On relative homotopy groups of the product filtration, the James construction, and a formula of Hopf*

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For Alex Heller on his 65^{th} birthday

Abstract

We compute the nth relative homotopy group of (Z_n, Z_{n-1}) when Z_* is a product of certain filtered spaces. A consequence is information on the homotopy of $\Omega\Sigma X$ when X is the classifying space of a crossed complex.

Introduction

For a filtered space $X_* = \{X_0 \subseteq X_1 \subseteq \cdots\}$, the relative homotopy groups $\pi_n X_* = \pi_n (X_n, X_{n-1}, a)$ with $a \in X_0$, $n \geqslant 2$, together with the fundamental groupoid $\pi_1 (X_1, X_0)$ and the usual actions and boundaries, form the *fundamental crossed complex*[1] $\Pi(X_*)$; see [2] and [6]-[9]. For $X = \mathrm{colim} X_*$ and $Y = \mathrm{colim} Y_*$ we obtain the usual *product filtration* $X_* \otimes Y_*$ of the topological product $X \times Y$ by

$$(X_* \otimes Y_*) \quad = \bigcup_{i+j=n} X_i \times Y_j. \tag{1}$$

Here the product is taken in a convenient category of spaces.

In this paper we deal with the following problem: Is it possible to compute the relative homotopy groups $\Pi(X \otimes Y)_n$ of the product filtration $X_* \otimes Y_*$ in terms of the relative homotopy groups $(\Pi X_*)_p$, $(\Pi Y_*)_q$ of X_* and Y_* respectively?

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[1]In this version we use ΠX_* for the fundamental crossed complex rather than πX_* as in the original version. This is in keeping with current usage.

If X and Y are CW-complexes with the skeletal filtration X^* and Y^*, we have the isomorphism of crossed complexes

$$\theta : \Pi X^* \otimes \Pi Y^* \cong \Pi (X^* \otimes Y^*), \tag{2}$$

where $X^* \otimes Y^* = (X \otimes Y)^*$ is the skeletal filtration of the product $X \times Y$ and $\Pi X^* \otimes \Pi Y^*$ denotes the tensor product of crossed complexes introduced in [8] - indeed this isomorphism is a basic motivation for the definition of the tensor product $C \otimes C'$ of crossed complexes C, C'. The generators of ΠX^* are the cells e in X and the isomorphism θ in (2) carries the generator $e \otimes f$ of the tensor product to the product cell $e \times f$.

The main purpose of this paper is the generalisation of (2) to a wider class of filtered spaces. We say that a filtered spaceX_* is *cofibred* if for all $n \geqslant 0$ the inclusion $X_n \to X_{n+1}$ is a closed cofibration. Moreover, X_* is *connected* if for $i \geqslant 0$ the induced function $\pi_0 X_0 \to \pi_0 X_i$ is surjective and if for $n \geqslant 1$ the pair (X_n, X_{n-1}) is $(n-1)$-connected.

Theorem 0.1 (Product Theorem) *For filtered spaces X_*, Y_* there is a unique natural transformation*

$$\theta : \Pi X_* \otimes \Pi Y_* \to \Pi (X_* \otimes Y_*)$$

which for CW-complexes coincides with θ in (2). Moreover, θ is associative. If X_* and Y_* are cofibred and connected, then so is $X_* \otimes Y_*$, and θ is an isomorphism.

We also give various applications of this result. In particular, we shall compute for a 2-type X the homotopy group $\pi_3 \Sigma X$. Recall that in [10], a Generalized Van Kampen Theorem is applied to yield for any connected space X an exact sequence

$$\pi_2 X \to \pi_3 \Sigma X \to \pi_1 X \bar{\otimes} \pi_1 X \to \pi_1 X \to 1,$$

where $-\bar{\otimes}-$ is the tensor product of groups each acting on the other defined in [10]. In this paper we shall determine $\pi_3 \Sigma X$ completely in terms of any crossed module representing the 2-type of X. For this we use the classifying space BC of a crossed complex C in the sense of Brown and Higgins [9]. If $C = (C_2 \to C_1)$ is just a crossed module we obtain from the computation of $\pi_3 (\Sigma BC)$ a formula for the second homology $H_2 (BC)$. This result uses the James construction on crossed complexes which was already considered in [3] and [4]. This formula also generalises a classical result of Hopf for $H_2 G = H_2 BG$.

1 Proof of the Product Theorem

We first show that there is at most one natural transformation θ as described in the Product Theorem. For this consider the closed *n-cell* $\mathbf{E}^n$ which is a CW-complex with the skeletal filtration, namely $E^0 = \{1\}$, $E^1 = \{0\} \cup \{1\} \cup e^1$, $E^n = e^0 \cup e^{n-1} \cup e^n$, $n \geqslant 2$. Now for any elements $[a] \in \Pi X_*$ and $[b] \in \Pi Y_*$ we obtain by naturality of θ the commutative diagram

$$\begin{array}{ccc} \Pi \mathbf{E}^p \otimes \Pi \mathbf{E}^q & \xrightarrow[\cong]{\theta} & \Pi (\mathbf{E}^p \otimes \mathbf{E}^q) \\ {\scriptstyle a_* \otimes b_*} \downarrow & & \downarrow {\scriptstyle (a \otimes b)_*} \\ \Pi X_* \otimes \Pi Y_* & \xrightarrow{\theta} & \Pi (X_* \otimes Y_*) \end{array} \tag{3}$$

where the top row is the canonical isomorphism in (2). For the identity 1_n of $\mathbf{E}^n$ we thus have the formula

$$\theta\left([a]\otimes[b]\right)=\left(a\otimes b\right)_*\theta\left([1_p]\otimes[1_q]\right), \tag{4}$$

which shows that there is at most one natural transformation θ as described in the Product Theorem.

It is laborious to show directly that this transformation is well defined by (4). It is shown in [9] that an easy construction of θ may be obtained by working in the category of ω-groupoids and using the equivalence of this monoidal closed category to that of crossed complexes [8]. The reason for this easy proof is that the former category is based on cubical sets, and cubes satisfy the formula $I^m\otimes I^n\cong I^{m+n}$.

Let K_* be a filtered CW-complex K, filtered by subcomplexes K_n which satisfy

$$K^n\subseteq K_n \text{ for } n\geqslant 0, \tag{5}$$

where K^n is the n-skeleton of K. Then clearly K_* is cofibred and connected. We now show the following:

Lemma 1.1 *The Product Theorem holds for filtered CW-complexes X_* and Y_* which satisfy condition (5).*

Proof Let X^* and Y^* be the skeletal filtrations of the CW-complexes X and Y respectively. By condition (5) we have filtered maps $i:X^*\to X_*$, $i:Y^*\to Y_*$, where X^* and Y^* denote the skeletal filtrations of X and Y. For a cell e in X, dim(e) denotes the dimension of e, while we write deg(e)$=n$ if e is a cell in $X_n\backslash X_{n-1}$, so that deg $e\leqslant$ dim e. Using the characteristic map $f_e:\mathbf{E}^d\to X^*$ of the cell e we obtain the generator

$$e=\left(f_e\right)_*[1_d]\in\Pi X^* \tag{6}$$

denoted also by e. We also have the induced morphism

$$i_*=\Pi\left(i\right):\Pi\left(X^*\right)\twoheadrightarrow\Pi\left(X_*\right) \tag{7}$$

which satisfies $i_*\left(e\right)=0$ if and only if deg $e<$ dim e. It follows easily from the exact sequences of the triples $\left(X_n,X^n,X^{n-1}\right)$ and $\left(X_n,X_{n-1},X^{n-1}\right)$ that i_* in (7) is surjective. An element a in ΠX^* is called *degraded* if $i_*a=0$. Clearly, if a is degraded, so is δa.

We now consider the product $P=X\times Y$ which is a filtered CW-complex with $P_n=\left(X_*\otimes Y_*\right)_n$ and which satisfies $P^n\subseteq P_n$ by the assumptions on X_* and Y_*. For product cells $e\times f$ in P we have

$$\begin{aligned}\deg\left(e\times f\right)&=\deg\left(e\right)+\deg\left(f\right),\\ \dim\left(e\times f\right)&=\dim\left(e\right)+\dim\left(f\right).\end{aligned} \tag{8}$$

It follows that if $e\times f$ is degraded, then one of e, f is degraded.

We now consider the following diagram, in which P_n^* is the skeletal filtration of P_n, so that for example $P_n^n = P^n$:

$$\begin{array}{ccccccccc}
\pi_{n+1}\left(P_n^{n+1}, P^n\right) & \xrightarrow{q_n} & \pi_{n+1}\left(P_n, P^n\right) & \xrightarrow{\partial} & \pi_n\left(P^n, P^{n-1}\right) & \xrightarrow{i} & \pi_n\left(P_n, P^{n-1}\right) & \xrightarrow{j} & \pi_n\left(P_n, P_{n-1}\right) \\
 & & & & \uparrow{\scriptstyle \theta^* \,\cong} & & \uparrow{\scriptstyle i'} & \searrow{\scriptstyle \tau} & \uparrow{\scriptstyle \theta_*} \\
 & & & & \left(\Pi X^* \otimes \Pi Y^*\right)_n & & \xrightarrow{i_* \otimes i_*} & & \left(\Pi X_* \otimes \Pi Y_*\right)_n \\
 & & & & \uparrow{\scriptstyle i_{n-1}} & & | & & \\
 & & & & \pi_n\left(P_{n-1}^n, P^{n-1}\right) & \xrightarrow{q_{n-1}} & \pi_n\left(P_{n-1}, P^{n-1}\right) & &
\end{array}$$

In this diagram, q_n is surjective, by the exact sequence of the triple $\left(P_n, P_n^{n+1}, P^n\right)$, and $\pi_{n+1}\left(P_n^{n+1}, P^n\right)$ is generated by degraded product cells $e\times f$ of dimension $n+1$. But $\left(\theta^*\right)^{-1}\delta q_n\,(e\times f)$ is a sum of terms involving $\delta e\otimes f$ and $e\otimes\delta d$, so that $\left(i_*\otimes i_*\right)\left(\theta^*\right)^{-1}\delta q_n = 0$. By exactness of the row at $\pi_n\left(P^n, P^{n-1}\right)$, there is a morphism τ, as in the diagram, such that $\tau i = \left(i_*\otimes i_*\right)\left(\theta^*\right)^{-1}$. Now $\left(i_*\otimes i_*\right)i_{n-1} = 0$, since $\pi_n\left(P_{n-1}^n, P^{n-1}\right)$ is generated by degraded product cells $e\times f$. Hence $\tau i' = 0$. By exactness of the sequence $\xrightarrow{i'}\cdot\xrightarrow{j}$, there is a morphism $\bar\theta : \pi_n\left(P_n, P_{n-1}\right)\to\left(\Pi X_*\otimes Y_*\right)_n$ such that $\bar\theta_j = \tau$. Then $\theta_*\bar\theta = 1$. But $\bar\theta$ is surjective, by commutativity of the diagram. So θ_* is an isomorphism.

Finally, that $\pi_i\left(P_n, P_{n-1}\right) = 0$ for $i<n$ is proved by a similar argument. This completes the proof of the lemma. □

Proof of the Product Theorem The fact that $X_*\otimes Y_*$ is cofibred is a consequence of the product theorem for cofibrations.

We say that a map $f : K_*\to X_*$ between filtered spaces is a *weak equivalence* if $f_n : K_n\to X_n$ is a weak homotopy equivalence in each degree $n\geqslant 0$ and if f_0 is surjective. Clearly a weak homotopy equivalence induces a map $f_* : \Pi K_*\to\Pi L_*$ which, restricted to each point $a\in K_0$, in an isomorphism. For weak equivalences f as above, and $g : L_*\xrightarrow{\sim} Y_*$, between cofibred filtered objects, the tensor product $f\otimes g : K_*\otimes L_*\xrightarrow{\sim} X_*\otimes Y_*$ is also a weak equivalence. This follows since $\left(X_*\otimes Y_*\right)_n$ can be obtained as a colimit of a diagram in which only products $X_i\times Y_j$ occur. Moreover, $X_*\otimes Y_*$ is cofibred by the union theorem for cofibrations; see [11]. Using the well-known method of CW-approximations we see that for a connected cofibred filtered space X_* there exists a filtered CW-complex K_* as in (5) together with a weak equivalence $f : K_*\xrightarrow{\sim} X_*$. In the same way we obtain a CW-approximation $g : L_*\xrightarrow{\sim} Y_*$ where L_* satisfies (5). So the Product Theorem is now a consequence of the special case in Lemma 1.1. □

We point out that in degree 1 the Product Theorem is also a consequence of the Seifert-Van Kampen theorem. For this we observe that we have for cofibred and connected filtered spaces X_*, Y_* the pushout diagram of pairs

$$\begin{array}{ccc}
\left(X_0\times Y_0, X_0\times Y_0\right) & \longrightarrow & \left(X_0\times Y_1, X_0\times Y_0\right) \\
\downarrow & & \downarrow \\
\left(X_1\times Y_0, X_0\times Y_0\right) & \longrightarrow & \left(P_1, P_0\right)
\end{array}$$

where $P_* = X_* \otimes Y_*$. By applying the fundamental groupoid functor π to this diagram we obtain the isomorphism

$$\theta : (C_1 \rightrightarrows C_0) \cong \Pi\,(P_1, P_0) \tag{9}$$

where $C = \Pi X_* \otimes \Pi Y_*$, $C_0 = P_0 = X_0 \times Y_0$. This is the degree-1 part of the isomorphism in the Product Theorem.

2 On the James construction

In this section we apply the Product Theorem to the James construction of a filtered space. For this we need the following notion of a 'free monoid':

Definition 2.1 *Let* $(\mathsf{C}, \otimes)$ *be a monoidal category with a terminal object* $*$ *satisfying* $X \otimes * = X = * \otimes X$ *for* $X \in \mathsf{C}$*. Let* $(X, *)$ *be a pointed object in* C*, i.e. an object* X *with a morphism* $0 : * \to X$*. Then we get for the n-fold tensor product* $X^{\otimes n}$ *the maps* $(1 \leqslant t \leqslant n)$

$$i_t : X^{\otimes(n-1)} \to X^{\otimes(n)}$$

given by $i_t = X^{\otimes(t-1)} \otimes 0 \otimes X^{\otimes(n-t)}$*. These maps define the diagram*

$$* \longrightarrow X \rightrightarrows X^{\otimes 2} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X^{\otimes 3} \cdots$$

the colimit of which in C *is written* $J(X)$*. In fact, if the bifunctor* $\otimes$ *preserves the colimits used for the definition of* $J(X)$*, then* $J(X)$ *becomes a monoid in* C *(with respect to* $\otimes$*), and the morphism* $X \to J(X)$ *makes* $J(X)$ *the free monoid on the pointed object* $(X, *)$*. In case* $\mathsf{C} = \mathsf{Top}$ *is a convenient category of topological spaces with* $\otimes$ *defined by the product, and* X *is a pointed space, then* $J(X)$ *is the classical James construction or infinite reduced product of* X*. The topological monoid* JX *is homotopy equivalent to the loop space*

$$JX \cong \Omega\Sigma X \tag{10}$$

provided X *is path-connected and* $* \to X$ *is a cofibration [11].*

Now let C be the category of filtered objects in Top (see [2], Chapter III, Section 1). Then the filtered product of our Introduction is a tensor product as in Definition 2.1 and the James construction JX_* of a pointed filtered space X_* is a filtered space with

$$\operatorname{colim}(JX_*) = JX, \quad X = \operatorname{colim} X_*. \tag{11}$$

For $x \in X_n \backslash X_{n-1}$ we write $\deg(x) = n$. Then $(JX_*)_n$ consists of all words $x_1 \dots x_i$ with $\deg(x_1) + \cdots + \deg(x_i) \leqslant n$, $x_j \neq *$, $i \geqslant 0$. On the other hand, for a pointed crossed complex A the free monoid JA is defined by the tensor product of Brown and Higgins [8] which was used in the Product Theorem. The next result is an application of this theorem.

Theorem 2.2 *For a pointed filtered space X_* there is natural transformation*

$$\eta : J(\Pi X_*) \to \Pi(JX_*).$$

Moreover, if X_* is cofibred and connected, then so is JX_* and the natural transformation η is an isomorphism.

Proof The natural transformation θ in the Product Theorem is essentially the identity if X_* or Y_* is the base point $*$, and so θ is compatible with the diagram in Definition 2.1. This yields the transformation η in Theorem 2.2.

Suppose now that X_* is cofibred and connected. By the Product Theorem, we know that $X_*^{\otimes n}$ is cofibred and connected. The construction of JX_* by successive coequalisers and unions then shows that JX_* is cofibred.

We now apply the Van Kampen Theorem for the fundamental crossed complex of a filtered space [7], Theorem C. This is stated in terms of a filtered space Y_* and an open cover $\mathsf{U} = \{U^\lambda\}$ of Y such that U is closed under finite intersection and each $U^\lambda_* = U^\lambda \cap Y_*$ is 'homotopy full'. This last condition is equivalent to the connected condition, as is shown by manipulations with homotopy exact sequences of triples. The Van Kampen theorem states that the diagram

$$\bigsqcup_{(\lambda,\mu)} \Pi(U^\lambda \cap U^\mu)_* \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_\lambda \Pi U^\lambda_* \xrightarrow{\ c\ } \pi Y_*,$$

in which a, b, c are induced by the maps $U^\lambda \cap U^\mu \to U^\lambda$, $U^\lambda \cap U^\mu \to U^\mu$, $U^\lambda \to Y$, is a coequaliser diagram of crossed complexes. A consequence (not stated in [7]) is that Y_* also is connected.

By applying homotopy colimit methods, one finds that the fundamental crossed complex preserves colimits obtained by pushouts of a cofibration, and by unions of cofibrations, for connected cofibred filtered spaces. This proves Theorem 2.2. □

In case X_* is the skeletal filtration of a reduced CW-complex X, i.e. one with $X^0 = *$, we see also that $(JX)_*$ is the skeletal filtration of the CW-complex JX. In this case Theorem 2.2 coincides with the special case given in Theorem C6 of Chapter III of [3]. Moreover , if X_* is simply the pair $* \subseteq BG$ where BG is the classifying space of a group G, then ΠX_* is the reduced crossed complex consisting of G concentrated in degree 1. In this case we get by Theorem 2.2 the isomorphism

$$\eta : J(G) \cong \Pi J_*(BG), \tag{12}$$

where $J_*(BG) = J(X_*)$ is the word-length filtration in $J(BG)$. This special case of Theorem 2.2 is proved in [4] by different methods. The paper [4] investigates the properties of the 'crossed tensor algebra' $J(G)$ of the non-abelian group G.

We now determine the first two terms of JC. First, if $\mu : M \to P$, $\nu : N \to P$ are crossed P-modules, then their coproduct in the category of crossed P-modules will be written $\kappa : M \circ_P N \to P$. This construction is studied in [5] (and written $M \circ N$) and in [12], where it is called the *Pfeiffer product* (and written $M \bowtie N$). It is shown in [5] that $M \circ_P N$ may be represented as a quotient of either of the semi-direct products $M \ltimes N$ or $M \rtimes N$ by the subgroup $\{M, N\}$ generated by the elements

$$(-m + m^n, -n + n^m)$$

for all $m \in M$, $n \in N$ (where M and N operate on each other *via* P), and that $\kappa(m,n) = (\mu m)(\nu n)$.

Proposition 2.3 *If C is a reduced crossed complex, then the first two terms of $J(C)$ in dimensions 2 and 1 form the crossed module*

$$C_2 \circ_{C_1} J_2(C_1) \xrightarrow{\kappa} C_1,$$

the coproduct of the crossed C_1-modules

$$\delta : C_2 \to C_1 \text{ and } \delta_J : J_2(C_1) \to J_1(C_1) = C_1.$$

Proof The proof is by verification of the universal property for this part of $J(C)$. So let A be a reduced crossed complex which is a monoid in the category of crossed complexes, i.e. is equipped with a multiplication $A \otimes A \to A$ with the usual monoid properties. Let $f : C \to A$ be a morphism of crossed complexes. Then $f_1 : C_1 \to A_1$ extends uniquely to a morphism of crossed complexes $J(C_1) \to A$. This, with $f : C \to A$, determines uniquely a morphism over f_1 of crossed modules

$$C_2 \circ_{C_1} J_2(C) \to A_2.$$

This completes the proof. □

Recall that if C is a reduced crossed complex then for $n \geqslant 2$ we define

$$H_n(C) = \mathrm{Ker}(\delta_n : C_n \to C_{n-1})/\mathrm{Im}(\delta_{n+1} : C_{n+1} \to C_n),$$

while $\pi_1 C = \mathrm{Cok}(\delta_n : C_2 \to C_1)$. It is known [4] that if G is a group regarded as a crossed complex with G concentrated in dimension 1, then

$$H_2(JG) = \mathrm{Ker}(G \bar{\otimes} G \to G). \tag{13}$$

The next result gives information on $H_2(JC)$ when C is the crossed module $i : K \hookrightarrow E$ with i the inclusion of a normal subgroup of E. We write this crossed module (crossed complex) as $K \lhd E$.

Theorem 2.4 *If $K \lhd E$ is an inclusion of a normal subgroup, then there is a commutative diagram with exact rows*

$$\begin{array}{ccccc}
H_2(JE) & \longrightarrow & H_2(J(K \lhd E)) & \twoheadrightarrow & (K \cap [E,E])/[K,E] \\
\downarrow & & \downarrow & & \| \\
H_2(E) & \longrightarrow & H_2(J(K \lhd E))/Q(E) & \twoheadrightarrow & (K \cap [E,E])/[K,E]
\end{array}$$

where $Q(E)$ is generated by classes of cycles e^2 for $e \in E$.

Proof Let $L = (K \cap [E,E])/[K,E]$. Let C be the crossed complex consisting of $i : K \lhd E$ as its crossed module part and trivial elsewhere. By Proposition 2.3, the 2-cycles of JC are represented by elements (k,x) of the semi-direct product $K \rtimes J_2(E)$, such that $ik = -\delta_J x$. Define the morphism

$$\phi : K \rtimes J_2(E) \to K \cap [E,E]$$

by $(k,x) \mapsto \delta_J x$. Then $\phi(-k+k^x, -x+x^k) \in [K,E]$, so that ϕ defines a morphism $\psi : K \circ_E J_2(E) \to L$.

Next we verify that ψ vanishes on boundaries. According to (1.4) of [4], $J_3(C)$ is generated by elements ke and z for $k \in K$, $e \in E$, $z \in J_3(E)$. Then according to (6) of (1.4) of [4],

$$\delta(ke) = (ik)e - (-k + k^e),$$

so that

$$\phi\delta(ke) = -i(-(-k + k^e)) \in [K, E].$$

On the other hand,

$$\psi(\delta z) = \delta^2 z = 0.$$

It follows that ψ defines a morphism $\tau : H_2(C) \to L$. This morphism is surjective since if $k \in K \cap [E, E]$, then $k = \delta x$ where $x \in J_2(E)$, and so $(-k, x)$ represents an element of $H_2(JC)$ mapped by τ to k. The morphism $\sigma : H_2(JE) \to H_2(JC)$ is induced by the inclusion $E \to C$. Clearly $\tau\sigma = 0$. Suppose now that $(k, x) \in K \rtimes J_2(E)$ represents a cycle in $J_2(C)$ such that $\delta x \in [K, E]$. We also write (k, x) as $k + x$. At this stage, we have to be careful about notation. We write k for an element of K considered as an element of $C_2 = K$, and we write ik for the same element considered as an element of $C_1 = E$. Then the condition on δx implies that there are a finite number of elements k_r of K and e_r of E such that

$$\delta x = \sum_r (-(ik_r)^{e_r} + ik_r).$$

Then $y = \sum_r k_r e_r$ is an element of $J_3(C)$ and

$$\delta y = \sum_r (ik_r)e_r + \sum_r (-k_r^{e_r} + k) = z + w,$$

say. But $ik = -\delta x = -iw$. So $k + x + \delta y = z$, and z is a cycle in $J_2(E)$. This completes the proof of exactness of the first row.

The exactness of the second row follows except for the identification of $H_2(JE)/Q(E)$ with $H_2(E)$. This identification is a consequence of the formula (13) and the description due to [15] of $H_2(E)$ as $\ker(E \wedge E \to E)$, where $E \wedge E$ is the quotient of $E \tilde{\otimes} E$ by the subgroup generated by all $e \otimes e$ for $e \in E$. See [10] for more information on this. □

3 Classifying spaces

Brown and Higgins in [9] study the *classifying space* BC of a crossed complex C. This is defined by $BC = |NC|$, where the *nerve* NC of C is the simplicial set such that $(NC)_n = \mathsf{Crs}(\Pi\Delta^n, C)$, the set of crossed complex morphisms $\Pi\Delta^n \to C$. For a crossed complex $C = G$ consisting of a group G concentrated in degree 1, this classifying space coincides with the usual classifying space BG of G, which we used in (12). A different, but homotopy equivalent, construction of the classifying space of a crossed module is given in [13].

It is known that for a reduced crossed complex C we have

$$\pi_1 BC \cong \pi_1 C, \quad \pi_n BC \cong H_n C \quad (n \geqslant 2).$$

Let $C_{\leqslant n}$ be the subcomplex of C which coincides with C in degree $\leqslant n$ and is zero in degree $> n$. Then BC is actually a filtered CW-complex $(BC)_*$ with $(BC)_n = B(C_{\leqslant n})$. Moreover, the filtered space $(BC)_*$ is connected and satisfies (see [1], p. 40)

$$\Pi(BC)_* \cong C. \tag{14}$$

Using this result we derive from Theorem 2.2 the following theorem:

Theorem 3.1 *For a pointed crossed complex C one has the natural isomorphism*

$$J(C) \cong \Pi J(BC)_*. \qquad \square$$

For a group $C = G$ this is just the formula described in (12).

The main property of the classifying space BC is the following homotopy classification formula [9], which generalises classical results on maps into an Eilenberg-MacLane space:

$$[X, BC] \cong [\Pi X^*, C]. \tag{15}$$

Here X is a CW-complex and the left-hand side is a set of homotopy classes in Top. The right hand side is a set of homotopy classes of maps in the category of crossed complexes; see [8].

By (15), we see that the path component of $BC_{\leqslant n}$ containing $*$ is actually an n-type. To this end recall that an *n-type* X is a path-connected CW-space with $\pi_i(X) = 0$ for $i > n$. Let $\mathsf{2\text{-}types}$ be the full subcategory of $\mathsf{Top}/\cong$ consisting of 2-types. It was shown by MacLane and Whitehead [14] that a 2-type is algebraically represented by a *crossed module*. Each crossed module C gives us a pointed crossed complex C which is concentrated in degree 1 and 2. Moreover, the classifying space B of Brown and Higgins actually yields an equivalence of categories

$$B : \mathrm{Ho}(\mathsf{Crs}^{(2)}) \xrightarrow{\sim} \mathsf{2\text{-}types}. \tag{16}$$

Here $\mathsf{Crs}^{(2)}$ is the category of crossed modules and $\mathrm{Ho}(\mathsf{Crs}^{(2)})$ is the localisation with respect to weak equivalences in $\mathsf{Crs}^{(2)}$. The equivalence (16) in fact goes back to [14]; compare also [13] and [2]. On the other hand, a 2-type X is represented by its k-invariant

$$k_X \in H^3(\pi_1 X, \pi_2 X) \tag{17}$$

and it is well known how to represent the cohomology class k_X by a crossed module C for which the sequence

$$\pi_2 X \rightarrowtail C_2 \xrightarrow{\partial} C_1 \twoheadrightarrow \pi_1 X \tag{18}$$

is exact. Any such C satisfies $B(C) \simeq X$.

As an application of Theorem 3.1, we get the following result on the homotopy groups $\pi_n \Sigma X$ of a suspended 2-type. Clearly $\pi_2 \Sigma X = (\pi_1 X)^{ab}$ is the abelianisation of the fundamental group.

Theorem 3.2 *Let X be a 2-type which is represented by a crossed module C, that is, $BC \simeq X$. Then there is a natural isomorphism*

$$\pi_3 \Sigma X \cong H_2(JC)$$

and a natural surjection $\pi_4 \Sigma X \twoheadrightarrow H_3(JC)$. Here JC is the James construction of the crossed module C in the category of crossed complexes and $H_n(JC)$ denotes the homology of the (reduced) crossed complex JC.

When $X = BG$ is the classifying space of a group G we obtain by Theorem 3.2 the isomorphism $\pi_3\Sigma BG \cong H_2(JG)$, where $H_2(JG) \cong \mathrm{Ker}(G\bar{\otimes}G \to G)$ as in the Introduction. This special case of Theorem 3.2 is considered in [4] and [10].

For the proof of Theorem 3.2 we use the following concept of 'certain exact sequence' in the sense of Whitehead [16]. Let X_* be a connected filtered space with $X_0 = *$. Then we have by (III.10.7) in [2] the exact sequence

$$\begin{aligned} \cdots \quad &\to \quad \Gamma_3 X_* \to \pi_3 X \to H_3(\Pi X_*) \\ &\to \quad \Gamma_2 X_* \to \pi_2 X \to H_2(\Pi X_*) \to 0, \end{aligned} \tag{19}$$

where $\Gamma_n X_* = \mathrm{Im}(\pi_n X_{n-1} \to \pi_n X_n)$.

Proof of Theorem 3.2 We consider the filtered space $X_* = J(BC)_*$ where $(BC)_*$ is filtered by $* \to BC_1 \to BC = (BC)_2$. Whence $X_1 = B(C_1)$ and thus $\Gamma_2 X_* = 0$. This implies the result in Theorem 3.2 by use of (19) and (10). □

Corollary 3.3 *Let $K \hookrightarrow E \twoheadrightarrow G$ be a short exact sequence of groups. Then there is an isomorphism of exact sequences*

$$\begin{array}{ccccc} \pi_3\Sigma BE & \longrightarrow & \pi_3\Sigma BG & \longrightarrow & (K \cap [E,E])\,/\,[K,E] \\ \downarrow\cong & & \downarrow\cong & & \| \\ Ker(E\bar{\otimes}E \to E) & \longrightarrow & Ker(G\bar{\otimes}G \to G) & \twoheadrightarrow & (K \cap [E,E])\,/\,[K,E] \end{array}$$

Proof This follows from (13), the first row of the exact sequence of Theorem 2.4, Theorem 3.2, and the fact that the canonical map $B(K \lhd E) \to BG$ is a homotopy equivalence. □

Whitehead's exact sequence [16] is the special case of (19) when the filtered space is the skeletal filtration of a reduced CW-complex. In particular, if X is a reduced CW-complex, then Whitehead's sequence for the 1-connected space ΣX yields the exact sequence

$$\cdots \to \pi_4\Sigma X \to H_3 X \to \Gamma H_1 X \xrightarrow{k} \pi_3\Sigma X \to H_2 X \to 0. \tag{20}$$

Here the isomorphism k is exactly the first k-invariant of the space ΣX, and k represents an element

$$k \in H^4(K(\pi_2, 2), \pi_3) = \mathrm{Hom}(\Gamma_2, \pi_3)$$

with $\pi_n = \pi_n\Sigma X$, $\pi_2 = H_2\Sigma X = H_1 X$.

Suppose now that $X = BC$ where C is a reduced crossed complex. Using the isomorphism in Theorem 3.2, the homomorphism

$$k : \Gamma H_1 X \to \pi_3\Sigma X \cong H_2(JC) \tag{21}$$

is obtained as follows. We have $H_1 X = (\pi_1 X)^{ab}$ with $\pi_1 X = \mathrm{Cok}(C_2 \to C_1)$. Whence elements $[c] \in H_1 X$ are represented by elements $c \in C_1$. Now k is induced by the quadratic map $H_1 X \to H_2(JC)$ which carries $[c]$ to the homology class $[c^2]$ of the cycle c^2 in JC. This result gives us by (20) the following possibility to compute the first two $\mathbb{Z}$-homology groups of a 2-type $X \simeq BC$.

Theorem 3.4 *Let C be a crossed module and let BC be the classifying space of C. Then $H_1(BC)$ is the abelianisation of $\pi_1 C = Cok(C_2 \to C_1)$. Moreover, the homology group $H_2(BC)$ is given by the formula*

$$H_2(BC) = H_2(JC)/Q(C),$$

where $Q(C)$ is the subgroup generated by all classes $\{c^2\}$, $c \in C_1$. □

Consider now a short exact sequence $K \hookrightarrow E \twoheadrightarrow G$. The morphism $E \to G$ induces a homotopy equivalence of classifying spaces $B(K \lhd E) \to BG$. So Theorem 3.4 with Theorem 2.4 yields an exact sequence $H_2(E) \to H_2(G) \to (K \cap [E,E])/[K,E] \to 1$. In the case E is free, so that $H_2(E) = 0$, this is the classical result of Hopf.

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