On the Freudenthal Suspension Theorem, the Blakers-Massey Theorem, and excision for triad homotopy groups

Ronald Brown

August 17, 2014

Abstract

We explain how the Freudenthal Suspension Theorem, the Blakers-Massey Theorem, and excision for triad homotopy groups, are all related to higher order Seifert-van Kampen Theorems.

1 The Freudenthal Suspension Theorem and the Blakers-Massey Theorem

I would like to explain why I see the Freudenthal Suspension Theorem (FST) as related to a Higher Seifert-van Kampen Theorem (HSvKT).

It was Blakers and Massey, [BM51, BM52, BM53], who explained the relation of the FST to the homotopy theory of triads $\mathsf{X} = (X; A, B)$ where X is a pointed space with subspaces A, B containing the base point. They gave an exact sequence

$$\cdots \to \pi_{n+1}(X; A, B) \to \pi_n(A, A \cap B) \xrightarrow{\varepsilon} \pi_n(X, B) \to \pi_n(X; A, B) \to \cdots \tag{1}$$

Thus in the case $X = A \cup B$ the groups $\pi_n(X; A, B)$ $(n \geqslant 3)$ and based set $(n = 2)$ give the *obstructions to excision* for relative homotopy groups.

A case in point is the triad $S\mathsf{X} = (SX; C^+X, C^-X)$ called the *suspension triad*. Then the above exact sequence is easily transformed into

$$\cdots \to \pi_{n+1}(S\mathsf{X}) \to \pi_n(X) \xrightarrow{\sigma} \pi_{n+1}(SX) \to \pi_n(S\mathsf{X}) \to \cdots \tag{2}$$

including the suspension morphism σ. So we want to know conditions for the triad groups to be trivial, and to determine at least the first non trivial group, often called the *critical group*.

The main theorem of Blakers-Massey is:

Theorem 1.1 *Suppose the triad* $\mathsf{X} = (X; A, B)$ *is such that:*

(i) *the interiors of* A, B *cover* X;

(ii) *A, B and $C = A \cap B$ are connected;*

(iii) *C is simply connected; and*

(iv) *(A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected, $m, n \geqslant 3$.*

Then $\mathsf{X} = (X; A, B)$ is $(m+n-2)$-connected; further, if C is simply connected then the morphism given by the generalised Whitehead product

$$\pi_m(A, C) \otimes \pi_n(B, C) \to \pi_{m+n-1}(X; A, B) \tag{3}$$

is an isomorphism.

Notice that since the tensor product is zero if one of its factors is zero, this result also gives criteria for the excision morphism ε to be injective or surjective in a certain range of dimensions. This excision isomorphism in a range of dimensions is also called the *excision theorem of Blakers and Massey*, though it would be better called a *connectivity Theorem*, and has been given quite separate proofs, for example in [Hat01, Theorem 4.23] and [tD08, [Theorem 6.4.1], assuming only $m, n \geqslant 2$ and without the assumption (iii). In algebraic topology, it is preferable to have *algebraic* results rather than just connectivity results.

The natural question is what happens to the algebraic isomorphism (3) if the conditions that $m, n \geqslant 3$ and C simply connected are weakened. For example in the case $m = n = 2$ we have additional structure that the morphisms $\pi_2(A, C) \to \pi_1(C), \pi_2(B, C) \to \pi_1(C)$ are crossed modules, and so the required relative homotopy groups are in general nonabelian. If $m > 2, n > 2$ then $\pi_m(A, C), \pi_n(B, C)$ are still $\pi_1(C)$-modules.

The extension to the non simply connected case was given by Brown and Loday in [BL84, BL87]; one simply replaces the usual tensor product by the nonabelian tensor product of groups[1] which act on each other and on themselves by conjugation. This result is a special case of Seifert-van Kampen Theorem for n-cubes of spaces. Notice that the assumption (i) of the theorem is reminiscent of such a type of theorem. The useful fact is that one gets such a theorem for a certain kind of *structured space* which allows for the development of algebraic structures which have structures in a range of dimensions.

Thus one of the intuitions is that the Blakers-Massey Theorem, and hence also the FST, is of the Seifert-van Kampen type, since we are assuming that X is the union of the interiors of A, B. In effect the triviality conclusion is because a tensor product of objects is trivial if one of the objects is trivial.

One of the intuitions behind the Brown-Loday approach derives from the well known fact that

> changes in homotopy in low dimensions usually
> affect high dimensional homotopy information.

[1]The writer compiled a bibliography on this nonabelian tensor product, and which currently has 131 items, with many authors. The url is `http://pages.bangor.ac.uk/~mas010/nonabtens.html`.

A standard example is to take $S^n \vee [0,1]$ and then identify $0,1$ to give $S^n \vee S^1$; there is a considerable change in π_n.

Now the groupoid SvKT, [Bro67], surprised me since it completely determined $\pi_1(X,x)$ as a vertex group of a groupoid $\pi_1(X,A)$, where A was a set of base points, chosen according to the geometry of the situation, and even when X was the union of many non-connected subspaces with non-connected intersection. To determine the group required simply combinatorial information and work[2]. This method worked, it seemed, because groupoids had structure in dimensions 0 as well as 1, and so could code relevant information. This groupoid $\pi_1(X,A)$ had a nice colimit presentation, not involving choices, except that of A, while there was not necessarily a "nice" presentation of $\pi_1(X,x)$. Could this be a model for higher homotopy results?

So it seemed desirable to have homotopical invariants which had structure in a range of dimensions, in order properly to model the gluing of spaces. Some traditional invariants, such as homotopy groups, might then be extracted from the bigger structure, possibly with difficulty.

In the case in point, we need to assign to a triad an algebraic structure covering the dimensions concerned. This was provided by Loday and Guin-Waléry, [GWL81], see also [Lod82], in dimension 3 with the notion of *crossed square*. The crossed square in question is given by

$$\begin{array}{ccc} \pi_3(X;A,B) & \to & \pi_2(B,C) \\ \downarrow & & \downarrow \\ \pi_2(A,C) & \to & \pi_1(C) \end{array}$$

with the additional structure of:

(i) the operation of $\pi_1(C)$ on all the groups in the square,and

(ii) a map
$$h : \pi_2(A,C) \times \pi_2(B,C) \to \pi_3(X;A,B)$$
given by a generalised Whitehead product,

(i) and (ii) all satisfying suitable axioms. This structure assigned to $(X;A,B)$ we write $\Pi_3(X;A,B)$. So Π_3 is regarded as a functor

$$(\text{triads}) \to (\text{crossed squares}).$$

What sort of colimits might this preserve? What are available?

It turns out that a triad is for these purposes more conveniently regarded as a square of spaces

$$\begin{array}{ccc} C & \to & B \\ \downarrow & & \downarrow \\ A & \to & X \end{array}$$

[2] The most general version of this theorem is in [BR84], with a cover by open sets such that A meets each 1-, 2-, 3-fold intersections of the sets of the cover.

where C is any pointed subspace of $A \cap B$, and which we abbreviate to

$$\begin{matrix} C & B \\ A & X\,. \end{matrix}$$

.

Here is a nice point: if X is the union of the interiors of A and B, with $C = A \cap B$, we can regard the following square as a pushout of squares of spaces:

$$\begin{matrix} \begin{matrix} C & C \\ C & C \end{matrix} & \to & \begin{matrix} C & B \\ C & B \end{matrix} \\ \downarrow & & \downarrow \\ \begin{matrix} C & C \\ A & A \end{matrix} & \to & \begin{matrix} C & B \\ A & X \end{matrix} \end{matrix}$$

Is this analogous to taking in one's own washing?

In any case, we are getting nearer to a SvKT situation. The next question is: what axioms is it reasonable to impose on crossed squares? Any change in the axioms to a nonequivalent system would necessarily give a change of colimits, so it is important to get this right.

A *crossed square* is a commutative diagram of morphisms of groups

$$\begin{matrix} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\nu} & P \end{matrix} \tag{4}$$

together with left actions of P on L, M, N and a function $h\colon M \times N \to L$ satisfying a number of axioms. These are that the morphisms in the square preserve the action of P, which acts on itself by conjugation; M, N act on each other and on L via P; $\lambda, \lambda', \mu, \nu$ and $\mu\lambda$ are crossed modules; and h satisfies axioms reminiscent of commutator rules, summarised by saying it is a *biderivation*. The further axioms are: λ, λ' are P-equivariant; and

$$h(mm', n) = h({}^{m}m', {}^{m}n)h(m, n), \quad h(m, nn') = h(m, n)h({}^{n}m, {}^{n}n'),$$

$$\lambda h(m, n) = m({}^{n}m^{-1}), \quad \lambda' h(m, n) = ({}^{m}n)n^{-1},$$

$$h(\lambda l, n) = l({}^{n}l)^{-1}, \quad h(m, \lambda' l) = ({}^{m}l)l^{-1},$$

$$h({}^{p}m, {}^{p}n) = {}^{p}h(m, n),$$

for all $l \in L, m, m' \in M, n, n' \in N, p \in P$.

Morphisms of crossed squares are defined in the obvious way, giving a category XSq of crossed squares.

This rather elaborate set of axioms raises a number of questions. One of them is: how do we know this is the "right" set of axioms, and what should be the criterion for that? Another is that if we do wish to use colimits of such gadgets, how, since there are a lot of axioms to verify, do we verify that some construction gives a colimit?

The solutions to these problems were in essence given by Loday and Guin-Waléry, [GWL81], as follows. A crossed square should be thought of as a *crossed module of crossed modules*. But what does this mean?

Well, crossed modules are equivalent to group objects in the category of groupoids, [BS76], where the crossed module $\mu : M \to P$ is changed to the groupoid with source and target maps $s, t : M \rtimes P \to P$, where $s : (m,p) \mapsto p, t : (m,p) \mapsto (\mu m)p$, and $\rtimes$ here denotes semidirect product. We thus require that the induced morphism

$$(L \rtimes N) \to (M \rtimes P)$$

also should be a crossed module. For this we want that we should have a group object in the category of double groupoids of the form

$$(L \rtimes N) \rtimes (M \rtimes P).$$

Thus the axioms for a crossed square should be such that the category of crossed squares is equivalent to the category of *cat^2-groups*, i.e. of group objects in the category of double groupoids. This is the result of Guin-Waléry and Loday.

Now for the Blakers-Massey Theorem for the triad group, we find ourselves considering a pushout of crossed squares of the form

$$\begin{array}{ccc} \begin{matrix} 1 & 1 \\ 1 & P \end{matrix} & \to & \begin{matrix} 1 & N \\ 1 & P \end{matrix} \\ \downarrow & & \downarrow \\ \begin{matrix} 1 & 1 \\ M & P \end{matrix} & \to & \begin{matrix} L & N \\ M & P \end{matrix} \end{array}$$

This pushout in this category ensures that L is determined by M, N, P and their crossed module properties, and that L has to be a "home" for an h-map in a universal way. Because of the biderivation properties of h, we write L as $M \otimes N$. It is quite a nice point that in this case the other properties of $h : M \times N \to M \otimes N$ follow from the biderivation properties: for the proof, one writes $h(m,n)$ as $m \otimes n$ and then expands $mm' \otimes nn'$ in two ways. See [BL87].

The functor which sends a pair of crossed modules $(M \to P, N \to P)$ to the crossed square

$$\begin{array}{ccc} M \otimes N & \to & N \\ \downarrow & & \downarrow \\ M & \to & P \end{array}$$

is left adjoint to the forgetful functor from crossed squares to such pairs of crossed modules.

Crossed squares are one of a number of algebraic models of homotopy 3-types. See for example the paper [AU06]. Baues in [Bau91] uses extensively the notion of *quadratic module*. Graham Ellis in his paper [Ell93] makes a comparison between crossed squares and quadratic modules which for our purposes we have slightly rewritten as follows:

(i) Crossed squares have an obvious extension to higher dimensions in which crossed n-cubes, as defined in [ES87], are the appropriate generalisation of crossed squares. The algebra involved is complicated, but complications are not unexpected in homotopy theory.

(ii) Crossed squares have a geometric interpretation in terms of relative and triadic homotopy groups; no such interpretation is available for quadratic modules.

(iii) Baues's theory is related to that of crossed squares via a quotient functor

$$q_3 : (\text{free crossed squares}) \to (\text{free quadratic modules}),$$

but it is not clear if this functor has a left adjoint, so we do not know if colimits of quadratic modules are relevant to homotopy theory considerations. In fact, the paper [AU06, Section 7] gives a functor

$$(\text{crossed squares}) \to (\text{quadratic modules}),$$

but again it is not clear if this functor has a left adjoint and so preserves colimits.

(iv) Non-free groups of operators can be handled.

2 Excision and crossed modules

A triad $\mathsf{X} = (X; A, B)$ is called *excisive* of X is the union of the interiors of A, B. Note that this is a typical situation for a Seifert-van Kampen type theorem.

We now have to discuss *induced crossed squares*. For this, we need also to refer to *induced crossed modules*. For a general discussion of this notion in the context of fibrations and cofibrations of categories, and *cocartesian morphisms* see [BHS11, Appendix B].

Suppose given a crossed module $\mu : M \to P$ and a morphism of groups $f : P \to Q$. We would like to get another crossed module $\lambda : f_*(M) \to Q$ with an appropriate universal property. This is most easily expressed by saying that the following diagram

$$\begin{tikzcd}
(1 \to P) \arrow[r, "f"] \arrow[d] & (1 \to Q) \arrow[d] \\
(M \to P) \arrow[r] & (f_*(M) \to Q)
\end{tikzcd}$$

should be a pushout of crossed modules.

We now explain how this notion arises homotopically.

Theorem 2.1 *Let $(X; A, B)$ be an excisive triad and let $C = A \cap B$. Suppose the based spaces C, A, B are path connected and the pair (B, C) is 1-connected. Let $\lambda : \pi_1(C) \to \pi_1(B)$ be the morphism induced by inclusion. Then the pair (X, A) is 1-connected and the natural morphism*

$$\lambda_*(\pi_2(B, C)) \to \pi_2(X, A)$$

is an isomorphism.

One applies a SvKT theorem to a pushout of pairs of spaces

$$\begin{tikzcd}
(C,C) \arrow[r] \arrow[d] & (B,B) \arrow[d] \\
(B,C) \arrow[r] & (X,A).
\end{tikzcd}$$

For more details, see see [BHS11]. The result was originally proved in [BH78].

It is convenient to extend the above result to adjunction spaces, especially to see its relation to Whitehead's theorem on free crossed modules. So we suppose (B, C) is a cofibred pair of based spaces, $f : C \to A$ is a map and $X = A \cup_f B$. So the following square

$$\begin{tikzcd}
(C,C) \arrow[r, "f"] \arrow[d] & (A,A) \arrow[d] \\
(B,C) \arrow[r] & (X,A)
\end{tikzcd}$$

and under the connectivity assumptions we have a pushout of crossed modules

$$\begin{tikzcd}
(1 \to \pi_1(C)) \arrow[r, "f_*"] \arrow[d] & (1 \to \pi_1(A)) \arrow[d] \\
(\pi_2(B,C) \to \pi_1(C)) \arrow[r] & (\pi_2(A \cup_f B, A) \to \pi_1(A)).
\end{tikzcd}$$

In the special case when C is a wedge of circles and B is the cone on C we obtain Whitehead's theorem on free crossed modules, [Whi49]; see [Bro80] for an exposition of Whitehead's proof[3].

[3]That proof uses transversalty and knot theory. The referee of this paper wrote: "The theorem is not new, the proof is not new, but the paper should be published since the original papers are notoriously difficult to read." In fact, every step of this exposition is in the original papers, but spread over three papers, so essentially the exposition is a repackaging of the proof for modern readers. See also [Hue12]

3 Excision and crossed squares

How can we extend Theorem 2.1 to crossed squares and triads? First we apply the SvKT for crossed squares and the methods of [BL87a] to obtain the following theorem.

Theorem 3.1 *We consider excisive triads* $\mathsf{A} = (A; A_+.A_-), \mathsf{B} = (B; B_+, B_-)$ *and a map* $f : \mathsf{A} \to \mathsf{B}$*, with* $\mathsf{X} = (X; A_+, A_-)$ *another triad, not excisive, but we assume that* (X, A) *is a cofibred pair of based spaces. Then we form* $Y = B \cup_f X$ *and we are attempting to describe* $\pi_3(Y; B_+, B_-)$ *in terms of the other information available. Under the appropriate connectivity assumptions, we already know the crossed squares* $\Pi(\mathsf{A}), \Pi(\mathsf{B})$ *in terms of nonabelian tensor products.*

If A, B *are connected excisive triads and* X *is a connected triad, then* $(B \cup_f X; B_+, B_-)$ *is connected and the following square is a pushout of crossed squares:*

$$\begin{tikzcd}
\Pi(A : A_+, A_-) \arrow[r, "f_*"] \arrow[d] & \Pi(B : B_+, B_-) \arrow[d] \\
\Pi(X : A_+, A_=) \arrow[r] & \Pi(B \cup_f X; B_+, B_-).
\end{tikzcd}$$

Note that under the assumptions of the theorem, this gives complete information on the 3-type of the space $Y = B \cup_f X$.

A detailed description of the induced crossed square $\Pi(Y; B_+, B_-)$ in terms of the other information now follows from Theorem B.3.2 of [BHS11], which derives from [BL84].

The relation of this result to those of Ellis in [Ell93] where he uses *free crossed squares* is that he is dealing with the particular situation $(A : A_+, A_-) = (S^2 : E^2_+, E^2_-)$, where E^2_+, E^2_- are the two hemispheres of S^2; then there is a homeomorphism $\eta : (A : A_+, A_-) \to (A : A_-, A_+)$, and this data is used to give other descriptions of the induced crossed square.

The following theorem shows how to get specific homotopy information from a crossed square.

A crossed square G has a classifying space BG: for more details see [Lod82], [Bro92] and also [AU06].

Theorem 3.2 *Let* G *be the crossed square* (4). *Then the homotopy groups of* BG *may be computed as the homology groups of the non-Abelian chain complex*

$$L \xrightarrow{(\lambda^{-1}, \lambda')} M \rtimes N \xrightarrow{\mu * \nu} P \tag{5}$$

where $\mu * \nu : (m, n) \mapsto (\mu m)(\nu n)$. *This implies that*

$$\pi_i BG \cong \begin{cases} P/(\mu M)(\nu N) & \text{if } i = 1, \\ (M \times_P N)/\{(\lambda l, \lambda' l) : l \in L\} & \text{if } i = 2, \\ (\operatorname{Ker} \lambda) \cap (\operatorname{Ker} \lambda') & \text{if } i = 3, \\ 0 & \text{if } i \geqslant 4. \end{cases} \tag{6}$$

The first Postnikov invariant of BG is the cohomology class determined by the crossed module

$$(M \rtimes N)/Im(\lambda^{-1}, \lambda') \xrightarrow{\mu * \nu} P.$$

Further, under the above isomorphisms, the composition $\eta^* : \pi_2 BG \to \pi_3 BG$ with the Hopf map $\eta : S^3 \to S^2$ is induced by the function $M \times_P N \to L$, $(m, n) \mapsto h(m, n)$, and the Whitehead product $\pi_2 \times \pi_2 \to \pi_3$ on BG is induced by the function $((m, n), (m', n')) \mapsto h(m', n)h(m, n')$.

References

[AU06] Arvasi, Z. and Ulualan, E. 'On algebraic models for homotopy 3-types'. *J. Homotopy Relat. Struct.* **1** (1) (2006) 1–27 (electronic).

[Bau91] Baues, H. J. *Combinatorial homotopy and 4-dimensional complexes*, *de Gruyter Expositions in Mathematics*, Volume 2. Walter de Gruyter & Co., Berlin (1991). With a preface by Ronald Brown.

[BM51] Blakers, A. L. and Massey, W. S., 'The homotopy groups of a triad. I', *Ann. of Math. (2)*, **53**, (1951) 161–205.

[BM52] Blakers, A. L. and Massey, W. S., 'The homotopy groups of a triad. II', *Ann. of Math. (2)*, **55**, (1952) 192–201.

[BM53] Blakers, A. L. and Massey, W. S., 'The homotopy groups of a triad. III', *Ann. of Math. (2)*, **58**, (1953) 409–417.

[Bro67] Brown, R. 'Groupoids and van Kampen's theorem'. *Proc. London Math. Soc.* **17** (3) (1967) 385–401.

[Bro80] Brown, R. 'On the second relative homotopy group of an adjunction space: an exposition of a theorem of J. H. C. Whitehead'. *J. London Math. Soc. (2)* **22** (1) (1980) 146–152.

[Bro84a] Brown, R. 'Coproducts of crossed P-modules: applications to second homotopy groups and to the homology of groups'. *Topology* **23** (3) (1984) 337–345.

[Bro89] Brown, R. 'Triadic Van Kampen theorems and Hurewicz theorems'. In 'Algebraic topology (Evanston, IL, 1988)', *Contemp. Math.*, Volume 96. Amer. Math. Soc., Providence, RI (1989), 39–57.

[Bro92] Brown, R. 'Computing homotopy types using crossed n-cubes of groups'. In 'Adams Memorial Symposium on Algebraic Topology, 1 (Manchester, 1990)', *London Math. Soc. Lecture Note Ser.*, Volume 175. Cambridge Univ. Press, Cambridge (1992), 187–210.

[BH78] Brown, R. and Higgins, P. J. 'On the connection between the second relative homotopy groups of some related spaces'. *Proc. London Math. Soc. (3)* **36** (2) (1978) 193–212.

[BHS11] Brown,R., Higgins, P.J. and Sivera, R. *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids*, EMS Tracts in Mathematics Vol. 15, (2011).

[BL84] Brown, R. and Loday, J.-L. 'Excision homotopique en basse dimension'. *C.R. Acad. Sci. Paris Sér. I* **298** (1984) 353–356.

[BL87] Brown, R. and Loday, J.-L. 'Van Kampen theorems for diagrams of spaces'. *Topology* **26** (3) (1987) 311–335. With an appendix by M. Zisman.

[BL87a] Brown, R. and Loday, J.-L. 'Homotopical excision, and Hurewicz theorems for n-cubes of spaces'. *Proc. London Math. Soc. (3)* **54** (1) (1987) 176–192.

[BR84] Brown, R. and Razak Salleh, A. 'A van Kampen theorem for unions of non-connected spaces'. *Arch. Math.* **42** (1984) 85–88.

[tD08] tom Dieck, T. *Algebraic topology.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich (2008).

[BS76] Brown, R. and Spencer, C. B. 'G-groupoids, crossed modules and the fundamental groupoid of a topological group'. *Nederl. Akad. Wetensch. Proc. Ser. A* **79**=*Indag. Math.* **38** (4) (1976) 296–302.

[Die89] Dieudonné, J. *A history of algebraic and differential topology. 1900–1960.* Birkhäuser Boston Inc., Boston-Basel, MA (1989).

[Ell93] Ellis, G. 'Crossed squares and combinatorial homotopy'. *Math. Z.* **214** (1) (1993) 93–110.

[ES87] Ellis, G. J. and Steiner, R. 'Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads'. *J. Pure Appl. Algebra* **46** (2-3) (1987) 117–136.

[GWL81] Guin-Waléry, D. and Loday, J.-L. 'Obstruction à l'excision en K-théorie algébrique'. In 'Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)', *Lecture Notes in Math.*, Volume 854. Springer, Berlin (1981), 179–216.

[Hat01] Hatcher, A. *Algebraic Topology* Cambridge University Press, 2002.

[Hue12] Huebschmann, J. 'Braids and crossed modules'. *J. Group Theory* **15** (2012) 57-83.

[Lod82] Loday, J.-L. 'Spaces with finitely many nontrivial homotopy groups'. *J. Pure Appl. Algebra* **24** (2) (1982) 179–202.

[MLW50] Mac Lane, S. and Whitehead, J. H. C. 'On the 3-type of a complex'. *Proc. Nat. Acad. Sci. U. S. A.* **36** (1950) 41–48.

[Whi49] Whitehead, J. H. C. 'Combinatorial homotopy. II'. *Bull. Amer. Math. Soc.* **55** (1949) 453–496.