HOPF FORMULAE FOR THE HIGHER HOMOLOGY OF A GROUP

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In this note we generalise Hopf's formula

$$H_2(G) \cong (R \cap [F, F])/[F, R]$$

for the second homology of a group G in terms of a free presentation $R \rightarrowtail F \twoheadrightarrow G$. We prove:

THEOREM 1. *Let $R_1, \ldots, R_n$ be normal subgroups of a group F such that $F/\prod_{1 \leqslant i \leqslant n} R_i \cong G$, and for each proper subset A of $\langle n \rangle = \{1, \ldots, n\}$ the groups $H_r(F/\prod_{i \in A} R_i)$ are trivial for $r = 2$ if $A = \varnothing$, and for $r = |A|+1$ and $|A|+2$ if $A \neq \varnothing$ (for example, the groups $F/\prod_{i \in A} R_i$ are free for $A \neq \langle n \rangle$). Then there is an isomorphism*

$$H_{n+1}(G) \cong \left\{\bigcap_{i=1}^{n} R_i \cap [F, F]\right\} \Big/ \{\prod_{A \subseteq \langle n \rangle} [\bigcap_{i \in A} R_i, \bigcap_{i \notin A} R_i]\}.$$

Here $|A|$ denotes the order of A, and $\prod_{i \in A} R_i$ is the subgroup of F generated by the subgroups R_i with $i \in A$ (in particular, $\prod_{i \in \varnothing} R_i$ is the trivial subgroup). Also $\bigcap_{i \in \varnothing} R_i$ is understood to mean F. Thus for $n = 2$ the formula reads

$$H_3(G) \cong \{R_1 \cap R_2 \cap [F, F]\}/\{[F, R_1 \cap R_2][R_1, R_2]\}.$$

Note that for any group G and $n \geqslant 1$, such an F and R_i can be found: let $F^1(G)$ be the free group on G; define inductively $F_i = F^1(F^{i-1}(G))$, and set $F = F^n(G)$; for $1 \leqslant i \leqslant n$ let $\varepsilon_i: F^n(G) \to F^{n-1}(G)$ denote the canonical homorphisms induced by applying F^{n-i} to the standard 'augmentation' map $F^1(F^{i-1}(G)) \to F^{i-1}(G)$ (where $F^0(G) = G$); and set $R_i = \operatorname{Ker} \varepsilon_i$.

An alternative method, analogous to methods in [**4, 5**], is best illustrated for $n = 2$. Choose any surjections $F_i \to G$ with F_i free, $i = 1, 2$. Let P be the pullback of these surjections and choose a surjection $F \to P$ with F free. Let R_i be the kernel of the composite $F \to P \to F_i$. In general, one constructs inductively an n-cube of groups F such that, for $A \subseteq \langle n \rangle$: (i) F_A is free if $A \neq \langle n \rangle$, (ii) F_A is G for $A = \langle n \rangle$, and (iii) the morphism $F_A \to \lim_{B \supset A} F_B$ is surjective. Such an n-cube might be called a *fibrant n-presentation* of G.

Again, suppose $G = F/HK$ where H and K are normal subgroups of F such that F, F/H and F/K are free. For example, we might be given a presentation $\langle X; U, V \rangle$ of

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G such that the normal closures of both U and V in the free group F on X are summands of F. Then $\langle X;\, U, V\rangle$ and the diagram of quotient morphisms

$$\begin{array}{ccc} F & \longrightarrow & F/H \\ \downarrow & & \downarrow \\ F/K & \longrightarrow & G \end{array}$$

could be called a 'double presentation' of G.

Our proof of Theorem 1 is topological and uses the *Hurewicz theorem for n-cubes of spaces* given in [2] which itself is an application of the *generalised Van Kampen Theorem* for diagrams of spaces proved in [1]. These theorems involve the 'fundamental cat^n-group of an n-cube of spaces'; we also need the equivalence between cat^n-groups and crossed n-cubes of groups proved in [3].

This presents a dilemma, because it is difficult to make the proof completely intelligible without saying a great deal. We hope therefore to say just enough to indicate the manner of the proof, and why these results do immediately imply Theorem 1.

The method is to introduce *hyper-relative homology groups* $H_i(Q;\, N_1, \dots, N_n)$ of a group Q relative to n normal subgroups $N_1, \dots, N_n$, $i \geqslant 1$. The definition is topological, in order to be able to apply the Hurewicz theorem. An algebraic definition is also possible, using bar resolutions. In the application, we will take $Q = F$, $N_i = R_i$.

To define the multirelative homology group, we define spaces $B(Q;\, N_1, \dots, N_n)$ inductively as follows. Let $B(Q) = K(Q, 1)$. Then $B(Q;\, N_1)$ is defined to be the mapping cone of the induced map $B(Q) \to B(Q/N_1)$. Thus $B(Q;\, N_1)$ contains $B(Q/N_1)$ as a closed subspace.

Suppose that $B(Q;\, N_1, \dots, N_{n-1})$ has been defined so as to be a functor of $(Q;\, N_1, \dots, N_{n-1})$. Then $B(Q;\, N_1, \dots, N_n)$ is defined to be the mapping cone of the induced map

$$B(Q;\, N_1, \dots, N_{n-1}) \to B(Q/N_n;\, N_1 N_n/N_n, \dots, N_{n-1} N_n/N_n).$$

Next we define the multirelative homology groups for $i \geqslant 1$ by

$$H_i(Q;\, N_1, \dots, N_n) = H_{n+i}(B(Q;\, N_1, \dots, N_n)).$$

PROPOSITION 2. *Let $n \geqslant 1$ and let $N_1, \dots, N_n$ be normal subgroups of the group Q. Then there is an isomorphism*

$$H_1(Q;\, N_1, \dots, N_n) \cong \left\{\bigcap_{i=1}^{n} N_i\right\} \Big/ \left\{ \prod_{A \subseteq \langle n\rangle} \left[\bigcap_{i \in A} N_i,\ \bigcap_{i \notin A} N_i\right]\right\}.$$

This proposition follows from Theorem 6.2 (the Hurewicz theorem) of [2] and the equivalence between cat^n-groups and crossed n-cubes of groups given in [3]. Here is some explanation.

Define an n-cube of spaces $\mathbf{X}$ by $X_A = B(Q/\prod_{i\in A} N_i)$, for $A \subseteq \langle n\rangle$, with maps $X_A \to X_{A\cup\{i\}}$ induced by the quotient map of groups. Let $\mathbf{M} = \prod \mathbf{X}$, the fundamental crossed n-cube of groups of $\mathbf{X}$. Then for $A \subseteq \langle n\rangle$, the group M_A is $\bigcap_{i\notin A} N_i$ with the maps of the cube being inclusion and the h-maps being commutators. Now the space $Y = B(Q;\, N_1, \dots, N_n)$ is the (multiple) mapping cone of the n-cube $\mathbf{X}$ (see below). Theorem 6.2 of [2] gives immediately that Y is n-connected, and also implies our stated description of $H_1(Q;\, N_1, \dots, N_n) = H_{n+1}(Y)$. In order to understand this it is best to restate Theorem 6.2 of [2] in terms of crossed n-cubes of groups [3] rather than cat^n-groups, as follows.

If $\mathbf{C}$ is a crossed n-cube of groups, then the crossed n-cube of groups triv$(\mathbf{C})$ is obtained from $\mathbf{C}$ by trivialising all groups C_A for $A \neq \langle n \rangle$. Hence (triv$(\mathbf{C}))_{\langle n \rangle}$, which is both the big and little group of the cat^n-group associated to triv$(\mathbf{C})$, is obtained from $C_{\langle n \rangle}$ by factoring out the images of all h-maps of $\mathbf{C}$ with codomain $C_{\langle n \rangle}$. Replacing $\mathbf{C}$ by the particular crossed n-cube of groups $\mathbf{M}$ gives Proposition 2.

In order to see how this gives Theorem 1, we first explain how the Hopf formula for H_2 follows from the relative Hurewicz theorem.

The exact homology sequence of the cofibration sequence

$$B(Q) \to B(Q/N) \to B(Q; N)$$

gives, if $G = Q/N$ and $H_2(Q) = 0$,

$$0 \to H_2(G) \to H_1(Q; N) \to H_1(Q) \to H_1(G) \to 0.$$

The relative Hurewicz theorem in this dimension gives that $H_1(Q; N)$ is isomorphic to $\pi_2(B(Q/N), B(Q))$ with the action of $Q = \pi_1(B(Q))$ killed. The exact homotopy sequence of the pair implies that this relative homotopy group is isomorphic to N. So we obtain the exact sequence

$$0 \to H_2(G) \to N/[N, Q] \to Q/[Q, Q] \to G/[G, G] \to 0$$

and this implies the Hopf formula (with $Q = F$, $N = R$).

In the general case, the assumptions of Theorem 1 and various exact homology sequences give the exact sequence

$$0 \to H_{n+1}(G) \to H_1(F; R_1, \ldots, R_n) \to H_1(F) \tag{1}$$

which with Proposition 2 gives Theorem 1. That is, we see that a generalised Hurewicz theorem yields a generalised Hopf formula. However, the derivation of (1) needs more explanation on the construction of the multiple mapping cone and on the associated exact sequences. This we now give.

We need some facts on n-cubes of cofibrations. Such n-cubes are the dual of n-cubes of fibrations which are dealt with in [5], so we just state the results here.

All spaces and maps will be pointed.

DEFINITION. An *n-cube of spaces* $\mathbf{X}$ is a commutative diagram consisting of spaces X_A ($A \subseteq \langle n \rangle$) and maps $X_A \to X_{A \cup \{i\}}$ ($i \notin A$). The n-cube of spaces $\mathbf{X}$ is *cofibrant* if the canonical maps

$$i_A\colon \operatorname*{colim}_{S \subset A} X_S \to X_A \quad (A \subseteq \langle n \rangle)$$

are closed cofibrations.

An *equivalence* $\mathbf{X} \to \mathbf{Y}$ of n-cubes of spaces is a map of n-cubes such that each $X_A \to Y_A$ is a homotopy equivalence.

An *n-cube of cofibrations* $\mathbf{X}$ is a commutative diagram consisting of spaces $X^{B,A}$ (B, A disjoint subsets of $\langle n \rangle$) and cofibration sequences

$$X^{B \setminus \{i\}, A} \to X^{B \setminus \{i\}, A \cup \{i\}} \to X^{B, A} \quad (i \in B). \tag{2}$$

(In a cofibration sequence $X \to Y \to Z$, $X \to Y$ is a closed cofibration and Z is the quotient space Y/X.) Thus a 0-cube of cofibrations is a space, a 1-cube of cofibrations is a cofibration sequence and a 2-cube of cofibrations is a commutative diagram

$$\begin{array}{ccccc}
X^{\varnothing,\varnothing} & \longrightarrow & X^{\varnothing,\{1\}} & \longrightarrow & X^{\{1\},\varnothing} \\
\downarrow & & \downarrow & & \downarrow \\
X^{\varnothing,\{2\}} & \longrightarrow & X^{\varnothing,\langle 2\rangle} & \longrightarrow & X^{\{1\},\{2\}} \\
\downarrow & & \downarrow & & \downarrow \\
X^{\{2\},\varnothing} & \longrightarrow & X^{\{2\},\{1\}} & \longrightarrow & X^{\langle 2\rangle,\varnothing}
\end{array}$$

whose rows and columns are cofibration sequences. The following are essentially the duals of Propositions 1, 2 of [**5**].

PROPOSITION 3. *A cofibrant n-cube of spaces* X *extends to an n-cube of cofibrations* X *such that* $X^{\varnothing, A} = X_A$.

PROPOSITION 4. *For every n-cube of spaces* X *there is a natural cofibrant n-cube of spaces* $\hat{X}$ *and an equivalence* $\hat{X} \to X$.

Propositions 3 and 4 give for every n-cube of spaces X an n-cube of cofibrations $\hat{X}$; the space $\hat{X}^{\langle n\rangle,\varnothing}$ is called the *multiple mapping cone* of the n-cube X.

We shall use repeatedly the homology exact sequence arising from the cofibration sequence (2). Our aim is the following result.

PROPOSITION 5. *Suppose that* X *is an n-cube of cofibrations, and that*

$$H_r X^{\varnothing, A} = 0 \quad \textit{if} \quad \begin{cases} r = 2 \quad \textit{and} \quad A = \varnothing, \\ r = |A|+1 \quad \textit{or} \quad |A|+2 \quad \textit{and} \quad A \neq \varnothing \quad \textit{or} \quad \langle n\rangle. \end{cases} \tag{3}$$

Then there is an exact sequence

$$0 \to H_{n+1} X^{\varnothing,\langle n\rangle} \to H_{n+1} X^{\langle n\rangle,\varnothing} \to H_1 X^{\varnothing,\varnothing}. \tag{4}$$

Proof. The proof is a diagram chase. The details are as follows.

Claim 1. If $B \cup A \neq \langle n\rangle$, then

$$H_r X^{B,A} = 0 \quad \text{if} \quad \begin{cases} r = |B \cup A|+2 \quad \text{and} \quad A = \varnothing, \\ r = |B \cup A|+1 \quad \text{or} \quad |B \cup A|+2, \quad \text{and} \quad A \neq \varnothing. \end{cases}$$

Proof. This is proved by induction on $|B|$. It is true by (3) if $B = \varnothing$. Suppose $B \neq \varnothing$. Choose $i \in B$, apply the inductive hypothesis to $X^{B\setminus\{i\}, A}$ and $X^{B\setminus\{i\}, A \cup \{i\}}$, and use the homology exact sequence of (2).

Claim 2. If $A \cup B = \langle n\rangle$ and $A \neq \varnothing$ ($A \cap B = \varnothing$) then

$$H_{n+1} X^{\varnothing,\langle n\rangle} \cong H_{n+1} X^{B,A}.$$

Proof. This follows from Claim 1 by induction on $|B|$, since if $B \neq \varnothing$, say $i \in B$, then

$$H_{n+1} X^{B\backslash\{i\},\, A \cup \{i\}} \cong H_{n+1} X^{B,\, A}.$$

Claim 3. If $B \neq \langle n \rangle$ and $i \in B$ then the map

$$H_{|B|+1} X^{B,\, \varnothing} \to H_{|B|} X^{B\backslash\{i\},\, \varnothing}$$

is injective.

Proof. Apply Claim 1 to show that $H_{|B|+1} X^{B\backslash\{i\},\, \{i\}} = 0$.

Conclusion. The exact sequence (4) now follows from the exact sequence

$$H_{n+1} X^{\langle n-1 \rangle,\, \varnothing} \to H_{n+1} X^{\langle n-1 \rangle,\, \{n\}} \to H_{n+1} X^{\langle n \rangle,\, \varnothing} \to H_{n+1} X^{\langle n-1 \rangle,\, \varnothing},$$

since $H_{n+1} X^{\langle n-1 \rangle,\, \varnothing} = 0$ by Claim 1, $H_{n+1} X^{\langle n-1 \rangle,\, \{n\}} \cong H_{n+1} X^{\varnothing,\, \langle n \rangle}$ by Claim 2, and $H_{n+1} X^{\langle n-1 \rangle,\, \varnothing} \to H_1 X^{\varnothing,\, \varnothing}$ is injective by repeated application of Claim 3. (We are grateful to the referee for some neat improvements in the arrangement of this proof.)

The exact sequence (4) now yields the exact sequence (1).

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