GROUPOIDS AS COEFFICIENTS

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Introduction

In a previous paper ([**1**]) we showed how a fibration of groupoids gave rise to a six-term exact sequence (three terms of which were groups and three were sets) and that this exact sequence included the bottom end of the topologists' exact sequence of a fibration of spaces, and also in the non-abelian cohomology of a group G the six-term exact sequence associated with an exact sequence of coefficient G-modules

$$1 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 1.$$

This latter exact sequence is described for example by Serre in [**9**]. However a curious feature emerges, because Serre also describes a five-term exact sequence (two terms of which are groups and three are sets) associated with a sub-G-module A of B, where A is not necessarily normal in B. One object of this paper is to show how this exact sequence fits into the groupoid theory.

In order to do this we generalize the theory of $H^i(G; A)$ $(i = 0, 1)$, where G is a group and A is a group which is a G-module, to the case where G and A are both groupoids. This seems a natural step to take, for since the *theory* involves groupoids it seems reasonable to start with groupoids. The general theory applies to the example by taking the exact sequence of a covering morphism of groupoids constructed from the sub-G-module.

It is hoped that the constructions used on the way, particularly the split extension of groupoids which is due to A. Fröhlich, will prove more generally useful. I would like to thank Professor Fröhlich for permission to use some material from some duplicated notes of his.

1. G-modules

The notation for groupoids is as in [**1**]. Further, for any groupoid G, we identify $\mathrm{Ob}(G)$ with the discrete subgroupoid of G on $\mathrm{Ob}(G)$.

1.1. Definition. A groupoid G *acts* on a groupoid A via a morphism $\omega\colon A \to \mathrm{Ob}(G)$ if for each g in $G(x, y)$ and α in the groupoid $\omega^{-1}(x)$ there is given an element $g.\alpha$ (also written $g\alpha$, or ${}^g\alpha$) in the groupoid $\omega^{-1}(y)$; this operation must satisfy the usual axioms:

(i) $(hg)\alpha = h(g\alpha)$,

(ii) $g(\beta\alpha) = (g\beta)(g\alpha)$,

(iii) $1.\alpha = \alpha$.

for elements h, g of G, β, α of A, and identity 1 of G; in these axioms it is understood that they hold when and only when both sides are defined.

Notice that because $\mathrm{Ob}(G)$ is discrete and $\omega\colon A \to \mathrm{Ob}(G)$ is a morphism, the groupoid A is the sum

$$A = \bigsqcup_x A_x$$

of the groupoids $A_x = \omega^{-1}(x)$ for all x in $\mathrm{Ob}(G)$. An element g in $G(x,y)$ defines a morphism $g_\sharp\colon A_x \to A_y$ of groupoids such that

$$1_\sharp = 1, \quad (hg)_\sharp = h_\sharp g_\sharp$$

when defined. Thus an operation of G on A defines a functor $A'\colon G \to \mathcal{G}d$, where $\mathcal{G}d$ is the category of groupoids, by $A'(x) = A_x$ for $x \in \mathrm{Ob}(G)$ and $A'(g) = g_\sharp$ for $g \in G$. Conversely, a functor $A'\colon G \to \mathcal{G}d$ defines an operation of G on the sum of the groupoids $A'(x)$, $x \in \mathrm{Ob}(G)$, in an obvious way.

If A is discrete, the above coincides with the usual operation of a groupoid on a set (as considered in [**1**]). An important example of an operation of a groupoid on a set is the operation of a groupoid G on its set $\mathrm{Ob}(G)$ of objects via the identity $\mathrm{Ob}(G) \to \mathrm{Ob}(G)$; this is defined by $g(x) = y$ whenever $g \in G(x,y)$.

Again if G operates on the groupoid A via $\omega\colon A \to \mathrm{Ob}(G)$, then G also operates on $\pi_0 A$ via $\pi_0\omega\colon \pi_0 A \to \mathrm{Ob}(G)$. This is because if $g \in G(x,y)$, then g defines $g_\sharp\colon A_x \to A_y$ and $g_\sharp$ induces $g_*\colon \pi_0 A_x \to \pi_0 A_y$.

We shall need the notion of a *trivial* action of a groupoid G on A. If G were a group, the action of G would be trivial if for each g in G, $g_\sharp$ is always the identity. This cannot be transferred directly to the case where G is a groupoid, since if G acts on A and $g \in G(x,y)$, where x, y are distinct, then $A_x \neq A_y$ and so $g_\sharp\colon A_x \to A_y$ cannot be the identity. We therefore make the following definition: G acts *trivially* on A if for all objects x, y of G, any two elements g, h of $G(x,y)$ induce the same isomorphism $A_x \to A_y$. This is equivalent to saying that the group $G\{x\}$ acts trivially on A_x for each $x \in \mathrm{Ob}(G)$.

1.2. PROPOSITION. *Let the groupoid G act on the groupoid A. Then A contains a unique maximal subgroupoid on which G acts trivially.*

Proof. Let A_x^G be the subgroupoid of A_x of elements fixed under $G\{x\}$. Let A^G be the union of these groupoids A_x^G. We prove that G acts on A^G.

Let $\alpha \in A^G_x$, $g \in G(x,y)$. Then $g\alpha \in A_y$, and if $h \in G\{y\}$, we have $g^{-1}hg \in G\{x\}$, whence

$$g^{-1}hg\alpha = \alpha.$$

This implies $hg\alpha = g\alpha$, and so $g\alpha \in A^G_y$.

Clearly G acts trivially on A^G, and A^G contains any subgroupoid of A on which G acts trivially.

Note that A^G can be empty, in contrast to the case when A is a group. An example of this is the action of $\mathbf{Z}_2$ on $\mathscr{I}$ which interchanges the two objects of $\mathscr{I}$.

It is useful to have a condition for A^G to be non-empty.

The action G on A is via a morphism $\omega\colon A \to \mathrm{Ob}(G)$. An *equivariant section* of A (more precisely, of ω) is a morphism $\lambda\colon \mathrm{Ob}(G) \to A$ of groupoids such that $\omega\lambda = 1$ and λ commutes with the action of G; this latter condition is that if $g \in G(x,y)$, then $g.\lambda(x) = \lambda(y)$.

1.3. PROPOSITION. *The groupoid A^G is non-empty if A has an invariant section. The converse holds if G is connected.*

Proof. Let $\lambda\colon \mathrm{Ob}(G) \to A$ be an invariant section. Let $x \in \mathrm{Ob}(G)$, and let $a = \lambda(x)$. If $g \in G\{x\}$, then

$$g.1_a = g.\lambda(x) = \lambda(x) = 1_a.$$

So A^G is non-empty.

For the converse, suppose that G is connected and $a \in \mathrm{Ob}(A^G)$. Let $\omega(a) = x$. We define an invariant section 'through a' by $\lambda(y) = g.a$ for some $g \in G(x,y)$—since $a \in \mathrm{Ob}(A^G)$, $g.a$ is independent of the choice of g in $G(x,y)$.

Given an invariant section λ of A, let $A^G\{\lambda\}$ be the set of sections $\varphi\colon x \to \varphi(x)$ such that (i) $\varphi(x) \in A\{\lambda(x)\}$, $x \in \mathrm{Ob}(G)$, and (ii) if $g \in G(x,y)$, then $g.\varphi(x) = \varphi(y)$. Clearly $A^G\{\lambda\}$ forms a group under multiplication of values; it is this group which replaces the usual group of fixed elements defined when G and A are both groups.

1.4. PROPOSITION. *If G is connected, and λ is an invariant section of A, then for any x in $\mathrm{Ob}(G)$ there is an isomorphism*

$$\varepsilon\colon A^G\{\lambda\} \to A\{\lambda(x)\}^{G\{x\}}.$$

Proof. We define ε by the evaluation $\varphi \mapsto \varphi(x)$.

In order to define an inverse ε' to ε, suppose that $\alpha \in A\{\lambda(x)^{G\{x\}}\}$, and $y \in \mathrm{Ob}(G)$. Since G is connected, there is an element g in $G(x,y)$ and $g.\alpha$ is independent of the choice of g. So we can define $\varphi = \varepsilon'(\alpha)$ by $\varphi(y) = g.\alpha$ for any $g \in G(x,y)$.

2. Split extensions

If a group G acts on a group A, then the split extension $G \ltimes A$ is defined. The (known) use of this construction in defining crossed morphisms was explained in [1].

In this section we define the split extension $G \ltimes A$ for the case that G is a groupoid acting on the groupoid A (via $\omega\colon A \to \mathrm{Ob}(G)$). This definition is due to Fröhlich (in some duplicated notes on 'Groupoids, groupoid spaces and cohomology' (1965)).

The split extension is closely linked with covering groupoids, and so to fix ideas we first *assume that A is a set.* We then let

$$\mathrm{Ob}(G \ltimes A) = A$$

and if $a, b \in A$, we let $(G \ltimes A)(a,b)$ be the set of pairs (g,b) such that $g \in G(\omega(a),\omega(b))$ and ${}^{g}a = b$. (In this and some later sections it is convenient to use exponential notation for operations; for ease of printing, we write ${}^{-g}a$ when a is operated on by g^{-1}). The multiplication is given by $(h,c)(g,b) = (hg,c)$, for (g,b) as above and $h \in G(\omega(b),\omega(c))$, with ${}^{h}b = c$.

This construction goes back, as far as I know, to Ehresmann ([3]). In [4], Appendix 1, §1, it is observed that this construction gives an equivalence from the category of operations of G on sets to the category of covering groupoids of G (see also [5], p. 101).

Mackey ([7], [8]) exploits this construction in the theory of ergodic actions, by replacing an ergodic action of a locally compact group G on a Borel space S by the groupoid $G \ltimes S$; with a suitable Borel structure this groupoid becomes an example of what Mackey calls an 'ergodic groupoid'†.

Higgins ([5]) makes extensive use of covering groupoids, particularly the groupoid $\mathrm{Tr}(G : H)$, which is $G \ltimes (G/H)$, where G is a group, H is a subgroup, and G operates on G/H in the usual way.

It has been observed by R. M. F. Moss that the Todd–Coxeter method of enumerating cosets G/H when G is a finite group (see, for example, [2], [6]) is really a method of constructing a multiplication table for the groupoid $\mathrm{Tr}(G : H)$.

We now give a construction which includes this construction of covering groupoids and also the split extensions of groups.

Let the groupoid G act on the groupoid A via $\omega\colon A \to \mathrm{Ob}(G)$. We first let

$$\mathrm{Ob}(G \ltimes A) = \mathrm{Ob}(A).$$

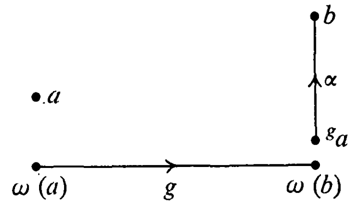
Next let $a, b \in \mathrm{Ob}(A)$. We define

$$(G \ltimes A)(a,b)$$

to be the set

$$\{(g,\alpha)\colon g \in G(\omega(a),\omega(b)),\ \alpha \in A({}^{g}a, b)\}. \tag{2.1}$$

† *Note added in proof.* For a more recent account of this theory, see [10].



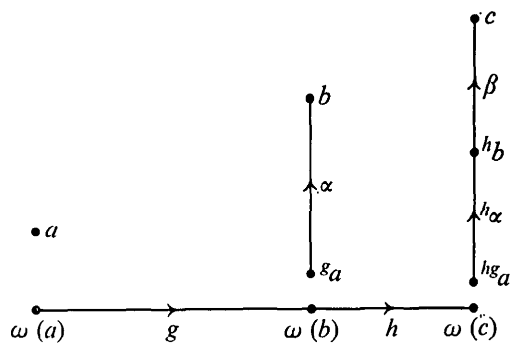
The above diagram is useful in keeping a picture of these pairs.

Suppose further that $(h,\beta) \in (G \,\tilde{\times}\, A)(b,c)$. Then we let

$$(h,\beta)(g,\alpha) = (hg, \beta\, {}^{h}\alpha) \tag{2.2}$$

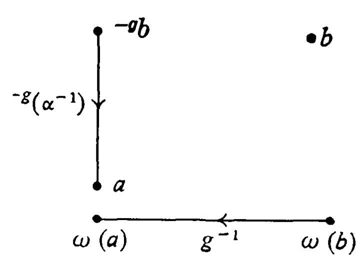
Notice that this formula is exactly the same as the formula for the multiplication in the split extension of groups. The following diagram gives a picture of the above product. The odd thing is that this picture

(2.3)



is clearer for the case where G and A are groupoids than it would be for the case of groups—the formula (2.2) is forced, in the groupoid case, simply by the necessity for the product to be well defined.

It is clear from diagram (2.3) that (g,α) has left-identity $(1_{\omega(b)}, 1_b)$ and right-identity $(1_{\omega(a)}, {}^{g}1_a)$. Further the inverse of (g,α) must be of the form (g^{-1},α'), where α' lies over $\omega(a)$. This suggests the choice $\alpha' = {}^{-g}(\alpha^{-1})$,



and it is easily checked that this works. Since associativity is easily verified, $G \,\tilde{\times}\, A$ is a groupoid.

The notation $G \tilde{\times} A$ does not conflict with that used earlier in this section, for A, if discrete, may be identified with its set $\mathrm{Ob}(A)$ of objects, and then the two definitions of $G \tilde{\times} A$ are identical.

Going back to the general case, the *projection* $p\colon G \tilde{\times} A \to G$ is defined to be ω on objects and $(g, \alpha) \mapsto g$ on elements. The *injection* $i\colon A \to G \tilde{\times} A$ is defined to be the identity on objects and on elements to be $\alpha \mapsto (1_{\omega(\alpha)}, \alpha)$. Clearly the image of i is the kernel of p.

When A is discrete, $p\colon G \tilde{\times} A \to G$ is a covering morphism. In general we have

2.4. PROPOSITION. *The projection* $p\colon G \tilde{\times} A \to G$ *is a fibration of groupoids.*

Proof. Let g in $G(x,y)$ and a in $\mathrm{Ob}(A)$ satisfy $\omega(a) = x$. Then g is covered by $(g, {}^{g}1_a)$ in $\mathrm{St}_{G \tilde{\times} A}\, a$.

The action of G on A determines two actions of G on sets, namely the action of G on $\pi_0 A$, and the action of G on $\mathrm{Ob}(A)$. These are both tied in with the fibration $p\colon G \tilde{\times} A \to G$. In order to explain this for $\pi_0 A$, recall from § 2 of [**1**] that the fibration p has a factorization

$$\begin{array}{ccc} G \tilde{\times} A & \xrightarrow{\kappa} & \tilde{G} \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \rho} & \\ G & & \end{array}$$

where $\tilde{G} = (G \tilde{\times} A)/\mathrm{Ker}\, p$, and ρ is a covering.

2.5. PROPOSITION. *There is an isomorphism*

$$(G \tilde{\times} A)/\mathrm{Ker}\, p \to G \tilde{\times} \pi_0 A$$

commuting with projection onto G.

Proof. Such an isomorphism is given by

$$\mathit{cls}(g, a) \mapsto (g, \mathit{cls}\, \alpha).$$

The action of G on $\mathrm{Ob}(A)$ gives a formal characterization of $G \tilde{\times} A$, which is due to Fröhlich.

2.6. PROPOSITION. *The morphisms*

$$A \xrightarrow{i} G \tilde{\times} A \xrightarrow{p} G$$

have the following properties.

(i) *i is an isomorphism onto* $\ker p$*;*

(ii) *$G \mathbin{\tilde{\times}} A$ contains $G \mathbin{\tilde{\times}} \mathrm{Ob}(A)$ as a wide subgroupoid; $p \,|\, G \mathbin{\tilde{\times}} \mathrm{Ob}(A)$ is the covering projection, and $\mathrm{Ob}(i)$ is the identity;*

(iii) *if $\alpha \in G({}^g a, b)$, where $g \in G(\omega(a), \omega(b))$, then*

$$(g^{-1}, {}^{-g}b)i(\alpha)(g, {}^g a) = i({}^{-g}\alpha);$$

these properties characterize the triple $(G \mathbin{\tilde{\times}} A, i, p)$ up to isomorphism. Further each element λ of $G \mathbin{\tilde{\times}} A$ has a unique representation

$$\lambda = i(\alpha)\mu,$$

with $\alpha \in A$ and $\mu \in G \mathbin{\tilde{\times}} \mathrm{Ob}(A)$.

Proof. The proof of (i) and (ii) is clear, and (iii) is proved by a direct computation. In order to prove that these properties characterize $G \mathbin{\tilde{\times}} A$, suppose given morphisms

$$A \xrightarrow{i'} E \xrightarrow{p'} G$$

satisfying the analogue of (i), (ii), and (iii). We wish to define an isomorphism $\theta\colon G \mathbin{\tilde{\times}} A \to E$ such that $\theta i = i'$, $p'\theta = p$.

First, since $\mathrm{Ob}(E) = A$, let θ be the identity on objects.

Next let $(g, \alpha) \in (G \mathbin{\tilde{\times}} A)(a, b)$. We define

$$\theta(g, \alpha) = i'(\alpha)(g, {}^g a) = (g, b)i'({}^{-g}\alpha),$$

(the last equality following from (iii)). I say that θ is injective on elements: because $\mathrm{Ob}(\theta) = 1$ and, if $\theta(g, \alpha)$ is an identity, then

$$g = p'\theta(g, \alpha)$$

is also an identity, whence $i'(\alpha)$ is an identity and so α is an identity. Further θ is surjective on elements: because suppose that $\xi \in E(a, b)$. Let $p'(\xi) = g$. Then $p'(\xi(g^{-1}, a)) = p'(1_b)$, and so $\xi(g^{-1}, a) = i'(\alpha)$ for $\alpha \in A({}^g a, b)$, whence $\xi = \theta(g, \alpha)$.

Finally we must prove that θ is a morphism. So let $(h, \beta) \in (G \mathbin{\tilde{\times}} A)(b, c)$, $(g, \alpha) \in (G \mathbin{\tilde{\times}} A)(a, b)$. Then

$$\theta(hg, \beta{}^h\alpha) = i'(\beta{}^h\alpha)(hg, {}^{hg}a)$$

and

$$\begin{aligned}\theta(h, \beta)\theta(g, \alpha) &= i'(\beta)(h, {}^h b)(g, b)i'({}^{-g}\alpha)\\ &= i'(\beta)(hg, {}^h b)i'({}^{-g}\alpha)\\ &= i'(\beta)i'({}^h\alpha)(hg, {}^{hg}a).\end{aligned}$$

The last part of the proposition is easy to prove.

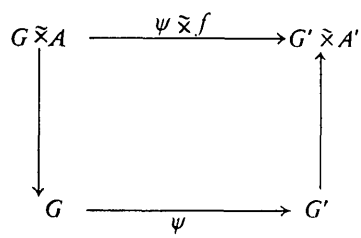
We wish to consider $G \mathbin{\tilde{\times}} A$ as a functor of G and of A. Suppose then that G acts on A via ω, and that G' acts on A' via ω'. A *morphism* of these operations is a pair (ψ, f) such that $\psi: G \to G'$, $f: A \to A'$ are morphisms of groupoids and we have the axioms

(i) $\omega' f = \mathrm{Ob}(\psi)\omega$,

(ii) $\psi(g).f(x') = f(g.x')$ whenever both sides are defined.

In this way we obtain a category $\mathcal{O}p$ of groupoid operations.

It is easy to check that a morphism (ψ, f) of operations induces a commutative square of morphisms



where $\psi \tilde{\times} f$ is $a \mapsto f(a)$ on objects and is $(g, \alpha) \mapsto (\psi(g), f(\alpha))$ on elements. In this way $\tilde{\times}$ is a functor $\mathcal{O}p \to \mathcal{G}d$.

As a special case, let $G = G'$, $\psi = 1$. If $(1, f)$ is a morphism of operations, then we say that $f: A \to A'$ is a morphism of G-modules. We need to record the fact that certain properties of f are preserved under $f \mapsto 1 \tilde{\times} f$.

2.7. Proposition. *Let $f: A \to A'$ be a morphism of G-modules. If f satisfies any of the following properties, then so correspondingly does* $1 \tilde{\times} f$, *namely* (i) *injective;* (ii) *fibration;* (iii) *connected fibres;* (iv) *quotient mapping;* (v) *discrete kernel;* (vi) *covering morphism.*

Proof. Note that (i) and (v) are obvious; that (ii) and (iii) together are equivalent to (iv); and that (ii) and (v) together are equivalent to (vi). So we need prove only (ii) and (iii).

Suppose that f is a fibration. Let $a \in \mathrm{Ob}(A) = \mathrm{Ob}(G \tilde{\times} A)$, and let $(g, \beta) \in (G \tilde{\times} A')(f(a), b)$. Then $\beta \in A'(gf(a), b) = A'(f(ga), b)$. Since f is a fibration, there is an α in $A(ga, a')$ such that $f(\alpha) = \beta$. Then (g, α) is a lift of (g, β) starting at a. Thus $1 \tilde{\times} f$ is a fibration.

Suppose that f has connected fibres. If $a, a' \in \mathrm{Ob}(A)$ satisfy $f(a) = f(a')$, then there is an α in $A(a, a')$, whence $\omega(a) = \omega(a')$ and

$$(1_{\omega(a)}, \alpha) \in (G \tilde{\times} A)(a, a').$$

We now give a useful example of a G-module A, where G is a group and A is a groupoid.

2.8. Example. Let C be a group acting on a set S, and let G be a group acting on both C and S in such a way that

(i) $${}^{g}cs = {}^{g}c\,{}^{g}s \quad \text{for all } c \in C, s \in S, g \in G.$$

Then the split extension $C \mathbin{\tilde{\times}} S$ admits an action of G by

(ii) $$^{g}(c,s) = (^{g}c, {}^{g}s).$$

That this action satisfies (i) and (iii) of Definition 1.1. is clear, and the verification of 1.1 (ii) is trivial. So we have a G-module $C \mathbin{\tilde{\times}} S$, and the projection $C \mathbin{\tilde{\times}} S \to C$ is a covering morphism of G-modules.

A special case of this example will be used later, namely when there is given a G-submodule D of C and S is the set C/D of left-cosets with the natural action of C, so that $C \mathbin{\tilde{\times}} S = \mathrm{Tr}(C : D)$.

A generalization of this example is to consider a groupoid G acting on a groupoid C via $\omega\colon C \to \mathrm{Ob}(G)$. Suppose also that C acts on a groupoid S via $\zeta\colon C \to \mathrm{Ob}(C)$ while G acts on C via $\eta = \omega\zeta\colon S \to \mathrm{Ob}(G)$ in such a way that $^{g}cs = {}^{g}c\,{}^{g}s$ whenever $c \in C$, $s \in S$, $g \in G$, and both sides are well defined. We can then define an action of G on $A = C \mathbin{\tilde{\times}} S$ by (ii) above. Thus if $a, b \in \mathrm{Ob}(S)$, $c \in C(\zeta(a), \zeta(b))$, $s \in S(ca, b)$, $g \in G(\eta(a), \eta(^{g}a))$, then

$$^{g}(c,s) \in A(^{g}(ca), {}^{g}b)$$

as required, and it is straightforward to verify the axioms for an action.

3. Crossed morphisms

Let A be a G-module via $\omega\colon A \to \mathrm{Ob}(G)$, and consider again the projection $p\colon G \mathbin{\tilde{\times}} A \to G$ of the split extension. We are interested in *sections* of p, by which we mean morphism $s\colon G \to G \mathbin{\tilde{\times}} A$ such that $ps = 1_G$. Such a section is on objects of the form $x \to \bar{s}(x)(\bar{s}(x) \in \mathrm{Ob}(A))$ and on elements of the form $g \to (g, \bar{s}(g))$. The properties satisfied by $\bar{s}$ lead to

3.1. Definition. A *crossed-morphism* $\bar{s}\colon G \to A$ consists of a function $\bar{s}\colon \mathrm{Ob}(G) \to \mathrm{Ob}(A)$ and a function $\bar{s}$ on elements such that

(i) if $x \in \mathrm{Ob}(G)$, then $\omega\bar{s}(x) = x$,

(ii) if $g \in G(x,y)$ then $\bar{s}(g) \in A(^{g}\bar{s}(x), \bar{s}(y))$,

(iii) $\bar{s}(hg) = \bar{s}(h).{}^{h}\bar{s}(g)$ whenever hg is defined.

Clearly there is a bijection between the sections s of $p\colon G \mathbin{\tilde{\times}} A \to G$ and the crossed morphisms $\bar{s}\colon G \to A$, assigning to each section s its principal part $\bar{s}$. Notice that, according to the discussion in § 2, we have

$$\bar{s}(1_x) = 1_{\bar{s}(x)}, \quad \bar{s}(g^{-1}) = {}^{-g}(\bar{s}(g)^{-1}).$$

This correspondence between sections of p and crossed morphism is well known for groups. One of its standard uses is that results on crossed morphisms can be deduced from corresponding results on morphisms. For example, if G is free, then a section $s\colon G \to G \mathbin{\tilde{\times}} A$ of $G \mathbin{\tilde{\times}} A \to G$ is entirely determined by its effect on a basis for G; it follows that the same is true for a crossed morphism $G \to A$.

The first question about crossed morphisms is whether they exist. Some conditions for existence are given in the next proposition.

3.2. PROPOSITION. *Each invariant section* λ *of* $\omega\colon A \to \mathrm{Ob}(G)$ *determines a crossed morphism* $\bar{s}_\lambda\colon G \to A$. *If* A *is totally disconnected then each crossed morphism* $\bar{s}\colon G \to A$ *determines an invariant section* $\lambda_{\bar{s}}$ *of* ω. *The function* $\lambda \mapsto \bar{s}_\lambda$ *is bijective if* A *is discrete.*

Proof. Let $\lambda\colon \mathrm{Ob}(G) \to A$ be an invariant section of ω. Let $\bar{s}_\lambda\colon G \to A$ be defined on objects by $x \mapsto \lambda(x)$ and on elements by $g \mapsto 1_{\lambda(y)}$ for $g \in G(x,y)$. Then if $h \in G(y,z)$, $g \in G(x,y)$,

$$\bar{s}_\lambda(h)\,{}^h\bar{s}_\lambda(g) = 1_{\lambda(z)}\,{}^h 1_{\lambda(y)} = 1_{\lambda(z)} 1_{\lambda(z)} = 1_{\lambda(z)} = \bar{s}_\lambda(hg).$$

Thus $\bar{s}_\lambda$ is a crossed morphism.

Suppose now that A is totally disconnected and that $\bar{s}\colon G \to A$ is a crossed morphism. If $g \in G(x,y)$ then $\bar{s}(g) \in A({}^g s(x), s(y))$ whence ${}^g s(x) = s(y)$. Thus $x \to 1_{\bar{s}(x)}$ is an invariant section of ω.

Clearly if A is discrete then $\bar{s} \mapsto \lambda_{\bar{s}}$ is an inverse of $\lambda \mapsto \bar{s}_\lambda$.

3.3. EXAMPLE. Let G be an infinite cyclic group with generator t, and let G operate on $\mathscr{I}$ so that t interchanges 0 and 1. Then $\mathscr{I}^G$ is empty, so that $\omega\colon \mathscr{I} \to \mathrm{Ob}(G)$ has no invariant section. However since G is free, we can define two crossed morphisms $\bar{s}\colon G \to \mathscr{I}$ as follows. Let $*$ denote the unique object of G. If $\bar{s}(*) = 0$, then $\bar{s}(t) \in \mathscr{I}({}^t0, 0) = \mathscr{I}(1,0)$, and so we have a unique crossed morphism $\bar{s}\colon G \to \mathscr{I}$ with $\bar{s}(t) = \iota^{-1}$. If $\bar{s}(*) = 1$, then $\bar{s}(t) \in \mathscr{I}({}^t1, 1) = \mathscr{I}(0,1)$, and so we have a unique crossed morphism $\bar{s}\colon G \to \mathscr{I}$ with $\bar{s}(t) = \iota$.

For either of these crossed morphism, $\bar{s}(t) = \bar{s}(t^{-1})$, and so the action of G on $\mathscr{I}$ and these crossed morphisms determine an action of $\mathbf{Z}_2$ on $\mathscr{I}$ and two crossed morphisms $\mathbf{Z}_2 \to \mathscr{I}$.

4. Cohomology

Let A be a G-module via $\omega\colon A \to \mathrm{Ob}(G)$. In this section we realize the crossed morphisms $G \to A$ as (in essence) the set of objects of a groupoid $Z^1(G; A)$. We then construct fibrations induced by fibrations $A \to B$ of G-modules and hence obtain exact sequences.

As explained in the last sections, the crossed morphisms $G \to A$ are in one-to-one correspondence with the sections of $p\colon G \mathbin{\tilde{\times}} A \to G$. These sections are the objects of the fibre of $p_*\colon (G(G \mathbin{\tilde{\times}} A)) \to (GG)$ over 1_G. So we define the groupoid

$$Z^1(G;\, A) \text{ to be } p_*^{-1}(1_G);$$

the objects of $Z^1(G; A)$ are to be the sections of p or the crossed morphisms to A as is convenient. Conditions for $Z^1(G; A)$ to be non-empty are given in the last section.

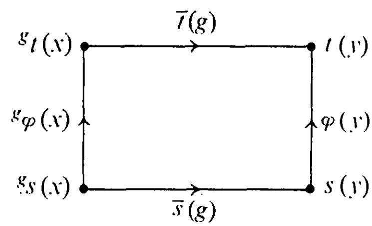
We now determine the elements of $Z^1(G; A)$, which we abbreviate for the present to Z^1.

4.1. Proposition. *Let s, t be two objects of Z^1. Then $Z^1(s,t)$ is bijective with the set of mappings φ: $\mathrm{Ob}(G) \to A$ such that*

(i) $\omega\varphi = 1$,

(ii) *if $x \in \mathrm{Ob}(G)$, then $\varphi(x) \in A(s(x), t(x))$,*

(iii) *if $g \in G(x, y)$ then the following diagram commutes*



Proof. The homotopies in Z^1 project to identities in (GG). Therefore a homotopy $s \simeq t$ in Z^1 assigns to each object x of G an element $(1_x, \varphi(x))$ of $G \ltimes A$ such that $\omega\varphi(x) = 1_x$ and, for any $g \in G(x, y)$,

$$(1_y, \varphi(y))s(g) = t(g)(1_x, \varphi(x)),$$

that is,

$$(1_y, \varphi(y))(g, \bar{s}(g)) = (g, \bar{t}(g))(1_x, \varphi(x))$$

and this is equivalent to (iii).

4.2. Proposition. *Let s be an object of Z^1, and φ: $\mathrm{Ob}(G) \to A$ a function such that* (i) $\omega\varphi = 1$, (ii) *if $x \in \mathrm{Ob}(G)$, then $\varphi(x) \in A(s(x), t(x))$ for some function t: $\mathrm{Ob}(G) \to \mathrm{Ob}(A)$. Then there is a unique object t of Z^1 such that φ is a homotopy $s \simeq t$.*

Proof. We define $\bar{t}$: $G \to A$ by the condition that if $g \in G(x, y)$ then the diagram of Proposition 4.1(iii) commutes. It is straightforward to check that $\bar{t}$ is a crossed morphism.

In particular let ω: $A \to \mathrm{Ob}(G)$ have an invariant section λ: $\mathrm{Ob}(G) \to A$ defining a section s_λ: $G \to G \ltimes A$. In § 1 we defined from λ a group $A^G\{\lambda\}$.

4.3. Corollary. *The group $Z^1\{s_\lambda\}$ is isomorphic to $A^G\{\lambda\}$.*

Proof. This follows from Proposition 4.1(iii): by taking $g \in G\{x\}$ we find that $\varphi(x) \in A\{\lambda(x)\}^G$, and by taking $g \in G(x, y)$ we find that $\varphi(y) = {}^g\varphi(x)$.

The *one-dimensional cohomology set of G with coefficients in A* is defined to be

$$H^1(G; A) = \pi_0 Z^1(G; A).$$

We determine this in some simple cases.

4.4. Proposition. *If A is discrete, then $H^1(G: A)$ is bijective with A^G.*

Proof. Since A is discrete, the crossed morphisms $G \to A$ are bijective with the set of invariant sections of $\omega: A \to \mathrm{Ob}(G)$. Further, by Proposition 4.1, $Z^1(G; A)$ is discrete.

If $f: B \to A$ is a morphism of G-modules, then f clearly induces a morphism of groupoids $Z^1(G; B) \to Z^1(G; A)$ and hence a function $f_*: H^1(G; B) \to H^1(G; A)$.

4.5. Proposition. *Let B be a full subgroupoid of A which is also a G-submodule. Then the function $i_*: H^1(G: B) \to H^1(G; A)$ induced by the inclusion $i: B \to A$ is injective. If, further, B is representative in A,* then *i_* is bijective.*

Proof. Let $\bar{s}, \bar{t}: G \to B$ be crossed morphisms and $\varphi: i\bar{s} \simeq i\bar{t}$ a homotopy. For each $x \in \mathrm{Ob}(G)$, $\varphi(x) \in A(s(x), t(x))$; since B is full, $\varphi(x) \in B(s(x), t(x))$ and so φ determines a homotopy $\bar{s} \simeq \bar{t}$. This proves i_* injective.

Now suppose that B is representative in A. Then for each $x \in \mathrm{Ob}(G)$ we can choose an element $\varphi(x)$ in A from $s(x)$ to some object $t(x)$ of B. If $g \in G(x, y)$, define $\bar{t}(g)$ by the equation

$$\bar{t}(g)^g\varphi(x) = \varphi(y)s(x).$$

Then $\bar{t}(g) \in A({}^g t(x), t(y))$; because B is G-invariant, ${}^g t(x) \in \mathrm{Ob}(B)$, and because B is full, $\bar{t}(g) \in B({}^g t(x), t(g))$. That $\bar{t}$ is a crossed morphism follows from Proposition 4.2. This proves i_* surjective.

4.6. Corollary. *Let each A_x, $x \in \mathrm{Ob}(G)$, be connected and let $\lambda: \mathrm{Ob}(G) \to A$ be an invariant section of $\omega: A \to \mathrm{Ob}(G)$. Let B be the full subgroupoid of A on the objects $\lambda(x), x \in \mathrm{Ob}(G)$. Then the inclusion $i: B \to A$ induces a bijection $i_*: H^1(G; B) \to H^1(G; A)$.*

We now consider exact sequences induced by a fibration of G-modules.

Let $j: B \to C$ be a fibration of G-modules, let $\mu: \mathrm{Ob}(G) \to B$ be an invariant section of $B \to \mathrm{Ob}(G)$; then $\nu = j\mu: \mathrm{Ob}(G) \to C$ is an invariant section of $C \to \mathrm{Ob}(G)$. Let

$$A = j^{-1}(\nu(\mathrm{Ob}(G)),$$

and let $i: A \to B$ be the inclusion.

4.7. PROPOSITION. *The groupoid A is a G-submodule of B; $\mu = i\lambda$ where λ: $\mathrm{Ob}(G) \to A$ is an invariant section; and there is a sequence*

$$(4.8)\quad 1 \longrightarrow A^G\{\lambda\} \longrightarrow B^G\{\mu\} \longrightarrow C^G\{\nu\} \xrightarrow{\partial} H^1(G;A) \longrightarrow H^1(G;B) \longrightarrow H^1(G;C)$$

which is exact in the usual sense.

Proof. By Proposition 2.9,

$$1 \mathbin{\tilde{\times}} j: G \mathbin{\tilde{\times}} B \to G \mathbin{\tilde{\times}} C$$

is a fibration. By [**1**], Proposition 2.14,

$$j_*: Z^1(G;B) \to Z^1(G;C)$$

is a fibration. I claim that the fibre of j_* over s_ν is $Z^1(G;A)$.

Let $\bar{s}$: $G \to B$ be a crossed morphism such that $j\bar{s} = \bar{s}_\nu$. Then clearly the image of $\bar{s}$ is contained in A. Again if φ: $\bar{s} \simeq \bar{t}$ is a homotopy of crossed morphisms such that $j\varphi$ is the constant homotopy $s_\nu \simeq s_\nu$, then again the image of φ must lie in A. Thus $j_*^{-1}(s_\nu) = Z^1(G;A)$.

The exact sequence follows immediately from [**1**], Theorem 4.3, and Corollary 4.3 above,

4.9. EXAMPLE. *Let G be a group and let C be a group which is a G-module. Let D be a sub-G-module of C. Then there is an exact sequence*

$$(4.10)\qquad 1 \longrightarrow D^G \longrightarrow C^G \xrightarrow{\partial} (C/D)^G \longrightarrow H^1(G;D) \xrightarrow{i_*} H^1(G;C)$$

Proof. Let j: $B \to C$ be the covering morphism $C \mathbin{\tilde{\times}} (C/D) \to C$. By Example 2.8, B is a G-module and j is a morphism of G-modules.

Let e denote the coset D in C/D. Then e is fixed under G and j maps $B\{e\}$ isomorphically to D. By Proposition 4.5, $H^1(G;B)$ is isomorphic to $H^1(G,B\{e\})$ and hence is isomorphic to $H^1(G;D)$.

The fibre A of j is the discrete groupoid C/D. By Proposition 4.4, $H^1(G;A) = (C/D)^G$. So the exact sequence (4.10) is a special case of (4.8).

The sequence (4.10) is the exact sequence of a subgroup ([**9**], Proposition 36, p. I-64). The additional information contained in [**9**], Corollaries 1 and 2, p. I-65 may be obtained by considering other fibres of $Z^1(G;B) \to Z^1(G;C)$.

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