CORRIGENDUM TO

RONALD BROWN AND OMAR ANTOLIN CAMARENA

(communicated by Fred Cohen)

Abstract
Omar Antolin Camarena pointed out a gap in the proofs in [BT&C, Bro06] of a condition for the Phragmen–Brouwer Property not to hold; this note gives some background and the correction.

1. Introduction

This note is concerned with a gap in the proofs in [BT&C, Bro06] that if a connected space $X$ does not have the Phragmen-Brouwer Property, then its fundamental group has the group of integers as a retract.

One source of confusion is that there are two forms of this Property, as follows.

Let $X$ be a topological space. We say $X$ separates if it has more than one component, and a subspace $D$ of $X$ separates $X$ if $X \setminus D$ separates. We say a subset $D$ of $X$ separates the points $a$ and $b$ of $X$ if $a$ and $b$ lie in distinct components of $X \setminus D$.

A topological space $X$ is said to have the Phragmen-Brouwer Property I, here abbreviated to PBI), if $X$ is connected and the following holds:

$\text{PBI): if } D \text{ and } E \text{ are disjoint, closed subsets of } X \text{, and if } a \text{ and } b \text{ are points in } X \setminus (D \cup E) \text{ which lie in the same component of } X \setminus D \text{ and in the same component of } X \setminus E \text{, then } a \text{ and } b \text{ lie in the same component of } X \setminus (D \cup E).$

Thus the PBI) is that: if $D$ and $E$ are disjoint closed subsets of $X$ and $a, b$ are points of $X$ not in $D \cup E$ such that neither $D$ nor $E$ separate $a$ and $b$, then $D \cup E$ does not separate $a$ and $b$.

We also consider:

$\text{PBI'): If } D, E \text{ are disjoint closed subsets of } X \text{ such that neither } D \text{ nor } E \text{ separate } X, \text{ then } D \cup E \text{ does not separate } X.$

It is proved in [Wil49], pp.47-49, that these two properties are equivalent if $X$ is connected and locally connected, and for the convenience of readers we reproduce that proof in Section 3.

Received , revised ; published on .

2000 Mathematics Subject Classification: 20L05, 57N05

Key words and phrases: fundamental groupoid, van Kampen theorem, Phragmen–Brouwer Property, Jordan Curve Theorem

© , Ronald Brown and Omar Antolin Camarena. Permission to copy for private use granted.
It is of interest and importance to relate these properties to the fundamental group of the space $X$, see also $[Eil37]$ Section3. Such a relation was also used in $[BT&G]$ $[Bro06]$ as a contribution to a proof of the Jordan Curve Theorem. To this end, we assume all spaces in the next Section are locally path connected, so that connected and path connected are equivalent. The result required to relate PBI) to properties of the fundamental group or groupoid is given in the next Section.

The gap in the proof was in considering a pushout of groupoids

$$
\begin{array}{ccc}
C & \rightarrow & B \\
i & \downarrow & \downarrow v \\
A & \rightarrow & G
\end{array}
$$

such that $C$ is totally disconnected, and $A, B, G$ are connected. Under these assumptions a vertex group $G(p)$ of $G$ is calculated$^1$ in $[BT&G] 8.4.1$, $[Bro06]$ Proposition 3.3] and from this it is proved that $G(p)$ contains as a retract a free group which is non trivial if $C$ has more than one component. This is sufficient to get the required criterion for PBI$'$. However, in the application to the Jordan Curve Theorem, we really need PBI), as is assumed in the above references, and in this case we do not know that $A, B$ are connected.

So we offer two ways of making the correction. The first is just to use the result already mentioned that properties PBI) and PBI$'$) are equivalent. The second is to give an appropriate retraction result on such a pushout of groupoids in the case where $A$ and $B$ are not necessarily connected. This we give in the next Section in Theorem 2.1.

Partially analogous results on fundamental groups with applications to the Jordan Curve Theorem are given in $[Mun75]$ Theorem 63.1], which inspired our results. His proof uses a covering space argument, avoiding the groupoid techniques introduced in $[Bro67]$.

2. A result on a pushout of groupoids

The groupoid version of the Seifert-van Kampen Theorem, proved originally in $[Bro67]$, and stated as Theorem 2.1 of $[Bro06]$, and 6.7.2 of $[BT&C]$, allowed for non connected spaces, and produces a pushout of groupoids, rather than the standard form which produces a pushout of groups$^2$. So the situation of spaces may be something like a union as follows:

---

$^1$Such a formula for a fundamental group appears without proof in the paper $[Kam33]$.  
$^2$The result for a union of a family of open sets, rather than just two, is given in $[BRa84]$
Some cases of pushouts of groupoids are handled in the above references, but for our purposes we need the following Theorem. Recall that by the rank of a connected free groupoid we mean the rank as a free group of any of its vertex groups.

**Theorem 2.1.** Let

\[
\begin{array}{c}
C \\ i
\end{array} \rightarrow \begin{array}{c}
\downarrow j \\
B
\end{array} \rightarrow \begin{array}{c}
v \\
\downarrow \\
A \\ u
\end{array} \rightarrow \begin{array}{c}
\downarrow \\
G
\end{array}
\]

be a pushout of groupoids. We assume $C$ is totally disconnected, that $i, j$ are the identity on objects, and that $G$ is connected. Then $G$ contains as a retract a free groupoid $F$ of rank

\[ k = n_C - n_A - n_B + 1, \]

where $n_P$ is the number of components of the groupoid $P$ for $P = A, B, C$.

Further, if $C$ contains distinct objects $a, b$ such that $A(ia, ib), B(ja, jb)$ are nonempty, then $F$ has rank at least 1.

**Proof.** Let $Fr : \text{DirectedGraphs} \rightarrow \text{Groupoids}$ be the free groupoid functor. From [BT&G] Chapter 8 it follows that any groupoid $G$ has a retraction $G \rightarrow Fr W$ where $W$ is a forest.

Let $Z$ be the set of objects of $C$ (and of $A, B$ and $G$) regarded as a directed graph with no edges, and pick spanning forests $X$ and $Y$ of the underlying directed graphs of $A$ and $B$. Then there are retractions

\[ C \rightarrow Fr Z, A \rightarrow Fr X, B \rightarrow Fr Y. \]

By "span" in a category we mean a pair of arrows $U \leftarrow W \rightarrow V$; this is the shape of a diagram whose pushout you can take. Then the following diagram in $\text{Groupoids}$ commutes and its rows are spans:

\[
\begin{array}{c}
Fr X \\ \downarrow \\
A
\end{array} \leftarrow \begin{array}{c}
Fr Z \\ \downarrow \\
C
\end{array} \rightarrow \begin{array}{c}
Fr Y \\ \downarrow \\
B
\end{array}
\]

So the span $Fr X \leftarrow Fr Z \rightarrow Fr Y$ is a retract of the span $A \leftarrow C \rightarrow B$. This implies that the pushout, say $F$, of $Fr X \leftarrow Fr Z \rightarrow Fr Y$ is a retract of $G$ (which is the pushout of $A \leftarrow C \rightarrow B$). Since the span of free groupoids is actually the image under $Fr$ of the obvious span $X \leftarrow Z \rightarrow Y$ of graphs, and since $Fr$ is a left adjoint, this pushout $F$ is actually just $Fr W$ where $W$ is the pushout in the category of directed graphs of $X \leftarrow Z \rightarrow Y$.

This graph $W$ is connected because $G$ is connected, so, denoting by $e(Q)$ and $v(Q)$ the number of vertices of a graph, the vertex groups in $Fr W$ are free of rank $k = e(W) - v(W) + 1$. We have $v(W) = v(X) = v(Y) = v(Z) = n_C$; and, since $Z$ has no edges, $e(W) = e(X) + e(Y)$. Also, since $X$
is a spanning forest we have \( e(X) = v(X) - n_A = n_C - n_A \), and similarly, \( e(Y) = n_C - n_A \). Putting this all together, the vertex groups in \( F \) have rank \( (n_C - n_A) + (n_C - n_B) - n_C + 1 = n_C - n_A - n_B + 1 \), as claimed.

For the last part of the theorem, we choose \( X, Y \) so that the elements \( \alpha, \beta \) of \( A(ia, ib), B(jb, ja) \) respectively are parts of \( FrX, FrY \) respectively. These map to elements \( \alpha', \beta' \) in \( F \) and the element \( \alpha' \beta' \) will be nontrivial in \( F \); so \( F \) has rank at least 1.

This following Corollary is an essential part of the proof of the Jordan Curve Theorem. It appears, without the retraction condition, as part of \([\text{Mun75}]\) Theorem 63.1, and also as \([\text{BT&G}]\) 9.2.1, \([\text{Bro06}]\) Proposition 4.1.

**Corollary 2.2.** If the space \( X \) is path connected and does not have the PBI), then its fundamental group at any point contains the infinite cyclic group as a retract.

**Proof.** The proof now follows the methods of \([\text{Bro06}]\). If \( X \) is path connected and does not have the PBI) then there are disjoint closed subsets \( D, E \) of \( X \) and points \( a, b \) of \( X \setminus (D \cup E) \) which lie in the same component of \( X \setminus D \) and in the same component of \( X \setminus E \), but \( a \) and \( b \) do not lie in the same component of \( X \setminus (D \cup E) \). Let \( U = X \setminus D, V = X \setminus E, W = U \cap V \). Then \( a, b \) lie in different components of \( W \). Let \( J \) be a minimal set containing \( a, b \) and one point in each path component of \( W \). The groupoid version of the Seifert-van Kampen Theorem gives as in \([\text{Bro06}]\) Theorem 2.1, \([\text{BT&G}]\) 6.7.2, a pushout of groupoids to which we can apply Theorem 2.1; the last part of that Theorem gives our conclusion.

---

### 3. Equivalence of forms of the Phragmen-Brouwer Property

The following argument is reproduced for the convenience of readers from \([\text{Wil49}]\), pp. 47–49. A stronger result is given in \([\text{Hun74}]\).

**Theorem 3.1.** PBI) implies PBI'), and if \( X \) is connected and locally connected, then PBI') implies PBI).

**Proof.** That PBI) implies PBI') is trivial.

We now assume \( X \) is locally connected. This is well known to imply that all components of \( X \) are open in \( X \). It follows that:

\(^(*)\) If \( F \) is the boundary of a component \( C \) of an open set of \( X \), and \( C \neq X \), then \( X \setminus F = C \cup (X \setminus \overline{C}) \) is a separation, by which we mean that the equality expresses \( X \setminus F \) as a union of sets neither of which meets the closure of the other.

Suppose then \( X \) is connected and locally connected and satisfies PBI') but not PBI). Then \( X \) contains two points \( a, b \) and two disjoint closed sets \( D, E \) such that neither of \( D, E \) separate \( a, b \) but \( D \cup E \) does separate \( a, b \). Let \( C_a \) be the component of \( X \setminus (D \cup E) \) containing \( a \); then \( b \) does not belong to \( C_a \).

Let \( F \) be the boundary of \( C_a \). Then \( X \setminus F = C_a \cup (X \setminus \overline{C_a}) \) is a separation. Define sets \( F_1, F_2 \) by \( F_1 = D \cap F, F_2 = E \cap F \). Then \( F = F_1 \cup F_2 \) and neither \( F_1, F_2 \) are empty. For if \( F_1 \) is empty
then $F \subseteq E$ and (*) implies that $X \setminus E = C_a \cup (X \setminus C_a \setminus E)$ is a separation; and so $E$ separates $a, b$, contrary to hypothesis.

It follows by similar reasoning that the component $C_b$ of $X \setminus F$ that contains $b$ has limit points in both $F_1$ and $F_2$. Let 

$$E_i = C_b \cap F_i, i = 1, 2, E^* = E_1 \cup E_2.$$ 

Let $K_a$ denote the component of $X \setminus E^*$ which contains $C_a$. As every limit point of $F$ is a limit point of $C_a$, so must every point of $E^*$ be a limit point of $K_a$. Consequently, $E^*$ is the common boundary of the disjoint sets $K_a, C_b$. To $E_1$ let us add every component of $X \setminus E^*$ that has its boundary entirely in $E_1$ and call the resulting set $E'_1$. Similarly, form a set $E'_2$ relative to $E_2$.

The set $X \setminus E'_2$ is connected. For let $C$ be a component of this set, and suppose $C \cap E_2 = \emptyset$. Then $C \subseteq X \setminus E^*$. Let $C'$ be the component of $X \setminus E'$ which contains $C$. If the boundary $E'$ of $C'$ were entirely in $E_1$ then we would have 

$$C \subseteq C' \subseteq E'_1.$$ 

If $E'$ were entirely in $E_2$ then we would have 

$$C \subseteq C' \subseteq E'_2$$ 

and consequently, since $E'_2 \subseteq X \setminus E'_1$ we would have 

$$C' \subseteq X \setminus E'_1 \text{ and } C \cap E_2 \neq \emptyset.$$ 

Finally, were $E_1; \cap E_1 \neq \emptyset, E' \cap E_2 \neq \emptyset$ we would again have $C' \subseteq X \setminus E'_1$ and hence $C \cap E_2 \neq \emptyset$. Consequently, $C' \cap E_2 \neq \emptyset$. But 

$$K_a^* \cup C_b \cup E'_2$$ 

is a connected subset of $X \setminus E'_1$ and as every component $C$ of $X \setminus E'_1$ has limit points in $E'_2$ the set $X \setminus E'_1$ is connected. Similarly, $X \setminus E'_2$ is connected. But $X \setminus (E'_1 \cup E'_2)$ is not connected since $K_a, K_b$ are disjoint components of the latter set. This contradicts the fact that $X$ was supposed to satisfy $\text{PBI}'$.

For further information on this area see [Wil49, GMI89].

References


Ronald Brown  
School of Computer Science, Bangor University, UK

Omar Antolin Camarena  
Department of Mathematics, Harvard University, Boston, Mass. USA