf{Aut}(M)$. See also Breen [2] for further uses of crossed squares. Related to this crossed square is a 2-*crossed module*

$$M \to P \ltimes Der^*(P, M) \to \mathsf{Aut}(\mathcal{M}). \qquad (2)$$

We shall later identify the group $P \ltimes Der^*(P, M)$ with the group of invertible *free derivations* $P \to M$.

A different context for these results was given in Brown and Gilbert [5] in terms of the monoidal closed category of crossed modules of groupoids. The technique of relating the structure of *braided regular crossed modules* which arose in this situation to simplicial groups allowed a construction of a set of equivalent algebraic models of homotopy 3-types. The extension from groups to groupoids is necessary for both these contexts since the object set of the internal hom is just the set of morphisms of crossed modules, and the object set of the automorphism structure has a group structure related to the fundamental group of the 3-type.

However for geometric applications it is natural to seek a similar family of results, in particular the 2-crossed module (2), for a crossed module of groupoids $\mathcal{C} = (C, G, \delta)$, leading to the 2-crossed module

$$M_2(\mathcal{C}) \xrightarrow{\zeta} FDer^*(\mathcal{C}) \xrightarrow{\Delta} \mathsf{Aut}(\mathcal{C}) \qquad (3)$$

with action of $\mathsf{Aut}(\mathcal{C})$ on the other groups and *Peiffer lifting*

$$\langle\, ,\rangle : FDer^{*}(\mathcal{C}) \times FDer^{*}(\mathcal{C}) \to M_2(\mathcal{C}).$$

The part involving Δ will be the main focus of our work on 2-dimensional holonomy.

This has interesting special cases. If $\mathcal{C}$ is essentially a groupoid G, then $FDer^{*}(\mathcal{C})$ can be identified with the group $M(G)$ of coadmissible sections of G and we have the crossed module

$$\Delta : M(G) \to \mathsf{Aut}(G)$$

which has occurred sporadically in the literature (see, for example, [3, Section 9.4, Ex. 5,6], which are attributed to [13]). Extensions by a group K of the type of this crossed module (cf. [9]) are relevant to lax actions of the group K on the groupoid G applied by Brylinski in [12].

From the point of view of the possibilities of 'higher-order symmetry', it is interesting to note that a set corresponds to a homotopy 0-type; the automorphisms of a set yield a group, which corresponds to a homotopy 1-type; the automorphisms of a group yield a crossed module, which corresponds to a homotopy 2-type; and the automorphisms of a crossed module yield a 2-crossed module, which corresponds to a homotopy 3-type.

One of our expository problems is that although the theory of [5] deals with this general case of crossed modules of groupoids, the interpretation for the groupoid case is not given.

We give below the full definitions and proofs of the algebraic structure except for the verification of the laws for the Peiffer lifting of the 2-crossed module, because we believe it will help the reader to see explicitly the algebra that is involved, and this will also make this work largely independent of the papers Brown and Higgins [10] and Brown and Gilbert [5]. In particular, this makes our work independent of the equivalence between crossed complexes and ω-groupoids which is used by Brown and Higgins in [10]. We quote from p. 2 of this paper on the formulae for the tensor product: "Given formulae (3.1), (3.11) and (3.14), it is possible, in principle, to verify all the above facts within the category of crossed complexes, although the computations, with their numerous special cases, would be long. We prefer to prove these facts using the equivalent category ω-Grd of ω-groupoids where the formulae are simpler and have clearer geometric content".

Thus we have in the above carried out a portion of this verification. For the verification of the laws for the braided part of the structure, we are however using facts from Brown and Gilbert [5].

In Section 5 we mention briefly analogous results for the category of 2-groupoids, whose equivalence with the category of crossed modules of groupoids is the 2-truncated case of the main result of [7]. A later paper will develop this theory in its own terms.

1. Crossed Modules of Groupoids

We recall the definition of crossed modules of groupoids. The basic reference is Brown and Higgins [7].

The source and target maps of a groupoid G are written α, β respectively. If G is *totally intransitive*, i.e. if $\alpha = \beta$, then we usually use the notation β. The composition in a groupoid G of elements a, b with $\beta a = \alpha b$ will be written additively, as $a + b$. The main reason for this is the convenience for dealing with combinations of inverses and actions.

DEFINITION 1.1. *Let G, C be groupoids over the same object set and let C be totally intransitive. Then an action of G on C is given by a partially defined function*

$$C \times G \circ\!\!\longrightarrow C$$

written $(c, a) \mapsto c^a$, which satisfies

(i) *c^a is defined if and only if $\beta(c) = \alpha(a)$, and then $\beta(c^a) = \beta(a)$,*
(ii) $(c_1 + c_2)^a = {c_1}^a + {c_2}^a$,
(iii) $c_1^{a+b} = (c_1^a)^b$ *and* $c_1^{e_x} = c_1$

for all $c_1, c_2 \in C(x, x)$, $a \in G(x, y)$, $b \in G(y, z)$.

DEFINITION 1.2. A *crossed module of groupoids* [7] consists of a morphism $\delta : C \to G$ of groupoids C and G which is the identity on the object sets such that C is totally intransitive, together with an action of G on C which satisfies

(i) $\delta(c^a) = -a + \delta c + a$,
(ii) $c^{\delta c_1} = -c_1 + c + c_1$

for $c, c_1 \in C(x, x)$, $a \in G(x, y)$.

It is a *pre-crossed module* if (ii) is not necessarily satisfied. A (pre-)crossed module will be denoted by $\mathcal{C} = (C, G, \delta)$. A *crossed module of groups* is a crossed module of groupoids as above in which C, G are groups [25].

The followings are standard examples of crossed modules of groups and of groupoids:

(1) Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion. The action of G on the right of H by conjugation makes (H, G, i) into a crossed module. This generalises to the case H is a totally intransitive normal subgroupoid of the groupoid G.
(2) Suppose G is a group and M is a right G-module; let $0 : M \to G$ be the constant map sending M to the identity element of G. Then $(M, G, 0)$ is a crossed module.

(3) Suppose given a morphism

$$\eta : M \to N$$

of left G-modules and form the semi-direct product $G \ltimes N$. This is a group which acts on M via the projection from $G \ltimes N$ to G. We define a morphism

$$\delta : M \to G \ltimes N$$

by $\delta(m) = (1, \eta(m))$ where 1 denotes the identity in G. Then $(M, G \ltimes N, \delta)$ is a crossed module.

(4) Labesse in [19] defines a *crossed group*. It has been pointed out to us by Larry Breen at Coimbra in 1999 that a crossed group is exactly a crossed module $(C, X \rtimes G, \delta)$ where G is a group acting on the set X, and $X \rtimes G$ is the associated actor groupoid; thus the simplicial construction from a crossed group described by Breen in [19] is exactly the nerve of the crossed module (see [4] for an exposition).

(5) Let (Y, Z) be a pair of topological spaces and let X be a subset of Z. Then there is a crossed module of groupoids (C, G, δ) in which G is the fundamental groupoid $\pi_1(Z, X)$ on the set X of base points and for $x \in X$, $C(x)$ is the relative homotopy group $\pi_2(Y, X, x)$.

Let $\mathcal{C}$, $\mathcal{C}'$ be crossed modules. A *morphism* $f : \mathcal{C} \to \mathcal{C}'$ of crossed modules is a triple $f = (f_0, f_1, f_2)$ such that (f_0, f_1) form a morphism of groupoids $G \to G'$ and f_2 is a family of morphisms $C \to C'$ such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{f_2} & C' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ G & \xrightarrow[f_1]{} & G' \end{array}, \qquad \begin{array}{ccc} C \times G & \xrightarrow{f_2 \times f_1} & C' \times G' \\ \downarrow & & \downarrow \\ C & \xrightarrow[f_2]{} & C' \end{array}$$

So we can define a category Cmod of crossed modules of groupoids.

2. Homotopies and Derivations

In this section, we combine the notion of coadmissible section, which is fundamental to the work of Ehresmann [15], with the derivations which occur in Whitehead's account [25] of automorphism of crossed modules and which are later developed by Lue, Norrie, Brown and Gilbert [5, 20, 23].

Whitehead in [26] explored homotopies of morphisms of his 'homotopy systems' and this was put in the general context of crossed complexes of groupoids by Brown and Higgins in [10]. So we are exploring the implications of the idea that a natural generalisation of the notion of coadmissible section for groupoids is that of coadmissible homotopy for crossed modules.

DEFINITION 2.1. Let $g : \mathcal{C} \to \mathcal{D}$ be a morphism of crossed modules $\mathcal{C} = (C, G, \delta)$, $\mathcal{D} = (D, H, \delta')$ with object sets X, Y. A *homotopy* on g is a pair (s, g) where $s = (s_0, s_1)$ and such that $s_0 : X \to H$, $s_1 : G \to D$ and

$$\beta(s_0 x) = g_0 x,$$
$$\beta(s_1 a) = \beta g_1(a),$$
$$s_1(a + b) = s_1(a)^{g_1 b} + s_1(b),$$

whenever $x \in X, a, b \in G$ and $a + b$ is defined. Such a function s_1 is called a g-derivation.

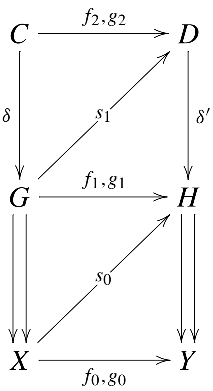
PROPOSITION 2.2. *Given a homotopy as above, the formulae*

$$f_0(x) = \alpha s_0(x),$$
$$f_1(a) = s_0(\alpha a) + g_1 a + \delta s_1(a) - s_0(\beta a),$$
$$f_2(c) = (g_2 c + s_1 \delta c)^{-s_0 \beta(c)}$$

for all $x \in X, a \in G, c \in C$ *define a morphism* $f : \mathcal{C} \to \mathcal{D}$ *of crossed modules.*

The proof is given in the Appendix.

We therefore call (s, g) a homotopy $f \simeq g$. This situation can be displayed as:



DEFINITION 2.3. For a homotopy $(s, 1) : f \simeq 1 : \mathcal{C} \to \mathcal{C}$, we also call s a *free derivation* on $\mathcal{C}$, and write $\Delta(s)$ for f.

Let $FDer(\mathcal{C})$ be the set of free derivations of $\mathcal{C}$.

PROPOSITION 2.4. *Let* $\mathcal{C} = (C, G, \delta)$ *be a crossed module over groupoids. Then a monoid structure on* $FDer(\mathcal{C})$ *is defined by the multiplication*

$$(s * t)_\varepsilon(z) = \begin{cases} (s_0 g_0(z)) + t_0(z) & \text{if } \varepsilon = 0, z \in X, \\ t_1(z) + (s_1 g_1(z))^{t_0(\beta z)} & \text{if } \varepsilon = 1, z \in G(x, y) \end{cases}$$

for $s, t \in FDer(\mathcal{C})$ *and* $g = \Delta(t)$. *Further* $\Delta(s * t) = \Delta(s) \circ \Delta(t)$ *and* $\Delta(1) = 1$, *so that* Δ *is a monoid morphism*

$$FDer(\mathcal{C}) \to \mathsf{End}(\mathcal{C}).$$

The proof is given in the Appendix.

Let $FDer^*(\mathcal{C})$ denote the group of invertible elements of this monoid $FDer(\mathcal{C})$. An invertible free derivation is also called a *coadmissible homotopy*.

THEOREM 2.5. *Let $s \in FDer(\mathcal{C})$ and let $f = \Delta(s)$. Then the following conditions are equivalent:*

(i) $s \in FDer^*(\mathcal{C})$,
(ii) $f_1 \in \mathsf{Aut}(G)$,
(iii) $f_2 \in \mathsf{Aut}(C)$.

The proof is given in the Appendix.

THEOREM 2.6. *There is an action of* $\mathsf{Aut}(\mathcal{C})$ *on* $s \in FDer^*(\mathcal{C})$ *given by*

$$s^f(z) = \begin{cases} {f_1}^{-1} s_0 f_0(z), & z \in X, \\ {f_2}^{-1} s_1 f_1(z), & z \in G \end{cases}$$

which makes

$$\Delta : FDer^*(\mathcal{C}) \to \mathsf{Aut}(\mathcal{C})$$

a pre-crossed module.

The proof is given in the Appendix.

The fact that $\Delta : FDer^*(\mathcal{C}) \to \mathsf{Aut}(\mathcal{C})$ is a pre-crossed module is a special case of results of Brown and Gilbert [5], which applies the monoidal closed structure of the category of crossed complexes introduced by Brown and Higgins in [10] (see our Section 4). The detailed description of $\Delta : FDer^*(\mathcal{C}) \to \mathsf{Aut}(\mathcal{C})$ is carried out in Brown and Gilbert [5, Proposition 3.3] only for the case $\mathcal{C}$ is a crossed module of groups.

We can also relate $FDer^*(\mathcal{C})$ and $Der^*(\mathcal{C})$, the group of all invertible derivations of $\mathcal{C}$ where a *derivation* is simply a pair (s_0, s_1) in which s_0 is $x \mapsto 1_x$. The multiplication of $Der^*(\mathcal{C})$ is

$$(s_1 * t_1)(a) = t_1 a + s_1(a + \delta t_1 a).$$

For a groupoid G over X let $M(G)$ denote the group of coadmissible sections of G with the usual multiplication $(s_0 * t_0)(x) = s_0(\alpha(t_0 x)) + t_0(x)$. Define an action of $t_0 \in M(G)$ on $s_1 \in Der^*(\mathcal{C})$ by

$$({s_1}^{t_0})(a) = s_1(t_0(\alpha a) + a - t_0(\beta a))^{t_0 \beta a}.$$

In view of the definition of the multiplication on $FDer^*(\mathcal{C})$ we have

THEOREM 2.7. *There is a natural isomorphism*

$$FDer^*(\mathcal{C}) \to M(G) \ltimes Der^*(\mathcal{C}).$$

The next theorem begins the analysis of the next level of the actor structure.

THEOREM 2.8. *Let* $\mathcal{C} = (C, G, \delta)$ *be a crossed module over groupoids with object set* X. *Let* $M_2(\mathcal{C})$ *be the group of sections* $s_2 : X \to C$ *of* β *under pointwise addition. Then there is a crossed module*

$$M_2(\mathcal{C}) \xrightarrow{\zeta} FDer^*(\mathcal{C}),$$

where $\zeta : M_2(\mathcal{C}) \to FDer^*(\mathcal{C})$ *is defined by* $s_2 \mapsto (\delta s_2, s_1)$ *where* $s_1 : a \mapsto -s_2(\alpha a)^a + s_2(\beta a)$, *and the action of* $FDer^*(\mathcal{C})$ *on* $M_2(\mathcal{C})$ *is given by*

$$(s_2)^{(t_0,t_1)} : x \mapsto s_2(x)^{t_0(x)}.$$

The proof is given in the Appendix.

3. Braided Regular Crossed Modules and 2-crossed Modules

In this section our object is to give the explicit relationship between braided regular crossed modules and 2-crossed modules. This indicates a possible further context for development of work on holonomy [11].

The following material can be found in Brown and Gilbert [5].

Here we find it convenient to write a crossed module $\mathcal{A}$ as (A_2, A_1, δ) instead of (C, G) – that is, we write A_2, A_1 for C, G respectively and we also write A_0 for $Ob(G)$. A *monoid bimorphism*

$$b : \mathcal{A}, \mathcal{A} \to \mathcal{A}$$

consists of a family of maps

$$b_{ij} : A_i \times A_j \to A_{i+j}$$

for $i, j = 0, 1, 2; i + j \leqslant 2$ satisfying the following axioms, in which $b_{11}(a, b)$ is written $\{a, b\}$.

3.1 b_{00} gives A_0 the structure of monoid with identity element written e and multiplication written $(x, y) \mapsto xy$.

3.2 For $i = 1, 2$, b_{0i} and b_{i0} give actions of this monoid on the left and right of the groupoid A_i written $(x, c) \mapsto x.c$ and $(c, y) \mapsto c.y$ and which commute $x.(c.y) = (x.c).y$. Further each action of an element of A_0 preserves the groupoid structure, so that $x.(a + b) = x.a + x.b$, $(a + b).y = a.y + b.y$ and α, β are equivariant with respect to these actions.

3.3 The actions are compatible with the actions of A_1 on A_2 in the sense that if $c \in A_2(x)$, $a \in A_1(x, y)$, and $z \in A_0$ then

$$z.(c^a) = (z.c)^{z.a} \in A_2(z.x),$$

$$(c^a).z = (c.z)^{a.z} \in A_2(x.z);$$

3.4 The boundary morphism $\delta : A_2 \to A_1$ is equivariant with respect to these actions of A_0.
3.5 $\{a, b\} \in A_2((\beta a)(\beta b)), \{O_e, b\} = O_{\beta b}, \{a, O_e\} = O_{\beta a}$;
3.6 $\{a, b + b'\} = \{a, b\}^{\beta a.b'} + \{a, b'\}$;
3.7 $\{a + a', b\} = \{a', b\} + \{a, b\}^{a'.\beta b}$;
3.8 $\delta\{a, b\} = -(\beta a.b) - a.\alpha b + \alpha a.b + a.\beta b$;
3.9 $\{a, \delta c'\} = -(\beta a.c') + (\alpha a.c')^{a.y}$ if $c' \in A_2(y)$;
3.10 $\{\delta c, b\} = -(c.\alpha b)^{x.b} + c.\beta b$ if $c \in A_2(p)$;
3.11 $x.\{a, b\} = \{x.a, b\}, \ \{a, b\}.x = \{a, b.x\}, \ \{a.x, b\} = \{a, x.b\}$,

for all $a, a', b, b' \in A_1$, $c, c' \in A_2$ and $x, y \in A_0$.

The crossed module $\mathcal{A}$ with this structure is called a *semiregular braided crossed module*, and it is called *regular* if the monoid A_0 is a group.

This structure is closely related to one given by Conduché in [14]. Recall from [14] that a 2-*crossed module* consists, in the first instance, of a complex of P-groups

$$L \xrightarrow{\partial} M \xrightarrow{\partial} P$$

(so that $\partial\partial = 0$) and P-equivariant homomorphisms, where the group P acts on itself by conjugation, such that

$$L \xrightarrow{\partial} M$$

is a crossed module, where M acts on L via P. We require that $(l^m)^p = (l^p)^{m^p}$ for all $l \in L, m \in M$, and $p \in P$. Further, there is a function $\langle\, ,\rangle : M \times M \to L$, called a *Peiffer lifting*, which satisfies the following axioms:

P_1: $\partial\langle m_0, m_1\rangle = m_0{}^{-1}m_1{}^{-1}m_0 m_1{}^{\partial m_0}$,
P_2: $\langle \partial l, m\rangle = l^{-1}l^m$,
P_3: $\langle m, \partial l\rangle = l^{-m}l^{\partial m}$,
P_4: $\langle m_0, m_1 m_2\rangle = \langle m_0, m_2\rangle\langle m_0, m_1\rangle^{m_2{}^{\partial m_0}}$,
P_5: $\langle m_0 m_1, m_2\rangle = \langle m_0, m_2\rangle^{m_1}\langle m_1, m_2^{\partial m_0}\rangle$,
P_6: $\langle m_0, m_1\rangle^p = \langle m_0{}^p, m_1{}^p\rangle$,

for all $p \in P, m_0, m_1, m_2 \in M, l \in L$.

THEOREM 3.1 (Brown and Gilbert [5]). *The categories of braided regular crossed modules and of* 2*-crossed modules are equivalent.*

Outline proof. Let $\mathcal{A} = (A_2, A_1, \delta)$ be a regular crossed module. Denote by K the costar in A_1 at the vertex $e \in A_0$, that is, $K = \{a \in A_1 : \beta a = e\}$. Then K has a group operation given for any $a, b \in K$ by

$$ab = b + (a.\alpha b).$$

The source map $\alpha : K \to A_0$ is a homomorphism of groups and is A_0-equivariant relative to the biaction of A_0 on A_1. Note that the new composition extends the group structure on the vertex group $A_1(e)$ so that $A_1(e)$ is a subgroup of K: it is plainly the kernel of α. Further, A_0 acts diagonally on K: for all $a \in K$ and $p \in A_0$ we set $a^p = p^{-1}.a.p$. (There should be no confusion with the given action of A_0 on A_2 which we denote in a similar way.) Then the homomorphism $\alpha : K \to A_0$ is A_0-equivariant relative to the diagonal action on K and the conjugation action of the group A_0 on itself. Now A_0 also acts diagonally on the vertex group $A_2(e)$ and so we have a complex of groups

$$A_2(e) \xrightarrow{\delta} K \xrightarrow{\alpha} A_0$$

in which δ and α are A_0-equivariant. We know that $\delta : A_2(e) \to A_1(e)$ is a crossed module: we claim that K acts on $A_2(e)$, extending the action of $A_1(e) \subseteq K$, so that $\delta : A_2 \to K$ is a crossed module.

We define an action $(c, a) \mapsto c!a$ by $c!a = (c.\alpha a)^a$ where $c \in A_2(e)$ and $a \in K$. This is indeed a group action and δ is K-equivariant. Moreover, the actions of $A_2(e)$ on itself via K and by conjugation coincide, for $\delta : A_2(e) \to A_1(e)$ is a crossed module and so for all $c, c' \in A_2(e)$,

$$c!\delta c' = (c.\alpha(\delta c'))^{\delta c'} = (c.e)^{\delta c'} = -c' + c + c'.$$

Therefore the map $\delta : A_2(e) \to K$ is a crossed module. Further the action of A_0 on $A_2(e)$ is compatible with that of K.

The final structural component of a 2-crossed module that we need is the Peiffer lifting, which is provided by the braiding. For suppose that $\mathcal{A}$ has a braiding $\{\, ,\}$: $A_1 \times A_1 \to A_2$. Then the map $K \times K \to A_2(e)$ given by $(a, b) \mapsto \{a^{-1}, b\}!a = \langle a, b\rangle$ is a Peiffer lifting. Therefore we have the 2-crossed module

$$A_2(e) \to K \to A_0.$$

The verification of the axioms is obtained in [5] by showing that this is the Moore complex of a simplicial group $S(\mathcal{A})$ whose Moore complex is of length 2 and applying the results of [14].

Now we show how a 2-crossed module give rises to a braided regular crossed module. So we begin with a 2-crossed module

$$L \xrightarrow{\partial} G \xrightarrow{\partial} P$$

and construct from it, in a functorial way, a regular, braided crossed module $\mathcal{A} = (A_2, A_1, \delta)$.

The group of object of A_0 is just the group P. The underlying set of elements of A_1 is $G \times P$ with source and target maps $\alpha(g, p) = \partial(g)p$ and $\beta(g, p) = p$. The groupoid composition in A_1 is given by $(g_1, p_1) + (g_2, p_2) = (g_1g_2, p_2)$ if $p_1 = \partial(g_2)p_2$. The underlying set of elements of A_2 is $L \times P$ with composition $(l_1, p) + (l_2, p) = (l_1l_2, p)$. The boundary map $\delta : A_2 \to A_1$ is given by $\delta(l, p) = (\partial l, p)$ and the action of A_1 on A_2 is given by $(l, p)^{(g,q)} = (l^g, q)$ if $p = \partial(g)q$.

This does define a crossed module over (A_1, A_0) and a biaction of A_0 on $\mathcal{A}$ is obtained if we define

$$p.(g, q) = (g^{p^{-1}}, pq), (g, q).p = (g, qp),$$
$$p.(l, q) = (l^{p^{-1}}, pq), (l, q).p = (l, qp),$$

where $(g, q), (l, q) \in A_2$ and $p \in A_0 = P$ and therefore $\mathcal{A}$ with this biaction is regular. The braiding on $\mathcal{A}$ is given by

$$\{(g_1, p_1), (g_2, p_2)\} = (\langle g_1{}^{-1}, g_2{}^{p_1}\rangle^{g_1}, p_1 p_2),$$

where $\langle\ ,\ \rangle : G \times G \to L$ is the Peiffer lifting. □

In the next section we apply this result to automorphisms of crossed modules of groupoids.

4. $\mathsf{AUT}(\mathcal{C})$ and 2-crossed Modules

In [10] the category of crossed complexes is given the structure of a monoidal closed category and this induces such a structure on the category Cmod of crossed modules of groupoids. So there is a tensor product $- \otimes -$ and internal hom $\mathsf{CMOD}(-, -)$ and for all crossed modules $\mathcal{A}, \mathcal{B}, \mathcal{C}$ a natural isomorphism

$$\theta : \mathsf{Cmod}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \to \mathsf{Cmod}(\mathcal{A}, \mathsf{CMOD}(\mathcal{B}, \mathcal{C})),$$

which, together with the associativity of the tensor product, implies the existence in Cmod of a natural isomorphism

$$\Theta : \mathsf{CMOD}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \to \mathsf{CMOD}(\mathcal{A}, \mathsf{CMOD}(\mathcal{B}, \mathcal{C})).$$

Further, the bijection

$$\theta : \mathsf{Cmod}(\mathsf{CMOD}(\mathcal{A}, \mathcal{B}) \otimes \mathcal{A}, \mathcal{B}) \to \mathsf{Cmod}(\mathsf{CMOD}(\mathcal{A}, \mathcal{B}), \mathsf{CMOD}(\mathcal{A}, \mathcal{B}))$$

shows that there is a unique morphism

$$\varepsilon_{\mathcal{A}} : \mathsf{CMOD}(\mathcal{A}, \mathcal{B}) \otimes \mathcal{A} \to \mathcal{B}$$

such that $\theta(\varepsilon_{\mathcal{A}})$ is the identity on $\mathsf{CMOD}(\mathcal{A}, \mathcal{B})$; $\varepsilon_{\mathcal{A}}$ is called the *evaluation morphism*. Then for all crossed modules $\mathcal{A}, \mathcal{B}, \mathcal{C}$ there is a morphism

$$\begin{aligned}
&(\mathsf{CMOD}(\mathcal{B}, \mathcal{C}) \otimes \mathsf{CMOD}(\mathcal{A}, \mathcal{B})) \otimes \mathcal{A} \\
&\quad \to \mathsf{CMOD}(\mathcal{B}, \mathcal{C}) \otimes (\mathsf{CMOD}(\mathcal{A}, \mathcal{B}) \otimes \mathcal{A}) \\
&\quad \to \mathsf{CMOD}(\mathcal{B}, \mathcal{C}) \otimes \mathcal{B} \to \mathcal{C}.
\end{aligned}$$

This corresponds under θ to a morphism

$$\gamma_{\mathcal{A}\mathcal{B}\mathcal{C}} : \mathsf{CMOD}(\mathcal{B}, \mathcal{C}) \otimes \mathsf{CMOD}(\mathcal{A}, \mathcal{B}) \to \mathsf{CMOD}(\mathcal{A}, \mathcal{C})$$

which is called *composition*.

We write $\mathsf{END}(\mathcal{C})$ for $\mathsf{CMOD}(\mathcal{C}, \mathcal{C})$. The terminal object in Cmod is written $*$. There is a morphism $\eta_{\mathcal{C}} : * \to \mathsf{END}(\mathcal{C})$ corresponding to the isomorphism $\lambda : * \otimes \mathcal{C} \to \mathcal{C}$. The main result we need is the following [17].

PROPOSITION 4.1. *The morphism* $\eta_{\mathcal{C}}$ *and the composition*

$$\mu = \gamma_{\mathcal{CCC}} : \mathsf{END}(\mathcal{C}) \otimes \mathsf{END}(\mathcal{C}) \to \mathsf{END}(\mathcal{C})$$

make $\mathsf{END}(\mathcal{C})$ *a monoid in* Cmod *with respect to* $\otimes$.

COROLLARY 4.2. *The crossed module* $\mathsf{END}(\mathcal{C})$ *may be given the structure of braided semiregular crossed module.*

Proof. As shown in [10, 5], for any crossed module $\mathcal{A}$, a morphism $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ corresponds exactly to a bimorphism $\mathcal{A}, \mathcal{A} \to \mathcal{A}$, and a monoid structure is equivalent to a monoid bimorphism. □

4.1. $\mathsf{AUT}(\mathcal{C})$ AND 2-CROSSED MODULES

Let $\mathcal{A} = \mathsf{AUT}(\mathcal{C})$ be the full subcrossed module of $\mathcal{E} = \mathsf{END}(\mathcal{C})$ on the object set $A_0 = \mathsf{Aut}(\mathcal{C})$ of automorphisms of the crossed module $\mathcal{C} = (C, G, \delta)$. Thus A_0 is the group of units of E_0. Now an element of A_2 is a section over an automorphism of $\mathcal{C} = (C, G, \delta)$ and consists of a pair (s_2, f) where s_2 is a section and $f \in A_0$. An element of A_1 is a homotopy over an automorphism of $\mathcal{C} = (C, G, \delta)$ and consists of a triple (s_0, s_1, f) where s_0 is a section, $f \in A_0$, and s_1 is an f-derivation $G \to C$ such that the endomorphism f^0 of $\mathcal{C} = (C, G, \delta)$ which gives the source object of (s_0, s_1, f) is actually an automorphism. Clearly f^0 is an automorphism of $\mathcal{C} = (C, G, \delta)$ if and only if

$$\begin{aligned} g(a) &= s_0(x) + f(a) + \delta s_1(a) - s_0(y), \\ g(c) &= (f(c) + s_1\delta(c))^{-s_0(\beta(c)}, \\ g(x) &= \alpha s_0(x) \end{aligned}$$

for all $a \in G(x, y)$, $c \in C(x)$, $x \in X$ defines an automorphism of $\mathcal{C} = (C, G, \delta)$.

Clearly $\mathcal{A}$ inherits from $\mathcal{E}$ the structure of a regular, braided, crossed module. To determine the biaction of A_0 and the braiding we have to understand the composition map γ explicitly. A direct calculation leads to the following non-trivial components for the bimorphism determining γ:

$$\begin{aligned} &A_0 \times A_0 \to A_0 : (f_1, f_2) \mapsto f_1 f_2, \\ &A_0 \times A_1 \to A_1 : (f_1, (s_0, s_1, f)) \mapsto (f_1 s_0, f_1 s_1, f_1 f), \\ &A_1 \times A_0 \to A_1 : ((s_0, s_1, f), f_2) \mapsto (s_0 f_2, s_1 f_2, f f_2), \\ &A_0 \times A_2 \to A_2 : (f_1, (s_2, f)) \mapsto (f_1(s_2), f_1 f), \\ &A_2 \times A_0 \to A_2 : ((s_2, f), f_2) \mapsto (s_2, f f_2), \\ &A_1 \times A_1 \to A_2 : ((s_0, s_1, f), (t_0, t_1, f')) \mapsto (s_1 t_0, f f'). \end{aligned}$$

These maps give biactions of A_0 on A_1, A_2 and a braiding $\{\,,\}: A_1 \times A_1 \to A_2$. The monoid structure on A_0 is the usual composition of maps.

THEOREM 4.3. *The regular crossed module* $\mathcal{A} = \mathsf{AUT}(\mathcal{C})$ *corresponds via the equivalence of Theorem* 3.1 *to the* 2*-crossed module*

$$M_2(\mathcal{C}) \xrightarrow{\zeta} FDer^*(\mathcal{C}) \xrightarrow{\Delta} Aut(\mathcal{C}).$$

Proof. The costar in the groupoid A_1 at the identity automorphism I of $\mathcal{C}$ may be identified as a set with $FDer^*(\mathcal{C})$ and the group structure is given by $(s_0, s_1) * (t_0, t_1) = (s_0 * t_0, s_1 * t_1)$ as in Proposition 2.4 and Theorem 2.5. The vertex group $A_2(I)$ is identified with the group $M_2(\mathcal{C})$ with $\zeta(s_2) = (\delta s_2, s_1)$ as required in Theorem 2.8. Note that $\mathsf{Aut}(\mathcal{C})$ acts on $FDer^*(\mathcal{C})$ by

$$(s_0, s_1)^f = (f^{-1}s_0 f, f^{-1}s_1 f)$$

proved in Theorem 2.6 and on $M_2(\mathcal{C})$ by $s_2{}^f = f^{-1}s_2 f$. The action of $FDer^*(\mathcal{C})$ on $M_2(\mathcal{C})$ is simply $s_2^{(t_0,t_1)} = s_2^{t_0}$ and the Peiffer lifting is given by

$$\begin{aligned} \langle (s_0, s_1), (t_0, t_1)\rangle &= \{(s_0, s_1)^{-1}, (t_0, t_1)\}!(s_0, s_1) \\ &= (\{(s_0{}^{-1}, (s_1{}^{-1})^{s_0{}^{-1}}), (t_0, t_1)\}.\Delta(s_0, s_1))^{(s_0,s_1)} \\ &= (s_1{}^{-1})^{s_0{}^{-1}} * (t_0)^{s_0} \\ &= s_1{}^{-1}(s_0{}^{-1} * t_0 * s_0). \end{aligned}$$

□

5. 2-groupoids

A commonly used 2-dimensional version of a groupoid is a 2-groupoid (see for example [18, 22]). This is in fact a 2-truncated case of the ∞-groupoids defined in 1981 in [8], where it is shown that the categories of ∞-groupoids and of crossed complexes are equivalent. In particular, the categories of 2-groupoids and of crossed modules of groupoids are equivalent. Hence the results of the previous sections may be transferred to the category of 2-groupoids. This gives the analogue of Theorem 4.3:

THEOREM 5.1. *A* 2*-groupoid* $\mathcal{G}$ *determines a* 2*-crossed module of the form*

$$M_2(\mathcal{G}) \xrightarrow{\zeta} FDer^*(\mathcal{G}) \xrightarrow{\Delta} \mathsf{Aut}(\mathcal{G}).$$

The precise description of these objects in 2-groupoid terms will be left to another paper.

Appendix

Proof of Proposition 2.2. We have to show that f_1 and f_2 are groupoid homomorphisms and $f_2(c^a) = f_2(c)^{f_1(a)}$, for $c \in C(x)$, $a \in G(x, y)$.

$$\begin{aligned}
f_1(a+b) &= s_0(x) + g_1(a+b) + \delta s_1(a+b) - s_0(z) \\
&= s_0(x) + g_1 a + g_1 b + \delta(s_1(a)^{g_1(b)} + s_1(b)) - s_0(z) \\
&= s_0(x) + g_1 a + g_1 b - g_1 b + \delta s_1(a) + g_1 b + \delta s_1(b) - s_0(z), \\
&\quad \text{by definition of } \delta \\
&= s_0(x) + g_1 a + \delta s_1(a) - s_0(y) + s_0(y) + g_1 b + \delta s_1(b) - s_0(z) \\
&= f_1(a) + f_1(b), \\
f_2(c+c') &= (g_2(c+c') + s_1\delta(c+c'))^{-s_0(x)} \\
&= (g_2 c + g_2 c' + s_1(\delta c + \delta c'))^{-s_0(x)} \\
&= (g_2 c + g_2 c' + s_1(\delta c)^{g_1 \delta c'} + s_1(\delta c'))^{-s_0(x)} \\
&= (g_2 c + g_2 c' - g_2 c' + s_1 \delta c + g_2 c' + s_1 \delta c')^{-s_0(x)} \\
&= (g_2 c + s_1 \delta c + g_2 c' + s_1 \delta c')^{-s_0(x)} \\
&= f_2(c) + f_2(c').
\end{aligned}$$

Let $c \in C(x)$, $a \in G(x, y)$. Then $\beta(c^a) = \beta a$, $\beta c^a = y$. So

$$\begin{aligned}
f_2(c^a) &= (g_2 c^a + s_1\delta(c^a))^{-s_0(\beta c^a) = -s_0(y)} \\
&= (g_2 c^a + s_1(-a + \delta c + a))^{-s_0(y)} \\
&= (g_2 c^a + s_1(-a)^{g_1(\delta c + a)} + s_1(\delta c)^{g_1 a} + s_1(a))^{-s_0(y)} \\
&\quad (\text{since } -(s_1(a))^{-g_1 a + g_1 \delta c + g_1 a} = (s_1(-a))^{g_1(\delta c + a)}), \\
&= (g_2 c^a - (s_1 a)^{-g_1 a + g_1 \delta c + g_1 a} + (s_1 \delta c)^{g_1 a} + s_1 a)^{-s_0(y)} \\
&= (g_2 c^a - s_1 a^{(\delta g_2(c^a))} + (s_1 \delta c)^{g_1 a} + s_1 a)^{-s_0(y)} \\
&= (-s_1(a) + g_2(c)^{g_1 a} + (s_1 \delta c)^{g_1 a} + s_1(a))^{-s_0(y)} \\
&= (-s_1(a) + (g_2 c + s_1 \delta c)^{g_1 a} + s_1(a))^{-s_0(y)} \\
&= (-s_1(a) + (f_2(c)^{s_0(x)})^{g_1 a} + s_1(a))^{-s_0(y)} \\
&= (f_2(c))^{s_0 x + g_1 a + \delta s_1 a - s_0 y} \\
&= f_2(c)^{f_1(a)}.
\end{aligned}$$

So f is an endomorphism of $\mathcal{C}$. □

Proof of Proposition 2.4. It is clear that $\beta(s{*}t)_0(x) = x$ and $\beta(s{*}t)_1(a) = \beta(a)$.

We have to show that $(s * t)_1$ is a derivation map. Let $a \in G(x, y)$, $b \in G(y, z)$. Then

$$\begin{aligned}
(s*t)_1(a+b) &= t_1(a+b) + (s_1 g_1(a+b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + (s_1(g_1(a) + g_1(b)))^{t_0(z)}
\end{aligned}$$

$$
\begin{aligned}
&= t_1(a)^b + t_1(b) + (s_1(g_1(a))^{g_1(b)} + s_1(g_1(b)))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + (s_1(g_1(a))^{t_0(y)+b+\delta t_1(b)-t_0(z)} + \\
&\quad +s_1(g_1(b)))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + s_1(g_1(a))^{t_0(y)+b+\delta t_1(b)} + s_1(g_1(b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + s_1(g_1(a)^{t_0(y)+b})^{\delta t_1(b)} + s_1(g_1(b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) - t_1(b) + (s_1(g_1(a))^{t_0(y)+b} + t_1(b) + \\
&\quad +s_1(g_1(b))^{t_0(z)} \\
&= t_1(a)^b + (s_1(g_1(a))^{t_0(y)+b} + t_1(b) + s_1(g_1(b))^{t_0(z)} \\
&= t_1(a) + s_1(g_1(a)^{t_0(y)})^b + t_1(b) + s_1(g_1(b))^{t_0(z)} \\
&= (s * t)_1(a)^b + (s * t)_1(b).
\end{aligned}
$$

For the associativity property, let $u, s, t \in FDer(\mathcal{C})$ and let $f = \Delta(s), g = \Delta(t), h = \Delta(u)$. Then

$$
\begin{aligned}
(u_0 * (s * t)_0)(x) &= (u_0(f_0 g_0(x)) + (t * s)_0(x) \\
&= u_0(f_0 g_0)(x) + (s_0 g_0)(x) + t_0(x) \\
&= u_0(f_0(g_0(x)) + s_0 g_0(x) + t_0(x) \\
&= (u * s)_0(g_0(x)) + t_0(x) \\
&= ((u * s)_0 * t_0)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(u_1 * (s * t)_1)(a) &= (s * t)_1(a) + u_1(fg(a))^{(s*t)_0(x)} \\
&= t_1 a + (s_1 ga)^{t_0(y)} + (u_1 fga)^{(s*t)_0(x)} \\
&= t_1 a + (s * u)_1(ga)^{t_0(y)} \\
&= ((u * s)_1 * t_1)(a).
\end{aligned}
$$

Let $s, t \in FDer(\mathcal{C})$ be as above and let $a \in G(x, y)$. Then

$$
\begin{aligned}
\Delta(s * t)_1(a) &= (s * t)_0(x) + a + \delta(s * t)_1(a) - (s * t)_0(y) \\
&= s_0 g_0(x) + t_0(x) + a + \delta(t_1(a) + s_1 g_1(a))^{t_0(y)} - \\
&\quad -(s_0 g_0(y) + t_0(y)) \\
&= s_0 g_0(x) + t_0(x) + a + \delta(t_1(a) + \delta(s_1 g_1(a)^{t_0(y)}) - \\
&\quad -(s_0 g_0(y) + t_0(y)) \\
&= s_0 g_0(x) + t_0(x) + a + \delta t_1(a) - t_0(y) + \delta s_1 g_1(a) + \\
&\quad +t_0(y) - t_0(y) - s_0 g_0(y) \\
&= s_0 g_0(x) + \delta_t(a) + \delta s_1 g_1(a) - s_0 g_0(y) \\
&= \Delta_s(\Delta_t)(a) \\
&= \Delta_s \circ \Delta_t(a).
\end{aligned}
$$

Let $c \in C(x)$, $a \in G(x, y)$.

$$\begin{aligned}
\Delta(s*t)(c) &= (c + (s*t)_1(\delta(c))^{-(s*t)_0(\beta c)} \\
&= (c + t_1(\delta(c)) + s_1 g_1(\delta(c))^{t_0(x)})^{-(s*t)_0(x)} \\
&= (c + t_1(\delta(c))^{-(s*t)_0(x)} + s_1\delta g_2(c))^{t_0(x)-(s*t)_0(x)} \\
&= (c + t_1(\delta(c))^{-t_0(x)-s_0 g_0(x)} + s_1\delta g_2(c))^{-s_0 g_0(x)} \\
&= ((c + t_1(\delta(c))^{-t_0(x)} + s_1\delta g_2(c))^{-s_0 g_0(x)} \\
&= (\Delta_t(c)) + (s_1\delta g_2(c))^{-s_0 g_0(x)}, \quad \text{since } \Delta_t(c) = g_2(c), \\
&= \Delta_s \circ \Delta_t(c).
\end{aligned}$$

So $\Delta(s*t) = \Delta(s) \circ \Delta(t)$.

The identity of $Fder^*(\mathfrak{C})$ is $c = (c_0, c_1)$ defined by $c_0(x) = 1_x$ and $c_1(a) = 1$ for $x \in X$ and $a \in G$. It is easy to see that $\Delta(c) = I$. □

Proof of Theorem 2.5. That (i)⇒(ii), (i)⇒(iii) follows from the fact that Δ is a morphism to $End(\mathfrak{C})$. We next prove (ii)⇒(i). Suppose then $f_1 \in Aut(G)$. We define $s^{-1} = ({s_0}^{-1}, {s_1}^{-1})$.

Let ${s_0}^{-1} : X \to G$, ${s_1}^{-1} : G \to C$ by

$${s_0}^{-1}(x) = -s_0({f_0}^{-1}(x)) \quad \text{and} \quad {s_1}^{-1}(a) = -s_1({f_1}^{-1}(a))^{{s_0}^{-1}(y)}.$$

Since $\beta {s_0}^{-1}(x) = x$ and $\alpha {s_0}^{-1}(x) = f_0(x)$, ${s_0}^{-1}$ is an inverse element of s_0. In fact,

$$\begin{aligned}
(s^{-1} * s)_0(x) &= {s_0}^{-1}({f_0}^{-1})(x) + s_0(x) \\
&= -s_0({f_0}^{-1}(f_0)(x)) + s_0(x) \\
&= -s_0(x) + s_0(x) \\
&= c_0(x) = 1_x
\end{aligned}$$

and also

$$\begin{aligned}
(s * s^{-1})_0(x) &= s_0({f_0}^{-1})(x) + {s_0}^{-1}(x) \\
&= s_0({f_0}^{-1})(x) - s_0({f_0}^{-1})(x) \\
&= c_0(y) = 1_y.
\end{aligned}$$

We have to show that ${s_1}^{-1}$ is a derivation map. Let $a, b, a + b \in G$ and let $a' = {f_1}^{-1}(a)$, $b' = {f_1}^{-1}(b)$, $\beta b = z$. Note that ${s_0}^{-1}\beta(a + b) = {s_0}^{-1}(z)$ and $-s_0(z) = -s_0 {f_0}^{-1}(z)$.

$$\begin{aligned}
{s_1}^{-1}(a + b) &= -(s_1 {f_1}^{-1}(a + b))^{{s_0}^{-1}\beta(a+b)}, \quad \text{by definition of } {s_1}^{-1}, \\
&= -(s_1({f_1}^{-1}a + {f_1}^{-1}b))^{{s_0}^{-1}(z)} \\
&= -(s_1(a' + b'))^{{s_0}^{-1}(z)} \\
&= -((s_1 a')^{b'} + s_1(b'))^{{s_0}^{-1}(z)}, \quad \text{since } s_1 \text{ is a derivation,}
\end{aligned}$$

$$\begin{aligned} &= (-s_1(b') - (s_1(a'))^{b'})^{-s_0(z)} \\ &= -(s_1(b') + (s_1(a'))^{b'+\delta s_1(b')})^{-s_0(z)} \\ &= -(s_1(b')^{-s_0(z)} + (s_1(a'))^{b'+\delta s_1(b')-s_0 f_0{}^{-1}(z)}. \end{aligned}$$

Since $f(b') = s_0 f_0{}^{-1}(y) + b' + \delta s_1(b') - s_0 f_0{}^{-1}(z) = b$ and $b' + \delta s_1(b') - s_0 f_0{}^{-1}(z) = b - s_0 f_0{}^{-1}(y)$,

$$\begin{aligned} &= -(s_1(b')^{-s_0(z)} + (s_1(a'))^{b-s_0 f_0{}^{-1}(y)} \\ &= -(s_1(b')^{-s_0(z)} + (s_1(a'))^{b-s_0{}^{-1}(y)} \\ &= -(s_1(a'))^{b})^{-s_0{}^{-1}(y)} - s_1(b')^{s_0{}^{-1}} \\ &= -(s_1(f_1{}^{-1}(a))^{b})^{-s_0{}^{-1}(y)} - s_1 f_1{}^{-1}(b')^{s_0{}^{-1}} \\ &= s_1{}^{-1}(a)^{b} + s_1{}^{-1}(b). \end{aligned}$$

One can easily show that $s * s^{-1} = c$ and $s^{-1} * s = c$.

$$\begin{aligned} (s * s^{-1})_1(a) &= s_1{}^{-1}(a) + s_1(f^{-1}(a))^{s_0{}^{-1}(y)} \\ &= -s_1 f^{-1}(a)^{s_0{}^{-1}(y)} + s_1(f^{-1}(a))^{s_0{}^{-1}(y)} \\ &= c_1(a) \end{aligned}$$

and

$$\begin{aligned} (s^{-1} * s)_1(a) &= s_1(a) + s_1{}^{-1}(f(a))^{s_0(y)} \\ &= s_1(a) - s_1(f^{-1}(f(a))^{s_0(y)})^{s_0{}^{-1}(y)} \\ &= s_1(a) - s_1(a) \\ &= c_1(a). \end{aligned}$$

Now we will prove (iii)⇒(i). We first recalculate $(s*t)_1$ in terms of f_2. Let $\Delta(t) = g$ and let $\Delta(s) = f$, $a \in G(x, y)$ as above.

$$\begin{aligned} (s * t)_1(a) &= t_1(a) + s_1 g_1(a)^{t_0(y)} \\ &= t_1(a) + s_1(t_0(x) + a + \delta t_1(a) - t_0(y))^{t_0(y)} \\ &= t_1(a) + s_1(t_0(x))^{a+\delta t_1(a)-t_0(y)} + s_1(a)^{\delta t_1(a)-t_0(y)} + \\ &\quad s_1(\delta t_1(a))^{t_0(y)}) + s_1(-t_0(y))^{t_0(y)} \\ &= t_1(a) + s_1(t_0(x))^{a+\delta t_1(a)} + s_1(a)^{\delta t_1(a)} + \\ &\quad s_1(\delta t_1(a)) + s_1(-t_0(y))^{t_0(y)} \\ &= t_1(a) + s_1(t_0(x)^{a})^{\delta t_1(a)} - t_1(a) + s_1(a) + t_1(a) + \\ &\quad +s_1(\delta t_1(a)) - s_1(t_0(y)), \\ &\quad \text{since } s_1(-t_0(y))^{t_0(y)} = -s_1 t_0(y), \\ &= t_1(a) - t_1(a) + s_1(t_0(x)^{a} + t_1(a) - t_1(a) + s_1(a) + \\ &\quad +t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y)) \end{aligned}$$

$$\begin{aligned} &= s_1(t_0(x))^a + s_1(a) + t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y)) \\ &= s_1(t_0(x))^a + s_1(a) + f_2(t_1(a))^{s_0(y)} - s_1(t_0(y)). \end{aligned}$$

Now, suppose that f_2 has inverse $f_2{}^{-1}$. Let $s^{-1} = (s_0{}^{-1}, s_1{}^{-1})$ be defined by

$$\begin{aligned} &s_0{}^{-1}(x) = -s_0 f_0{}^{-1}(x), \quad x \in X, \\ &s_1{}^{-1}(a) = f_2{}^{-1}(-s_1(a) - (s_1 s_0{}^{-1}(x))^a + (s_1 s_0{}^{-1}(y))^{-s_0(y)}, \quad a \in G(x, y). \end{aligned}$$

We prove that s^{-1} is an inverse element of s and is a derivation map. Clearly

$$s * s^{-1}(x) = c_0(x) \quad \text{and} \quad s^{-1} * s(x) = c_0(x)$$

by the argument as above.

Next we prove $(s * s^{-1})_1(a) = c_1(a)$, for $a \in G(x, y)$.

$$\begin{aligned} (s * s^{-1})_1(a) &= (s_1(s_0{}^{-1}(x))^a + s_1(a) + f_2(f_2{}^{-1}((-s_1(a) + \\ &\quad +(s_1 s_0 f_0{}^{-1}(x))^a - (s_1 s_0 f_0{}^{-1}(y))^{-s_0(y)})^{s_0(y)}) - s_1 s_0{}^{-1}(y) \\ &= (s_1(s_0 f_0{}^{-1}(x))^a + s_1(a) - s_1(a) + (s_1 s_0 f_0{}^{-1}(x))^a + \\ &\quad +(s_1 s_0 f_0{}^{-1}(y)) - s_1 s_0 f_0{}^{-1}(y) \\ &= c_1(a). \end{aligned}$$

Since $(s * s^{-1}) = 1$ and also $s^{-1} * s' = 1$. It follows that $s_1{}^{-1} * s_1 = s_1{}^{-1} * s_1 * s_1{}^{-1} * s_1{}' = s_1{}' * (s_1 * s_1{}^{-1}) * s_1{}' = s_1{}^{-1} * s_1{}' = 1$ and so $s_1{}^{-1} = s_1{}'$, i.e.,

$$(s^{-1} * s)_1(a) = c_1(a), \quad \text{for all} \quad a \in G.$$

We have to prove that $s_1{}^{-1}$ is a derivation map. Let $a \in G(x, y), b \in G(y, z)$. We write $f_2 s_1{}^{-1}(a) = (-s_1(a) - (s_1 s_0{}^{-1}(x))^a + (s_1 s_0{}^{-1}(y))^{-s_0(y)}$, and then

$$\begin{aligned} &f_2((s_1{}^{-1}(a))^b + s_1{}^{-1}(b)) \\ &= f_2(s_1{}^{-1}(a))^b + f_2(s_1{}^{-1}(b)) \\ &= (-s_1(b) + (f_2(s_1{}^{-1}(a))^{s_0(y)})^b + s_1(b))^{-s_0(z)} + f_2(s_1{}^{-1}(b)), \\ &= (-s_1(b)^{-s_0(z)} + (-s_1(a) - (s_1 s_0{}^{-1}(x))^a + s_1 s_0{}^{-1}(y))^{b-s_0(z)} + \\ &\quad +s_1(b)^{-s_0(z)}(-s_1 b)^{-s_0 z} - (s_1 s_0{}^{-1}(y)))^{b-s_0(z)} + s_1 s_0{}^{-1}(z)^{-s_0(z)} \\ &= (-s_1(b)^{-s_0(z)} - s_1(a)^{b-s_0(z)} - (s_1 s_0{}^{-1}(x))^{a+b-s_0(z)} + s_1 s_0{}^{-1}(z))^{-s_0(z)} \\ &= (-s_1(b) - s_1(a)^b - (s_1 s_0{}^{-1}(x))^{a+b} + s_1 s_0{}^{-1}(z))^{-s_0(z)} \\ &= (-(s_1(a)^b + s_1(b)) - (s_1 s_0{}^{-1}(x))^{a+b} + s_1 s_0{}^{-1}(z))^{-s_0(z)} \\ &= f_2(s_1{}^{-1}(a + b)) \end{aligned}$$

Hence $s_1{}^{-1}$ is a free derivation. □

Proof of Theorem 2.6. Now, we will show that

$$\begin{aligned} &FDer^*(\mathcal{C}) \times Aut(\mathcal{C}) \to FDer^*(\mathcal{C}) \\ &(s, f) \mapsto s^f \end{aligned}$$

is an action of $Aut(\mathcal{C})$ on $FDer^*(\mathcal{C})$.

$$s^f(z) = \begin{cases} {f_1}^{-1} s_0 f_0(z), & z \in X, \\ {f_2}^{-1} s_1 f_1(z), & z \in G. \end{cases}$$

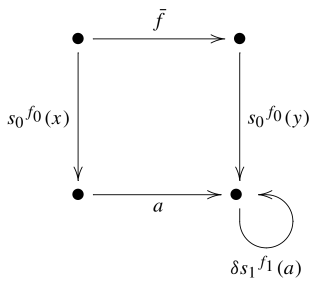
In fact this give rise to an action over a groupoid:

$$s^{fg}(z) = \begin{cases} (fg)_0{}^{-1} s_0 (fg)_0(z) & = {g_0}^{-1} {f_0}^{-1} s_0 f_0 g_0(z) = {g_0}^{-1} {s_0}^{f_0} g_0(z) \\ & = ({s_0}^{f_0})^{g_0}(z), \quad z \in X, \\ (fg)_1{}^{-1} s_1 (fg)_1(z) & = {g_1}^{-1} {f_1}^{-1} s_1 f_1 g_1(z) = {g_1}^{-1} {s_0}^{f_1} g_1(z) \\ & = ({s_1}^{f_1})^{g_1}(z), \quad z \in G \end{cases}$$

and

$$s^I(z) = \begin{cases} I^{-1} s_0 I(z) = s_0(z), & z \in X, \\ I^{-1} s_1 I(z) = s_1(z), & z \in G. \end{cases}$$

Let $s : f \simeq I$. Then $s^f : \bar{f} \simeq I$. We can show s^f as the following diagram:



Is $s^f \in FDer^*(\mathcal{C})$? Clearly one can see $\beta {s_0}^{f_0} x = x$ and $\beta {s_1}^{f_1}(a) = \beta(a)$. Also we should have to show that $s^f(a+b) = s^f(a)^b + s^f(b)$. We have

$$\bar{f}(a+b) = {s_0}^{g_0}(x) + (a+b) + \delta s^f(a+b) - {s_0}^{f_0}(z)$$

by definition of $\bar{f}$ and

$$\begin{aligned} \bar{f}(a) + \bar{f}(b) &= {s_0}^f(x) + a + \delta {s_1}^f(a) + b + \delta {s_1}^f(b) - {s_1}^f(z) \\ &= {s_0}^f(x) + a + b - b + \delta {s_1}^f(a) + b + \delta {s_1}^f(b) - {s_1}^f(z) \\ &= {s_0}^f(x) + a + b + \delta ({s_1}^f(a)^b + {s_1}^f(b) - {s_0}^f(z) \\ &= \bar{f}(a+b). \end{aligned}$$

So $s^f(a)^b + s^f(b) = s^f(a+b)$ and also we can obtain

$$I(a) = -s_0(x) + \bar{f}(a) + {s_0}^f(y) - \delta s^f(a).$$

$$\begin{aligned} f^{-1} \Delta s f(a) &= f^{-1}(s_0 f(x) + f(a) + \delta_s(fa) - {s_0}^f(y)) \\ &= f^{-1} s_0 f(x) + f^{-1} f(a) + f^{-1} \delta_s(fa) - f^{-1} {s_0}^f(y)) \\ &= {s_0}^f(x) + a + \delta f^{-1} s_1 f(a) - {s_0}^f(y) \\ &= {s_0}^f(x) + a + \delta {s_1}^f(a) - {s_0}^f(y) \\ &= \Delta(s^f)(a). \end{aligned}$$

Hence $\Delta(s^f)(a) = f^{-1}\Delta sf(a)$. □

Proof of Theorem 2.8. The group structure on $M_2(\mathcal{C})$ is pointwise multiplication. If $\zeta(s_2) = (s_0, s_1)$ then s_1 is a derivation since

$$\begin{aligned} s_1(a+b) &= (-s_2(x))^{(a+b)} + s_2(z) \\ &= (-s_2(x))^{(a+b)} + s_2(y)^{(b)} - s_2(y)^{(b)} + s_2(z) \\ &= ((-s_2(x))^{(a)} + s_2(y))^{(b)} - s_2(y)^{(b)} + s_2(z) \\ &= (s_1(a))^{(b)} + s_1(b). \end{aligned}$$

The action of $FDer^*(\mathcal{C})$ on $M_2(\mathcal{C})$ is

$$(s_2)^{(t_0,t_1)} = {s_2}^{t_0} : x \mapsto (s_2x)^{t_0x}$$

and we have to prove
(i) $\zeta((s_2)^{(t_0,t_1)}) = (t_0, t_1)^{-1} * \zeta(s_2) * (t_0, t_1)$,
(ii) $(s_2)^{\zeta(t_2)} = (t_2)^{-1} * (s_2) * (t_2)$.
Then $\zeta(s_2)^{(t_0,t_1)} = \zeta({s_2}^{t_0}) = (\delta({s_2}^{t_0}), t')$ say where if $a : x \to y$

$$\begin{aligned} t'(a) &= (-{s_2}^{t_0}(x))^a + {s_2}^{t_0}(y), \quad a \in G(x, y) \\ &= (-s_2(x)^{t_0(x)})^a + s_2(y)^{t_0(y)}, \\ &= -s_2(x)^{t_0(x)+a} + ({s_2}^{t_0})(y). \end{aligned}$$

On the other hand,

$$\begin{aligned} (t_0, t_1)^{-1} * \zeta(s_2) * (t_0, t_1) &= ({t_0}^{-1}, {t_1}^{-1}) * (\delta(s_2), s_1) * (t_0, t_1), \\ &= ({t_0}^{-1} * \delta(s_2) * t_0, {t_1}^{-1} * s_1 * t_1), \\ &= (\delta({s_2}^{t_0}), {t_1}^{-1} * s_1 * t_1). \end{aligned}$$

So we have to show that $({t_1}^{-1} * s_1 * t_1)(a) = -s_2(x)^{t_0(x)+a} + ({s_2}^{t_0})(y)$. Clearly $\zeta(s_2) * (t_0, t_1) = (\delta(s_2), s_1) * (t_0, t_1) = (\delta(s_2) * t_0, s_1 * t_1)$ and $(s_1 * t_1)(a) = t_1(a) + s_1(a) = t_1(a) + (-s_2(x)^a + s_2(y))^{t_0(y)}$. Then

$$\begin{aligned} &({t_1}^{-1} * s_1 * t_1)(a) \\ &\quad = (s_1 * t_1)(a) + ({t_1}^{-1}(a))^{(\delta(s_2)*t_0)(y)} \\ &\quad = t_1(a) + (-s_2(x)^a + s_2(y))^{t_0(y)} + ({t_1}^{-1}(a))^{(\delta(s_2)*t_0)(y)} \\ &\quad = t_1(a) - s_2(x)^{a+t_0(y)} + s_2(y)^{t_0(y)} + (-t_1(a)^{-t_0(y)})^{(\delta(s_2)*t_0)(y)} \\ &\quad = t_1(a) - s_2(x)^{a+t_0(y)} + s_2(y)^{t_0(y)} + (-t_1(a))^{-t_0(y)+(\delta(s_2))*t_0(y)} \\ &\quad = t_1(a) - s_2(x)^{a+t_0(y)} + s_2(y)^{t_0(y)} + (-t_1(a))^{(\delta({s_2}^{t_0})(y)} \\ &\quad = t_1(a) - s_2(x)^{a+t_0(y)} + s_2(y)^{t_0(y)} - s_2(y)^{t_0(y)} - t_1(a) + s_2(y)^{s_0(y)} \\ &\quad = -s_2(x)^{a+t_0(y)-\delta(t_1(a))} + s_2(y)^{t_0(y)} \\ &\quad = -s_2(x)^{t_0(x)+a} + {s_2}^{t_0}(y) \\ &\quad = t'(a). \end{aligned}$$

This proves (i).

To prove (ii) we note that

$$\begin{aligned}(s_2)^{\zeta(t_2)} &= (s_2)^{(\delta(t_2),t_1)}\\ &= ({s_2}^{\delta(t_2)})\\ &= ({t_2}^{-1} * s_2 * t_2)\end{aligned}$$

as is required. □

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