ON THE CONNECTION BETWEEN THE SECOND RELATIVE HOMOTOPY GROUPS OF SOME RELATED SPACES

by

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Introduction

The title of this paper is chosen to imitate that of the paper of van Kampen [15] which gave a basic computational rule for the fundamental group $\pi_1(Y,\zeta)$ of a based space (an earlier and more special result is due to Seifert [18]). Our object is to give some computational rules for the second relative homotopy group $\pi_2(X,Y,\zeta)$ of a based pair.

Until now the known results on $\pi_2(X,Y,\zeta)$ have been as follows:

1. There is an operation of $\pi_1(Y,\zeta)$ on $\pi_2(X,Y,\zeta)$ which, with the boundary map $\partial : \pi_2(X,Y,\zeta) \to \pi_1(Y,\zeta)$, gives $\pi_2(X,Y,\zeta)$ the structure of crossed module(that is we have the two rules $\partial(a^b) = b^{-1}\partial(a)b$, $a^{-1}a_1a = a_1^{\partial a}$, for all $a,a_1 \in \pi_2(X,Y,\zeta)$ and $b \in \pi_1(Y,\zeta)$).

2. The following sequence is exact:

$$\to \pi_2(Y,\zeta) \to \pi_2(X,\zeta) \to \pi_2(X,Y,\zeta) \xrightarrow{\partial} \pi_1(Y,\zeta) \to \pi_1(X,\zeta) \to \ \ldots$$

3. The relative Hurewicz Theorem. If X,Y are path-connected and $\pi_1(X,Y) = 0$, then $\pi_2(X,Y,\zeta)$ with the operations of $\pi_1(Y,\zeta)$ factored out, is isomorphic to $H_2(X,Y)$.

4. Homotopy excision. If $X = Y \cup Z$ where Y,Z are open, $Y,Z,Y \cap Z$ are path-connected and $\pi_1(Y, Y \cap Z)$, $\pi_1(Z,Y \cap Z)$ are 0, then $\pi_2(Z,Y \cap Z,\zeta) \to \pi_2(X,Y,\zeta)$ is surjective, and is bijective if further $\pi_2(Y,Y \cap Z,\zeta) = 0$.

5. If X is obtained from the path-connected space Y by adjoining 2-cells, then $\pi_2(X,Y,\zeta)$ is a free crossed module over $\pi_1(Y,\zeta)$. (This is in §16 of [24]; a simpler proof for the case when Y is the 1-skeleton of a CW-complex X is given in [9].)

Whitehead's result 5) is, as far as we know, the only deep result on $\pi_2(X,Y,\zeta)$ in the literature which uses crucially the crossed module structure. Applications of 5) are given in [8,10,16,17,24]. We use the crossed module structure to prove a result which contains both 4) and 5) as very special cases.

Our general method is a 2-dimensional version of the groupoid van Kampen Theorem given in [3] and in [4], Ch. 8. The possibility of such a result was suggested in the Introduction of [3], but the realisation of this possibility required two innovations. First, it was necessary to find the right definition of 2-dimensional groupoids - this was solved by Brown-Spencer in [6] with the "special double groupoids with special connection". We shall call these objects simply <u>double groupoids</u>. The close relation of double groupoids to crossed modules was shown in [6].

The second innovation was finding the right functor from spaces to double groupoids. This is solved here with the introduction of $\rho(X,Y,Z)$, the homotopy double groupoid of a triple. The main features of $\rho(X,Y,Z)$ are: (i) it has good subdivision properties, (ii) it admits a version of the homotopy addition lemma, (iii) it admits cancellation, that is, its various compositions are all groupoids. All these are crucial for our proof of the Union Theorem of §6, which constitutes the geometric core of this work. A final feature of $\rho(X,Y,Z)$ is that when $Z = \{\zeta\}$, a singleton, then $\rho(X,Y,Z)$ contains the crossed module $\partial : \pi_2(X,Y,\zeta) \to \pi_1(Y,\zeta)$; this enables us to deduce results on relative homotopy groups from results on $\rho(X,Y,Z)$.

The structure of the paper is as follows. In §1 we recapitulate from [6] the basic properties of double groupoids; we also introduce the useful notion of degenerate square in a double groupoid. §2 gives the basic properties of the homotopy double groupoid $\rho(X,Y,Z)$. In §3 we quote a special case of the Union Theorem of §6, and use it to prove an adjunction space theorem for $\rho(X,Y,Z)$. This implies, using the relationship of double groupoids to crossed modules proved in [6], that in many cases where X,Y are adjunction spaces the crossed module $\partial : \pi_2(X,Y,\zeta) \to \pi_1(Y,\zeta)$ can be described as a pushout of crossed modules.

§4 takes up the algebraic side of the story with a description of pushouts of crossed modules in terms of more basic constructions; in particular the universal crossed module (A^*, B^*, ∂^*) induced from a crossed module (A,B,∂) by a morphism $f : B \to B^*$ of groups plays a key role. §5 gives applications of the algebra to topology. For example, we see that both the homotopy excision theorem in dimension 2 and Whitehead's result on free crossed modules are special cases of results on universal crossed modules.

§6 states and proves the Union Theorem for $\rho(X,Y,Z)$ when X is covered by the interiors of sets of a family $\{X_\lambda\}_{\lambda \in \Lambda}$ satisfying certain connectivity assumptions. This theorem contains also a form of the groupoid van Kampen theorem which is more general than that given in [3]. The proof of the Union Theorem is, however, nearer to Crowell's proof in [11] than to the proof in [3].

It should be noted that $\rho(X,Y,Z)$ contains structure in dimensions 0,1 and 2; this seems to be essential for obtaining the results in terms of colimits rather than, as might be expected for invariants in algebraic topology, obtaining results in the weaker form of exact sequences or spectral sequences.

Higher dimensional Seifert-van Kampen theorems have also been considered in [1,2], but the results there have little overlap with ours.

1. Preliminaries on double groupoids

By "double groupoid" we shall always mean "special double groupoid with special connection" as defined in §3 of [6]. We recall that if $G = (G_2, G_1, G_0)$ is a double groupoid, then G_2 is the set of "squares", G_1 the set of "edges", and G_0 the set of "vertices" or "points". The pair (G_1, G_0) is a groupoid, written multiplicatively, with objects G_0, arrows G_1, initial and final maps $\delta_0, \delta_1 : G_1 \to G_0$ and identity map $e : G_0 \to G_1$. If $a, b \in G_1$ and $\delta_1 a = \delta_0 b$, then the composite of a and b is written ab.

The pair (G_2, G_1) has two groupoid structures, both with objects G_1 and arrows G_2. The "horizontal" groupoid, written additively, has initial and final maps $\partial_0, \partial_1 : G_2 \to G_1$ and also "degenerate" squares 0_p for $p \in G_1$, which act as zero elements. The "vertical" groupoid, written with operations $\circ$ and $^{-1}$, has initial and final maps $\varepsilon_0, \varepsilon_1 : G_2 \to G_1$ and "degenerate" squares 1_p for $p \in G_1$ acting as identity elements. The conventions adopted for drawing squares and their boundaries in G are

$$\begin{tikzcd} \cdot \arrow[r, dash, "\varepsilon_0 a"] \arrow[d, dash, "\partial_0 a"'] \arrow[dr, phantom, "a"] & \cdot \arrow[d, dash, "\partial_1 a"] \\ \cdot \arrow[r, dash, "\varepsilon_1 a"'] & \cdot \end{tikzcd}$$

With respect to all the above boundaries and degeneracies, G is the part in dimensions ≤ 2 of a (semi-) cubical complex.

The "connection" $\Gamma : G_1 \to G_2$ introduces extra degenerate squares $\Gamma(p)$ with boundaries given by the diagram

$$\begin{tikzcd} \cdot \arrow[r, dash, "p"] \arrow[d, dash, "p"'] \arrow[dr, phantom, "\Gamma(p)"] & \cdot \arrow[d, dash, "e_x"] \\ \cdot \arrow[r, dash, "e_x"'] & \cdot \end{tikzcd}$$

when $x = \delta_1 p$.

The laws which relate the three groupoid structures are as follows:

(i) $\partial_i(a \circ b) = (\partial_i a)(\partial_i b)$ $(i = 0,1)$

and $\varepsilon_i(a + b) = (\varepsilon_i a)(\varepsilon_i b)$ $(i = 0,1)$

whenever the left-hand sides are defined.

(ii) <u>The interchange law</u>, which asserts that if $a,b,c,d \in G_2$ with $\partial_1 a = \partial_0 b$, $\partial_1 c = \partial_0 d$, $\varepsilon_1 a = \varepsilon_0 c$ and $\varepsilon_1 b = \varepsilon_0 d$ then $(a + b) \circ (c + d) = (a \circ c) + (b \circ d)$.

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

(iii) <u>The transport law</u>, which asserts that if $p,q \in G_1$ and $\delta_1 p = \delta_0 q$, then $\Gamma(pq) = (\Gamma(p) + 1_q) \circ \Gamma(q) = (\Gamma(p) \circ 0_q) + \Gamma(q)$:

$$\text{i.e. } \Gamma(pq) = \begin{array}{c|c|c|c} & p & q & \\ \hline p & \Gamma(p) & 1_q & e \\ \hline q & 0_q & \Gamma(q) & e \\ \hline & e & e & \end{array}$$

The laws (i) and (ii) are easily seen to imply that $0_{pq} = 0_p \circ 0_q$ and $1_{pq} = 1_p + 1_q$ whenever pq is defined in G_1. Also for any $a \in G_2$ we may deduce that $(-a)^{-1} = -(a^{-1})$, $\partial_i(a^{-1}) = (\partial_i a)^{-1}$, $\varepsilon_i(-a) = -\varepsilon_i(a)$. Since G is a cubical complex there is for each $x \in G_0$ a unique doubly degenerate square $\circledcirc_x = 0_{e_x} = 1_{e_x}$, and the transport law implies that $\Gamma(e_x) = \circledcirc_x$. We shall often abbreviate e_x and $\circledcirc_x$ to e and $\circledcirc$ when no confusion can arise.

It is convenient to use matrix notation for compositions of squares. Thus if a,b satisfy $\partial_1 a = \partial_0 b$, we write $[a, b]$ for $a + b$, and if $\varepsilon_1 a = \varepsilon_0 c$, we write $\begin{bmatrix} a \\ c \end{bmatrix}$ for $a \circ c$. More generally we define a <u>subdivision</u> of a square a in G_2 to be a rectangular array (a_{ij}) $1 \le i \le m$, $1 \le j \le n$ of squares in G_2 satisfying

$$\partial_1 a_{i,j-1} = \partial_0 a_{ij} \qquad (1 \le i \le m,\ 2 \le j \le n)$$

and $$\varepsilon_1 a_{i-1,j} = \varepsilon_0 a_{ij} \qquad (2 \le i \le m,\ 1 \le j \le n)$$

such that

$$(a_{11} + a_{12} + \dots + a_{1n}) \circ (a_{21} + \dots + a_{2n}) \circ (a_{m1} + \dots + a_{mn}) = a.$$

We call a the composite of the array (a_{ij}) and write $a = [a_{ij}]$. The interchange law then implies that if in the array (a_{ij}) we partition the rows and columns into blocks $B_{k\ell}$ and compute the composite $b_{k\ell}$ of each block, then $a = [b_{k\ell}]$.

If $G = (G_2, G_1, G_0)$ and $H = (H_2, H_1, H_0)$ are double groupoids, then a morphism f from G to H is a triple of functions $f_i : G_i \to H_i$ $(i = 0,1,2)$ preserving all the structure, including the connection. So we have a category $\underset{\sim}{\mathrm{DG}}$ of double groupoids.

We now recall one of the main results of [6], namely the relationship between double groupoids and crossed modules.

A crossed module (A, B, ∂) consists of groups A, B; an operation of B on the right of the group A, written $(a,b) \mapsto a^b$, $a \in A$, $b \in B$; and a morphism $\partial : A \to B$ of groups. These must satisfy the rules (i) $\partial(a^b) = b^{-1}\partial(a)b$, (ii) $a^{-1}a_1a = a_1^{\partial(a)}$, for $a, a_1 \in A, b \in B$. A morphism $(f,g) : (A,B,\partial) \to (A',B',\partial')$ of crossed modules consists of morphisms $f : A \to A'$, $g : B \to B'$ of groups such that $g\partial = \partial' f$ and $f(a^b) = f(a)^{g(b)}$ for $a \in A$, $b \in B$. So we have a category $\underset{\sim}{C}$ of crossed modules.

Let G be a double groupoid, and let $x \in G_0$. We define groups A, B by

$$A = \{a \in G_2 : \partial_0 a = \partial_1 a = \varepsilon_1 a = e_x\}$$

$$B = \{p \in G_1 : \delta_0 p = \delta_1 p = \delta_1 p = x\}$$

and a morphism $\varepsilon : A \to B$ by $\varepsilon(a) = \varepsilon_0(a)$. It is proved in Proposition 1 of [6] that there is also an action of B on A giving (A, B, ε) the structure of crossed module. This crossed module we write $\gamma(G,x)$. If G_0 has only one element x, then we write $\gamma(G)$ for $\gamma(G,x)$.

Now let $\underset{\sim}{DG}^{\boldsymbol{\cdot}'}$ be the category of double groupoids with a single vertex, and of morphisms of double groupoids. We quote from Theorem A of [6] :

Theorem A. The rule $G \mapsto \gamma(G)$ defines an equivalence of categories $\gamma : \underset{\sim}{DG}^{\boldsymbol{\cdot}'} \to \underset{\sim}{C}$.

The main use we shall make of this result is:

Corollary. The functor $\gamma : \underset{\sim}{DG}^{\boldsymbol{\cdot}'} \to \underset{\sim}{C}$ preserves colimits, and in particular preserves coproducts and pushouts.

There are two further results on double groupoids which we need to supplement those of [6].

It is a familiar fact that a map f from the arrows of one groupoid to the arrows of another, which preserves composition whenever it is defined, extends uniquely to a morphism of groupoids, its value at an object x being $\delta_0 f(e_x)$. The corresponding statement for double groupoids is as follows:

Proposition 1. Let G,H be double groupoids and let $f_2 : G_2 \to H_2$ be a function satisfying

(i) $f_2(a + b) = f_2(a) + f_2(b)$ whenever $\partial_1 a = \partial_0 b$,

(ii) $f_2(a \circ b) = f_2(a) \circ f_2(b)$ whenever $\varepsilon_1 a = \varepsilon_0 b$,

and (iii) for each $p \in G_1$ there is a $q \in H_1$ such that $f_2(\Gamma(p)) = \Gamma(q)$.

Then there exist unique functions $f_1 : G_1 \to H_1$ and $f_0 : G_0 \to H_0$ such that (f_2, f_1, f_0) is a morphism $G \to H$ of double groupoids.

Proof. (G_2,G_1) and (H_2,H_1) are groupoids with respect to addition, so by (i) there is a unique function $f_1^+ : G_1 \to H_1$ such that (f_2, f_1^+) is a morphism of additive groupoids. In particular, $f_2(-a) = -f_2(a)$, $f_1^+(\partial_i a) = \partial_i f_2(a)$ $(i = 0,1)$ and $f_2(0_p) = 0_{q'}$ where $q' = f_1^+(p)$.

Similarly, by (ii), there is a unique function $f_1^\circ : G_1 \to H_1$ such that (f_2, f_1°) is a morphism of the other groupoid structure. In particular $f_2(a^{-1}) = f_2(a)^{-1}$, $f_1^\circ(\varepsilon_i a) = \varepsilon_i f_2(a)$ $(i = 0,1)$ and $f_2(1_p) = 1_{q''}$ where $q'' = f_1^\circ(p)$.

Given $p \in G_1$, let q be an element of H_1 satisfying $f_2(\Gamma(p)) = \Gamma(q)$ as in (iii). Then $q = \partial_0\Gamma(q) = \partial_0 f_2(\Gamma(p))$ $= f_1^+(\partial_0\Gamma(p)) = f_1^+(p) = q'$, and similarly $q = q''$. Hence q is unique and there is a unique function $f_1 = f_1^+ = f_1^\circ : G_1 \to H_1$ such that (f_2, f_1) preserves both groupoid structures and the connection. Also this f_1 preserves multiplication in G_1. For if $p, p' \in G_1$ and pp' is defined, then writing $q = f_1(p)$, $q' = f_1(p')$, we have

$f_1(pp') = f_1(\partial_0\, 0_{pp'}) = \partial_0 f_2(0_{pp'})$

$= \partial_0 f_2(0_p \circ 0_{p'}) = \partial_0(0_q \circ 0_{q'}) = \partial_0\, 0_{qq'} = qq'$. It follows that there is a unique function $f_0 : G_0 \to H_0$ making (f_1, f_0) a morphism of groupoids, and the triple (f_2, f_1, f_0) is now a morphism of double groupoids.

We now define a <u>degenerate square in G</u> to be any square d having a subdivision (d_{ij}) in which each d_{ij} is of the form 0_p, 1_p, $\Gamma(p)$, $-\Gamma(p)$, $\Gamma(p)^{-1}$ or $-\Gamma(p)^{-1}$ for some $p = p_{ij}$ in G_1. The degenerate squares, together with their edges and vertices, form a sub-double-groupoid D of G, with $D_1 = G_1$ and $D_0 = G_0$. A square $a \in G_2$ is said to have <u>commuting boundary</u> if $(\partial_0 a).(\varepsilon_1 a) = (\varepsilon_0 a).(\partial_1 a)$, and it is clear that these squares also, together with their edges and vertices, form a sub-double-groupoid C of G with $C_1 = G_1$ and $C_0 = G_0$. Since 0_p, 1_p and $\Gamma(p)$ all have commuting boundary, D is a sub-double-groupoid of C.

<u>Proposition 2</u>. Let G be a double groupoid and let $p, q, r, s \in G_1$ satisfy $pq = rs$. Then there is a unique degenerate element $\Delta \in G_2$ such that $\partial_0\Delta = p$, $\partial_1\Delta = s$, $\varepsilon_0\Delta = r$, $\varepsilon_1\Delta = q$.

<u>Proof</u>. For any p, q, r, s in G_1 satisfying $pq = rs$, define

$$\Delta = \Delta\left(p \overset{r}{\underset{q}{}} s\right) = \Gamma(p) + 1_q - \Gamma(s).$$

Then Δ is degenerate and, since $pq = rs$, its edges are as given in the Proposition. We now prove that Δ satisfies the following laws:

(i) $\Delta(e \,{}^{p}_{p}\, e) = 1_p, \quad \Delta(p \,{}^{e}_{e}\, p) = 0_p, \quad \Delta(p \,{}^{p}_{e}\, e) = \Gamma(p);$

(ii) $\Delta(p \,{}^{r}_{q}\, s) + \Delta(s \,{}^{u}_{t}\, v) = \Delta(p \,{}^{ru}_{qt}\, v);$

(iii) $\Delta(p \,{}^{r}_{q}\, s) \circ \Delta(t \,{}^{q}_{u}\, v) = \Delta(pt \,{}^{r}_{u}\, sv);$

(iv) $-\Delta(p \,{}^{r}_{q}\, s) = \Delta(s \,{}^{r^{-1}}_{q^{-1}}\, p);$

(v) $\Delta(p \,{}^{r}_{q}\, s)^{-1} = \Delta(p^{-1} \,{}^{q}_{r}\, s^{-1}).$

The proofs of (i) and (ii) are trivial. To prove equation (iii) we observe that since $q = tu\,v^{-1}$, and $r = pq\,s^{-1} = pt.u.(sv)^{-1}$, both sides of equation (iii) have the common subdivision

$$\begin{bmatrix} \Gamma(p) & 1_t & 1_u & -1_v & -\Gamma(s) \\ 0_t & \Gamma(t) & 1_u & -\Gamma(v) & 0_v \end{bmatrix}.$$

Finally (iv) follows from (i) and (ii), and (v) follows from (i) and (iii).

From equations (ii) - (v) it follows that any square a in G_2 having a subdivision $a = [a_{ij}]$ in which a_{ij} is of the form $\Delta, -\Delta, \Delta^{-1}$ or $-\Delta^{-1}$ is itself of the form $\Delta(p \,{}^{r}_{q}\, s)$ where p,q,r,s are the edges of a. From this and (i) we deduce that all degenerate squares are of the form $\Delta(p \,{}^{r}_{q}\, s)$ and are therefore uniquely determined by their edges.

2. The homotopy double groupoid of a triple of spaces

By a triple (X,Y,Z) of spaces is meant a space X together with a subspace Y of X and subspace Z of Y.

Let (X,Y,Z) be a triple such that the map $\pi_1(Z,\zeta) \to \pi_1(Y,\zeta)$ induced by the inclusion $Z \to Y$ is trivial for each $\zeta \in Z$. We shall construct a double groupoid $\rho = \rho(X,Y,Z)$ whose elements are certain relative homotopy classes of squares, paths and points in X,Y and Z.

The zero-dimensional part of ρ is $\rho_0 = \pi_0(Z)$. The one-dimensional part ρ_1 is the set of homotopy classes of paths $\sigma : (I,\dot{I}) \to (Y,Z)$ with respect to homotopies $h : (I,\dot{I}) \times I \to (Y,Z)$ of pairs. Similarly ρ_2 is the set of homotopy classes of squares

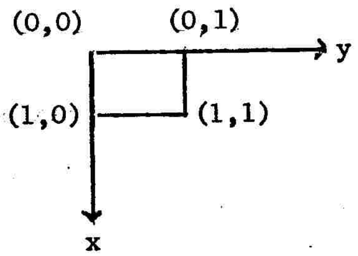
$$\alpha : (I^2,\dot{I}^2,\ddot{I}^2) \to (X,Y,Z)$$

with respect to homotopies

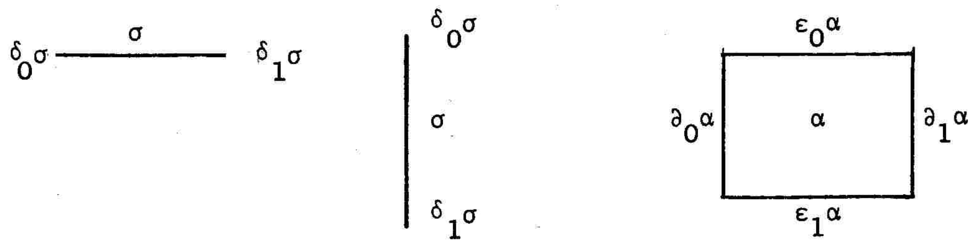
$$h : (I^2,\dot{I}^2,\ddot{I}^2) \times I \to (X,Y,Z)$$

of triples, where $\dot{I}^2$ and $\ddot{I}^2$ are respectively the boundary and the set of vertices of the unit square I^2. The three equivalence relations will be denoted by $\sim$ and the classes of ζ,σ,α in ρ_0,ρ_1,ρ_2 will be denoted by $\bar{\zeta},\bar{\sigma},\bar{\alpha}$. We also write $R_0 = Z$ and let R_1,R_2 be the sets of paths and squares defined above, so that $\rho_i = R_i/\sim$ $(i = 0,1,2)$. Then $R = (R_0,R_1,R_2)$ is a cubical complex, and we adopt notations for the boundaries and degeneracies conforming with those of §1.

Thus the unit square is denoted



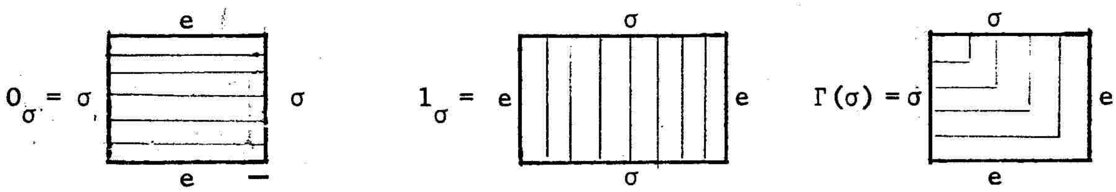
and boundaries of paths and squares are



The degenerate path e_ζ for $\zeta \in R_0 = Z$ is the constant path at ζ and the degenerate squares 0_σ and 1_σ for $\sigma \in R_1$ are given by $0_\sigma(x,y) = \sigma(x), 1_\sigma(x,y) = \sigma(y)$. There is also a degeneracy $\Gamma: R_1 \to R_2$ defined by

$$\Gamma(\sigma) : \quad (x,y) \mapsto \begin{cases} \sigma(x) & \text{if } 0 \le y \le x \le 1 \\ \sigma(y) & \text{if } 0 \le x \le y \le 1 . \end{cases}$$

We shall depict these degenerate squares as



the internal lines being lines of constancy.

It is clear that each of these boundary and degeneracy maps is compatible with the equivalences $\sim$ and therefore $\rho(X,Y,Z)$ inherits the structure of a cubical complex with a "connection" $\Gamma : \rho_1 \to \rho_2$ satisfying $\partial_0\Gamma(s) = \varepsilon_0\Gamma(s) = s$ and $\partial_1\Gamma(s) = \varepsilon_1\Gamma(s) = e_z$, where $z = \delta_1 s$.

The three groupoid operations on ρ are defined in terms of the usual composition of paths and horizontal and vertical compositions of squares. A key idea of this paper is that a subdivided square is the composite of the squares of its subdivision: the precise definition is as follows. For positive integers m,n, let $\phi_{m,n}: I^2 \to [0,m] \times [0,n]$ be the map $(x,y) \mapsto (mx, ny)$. An <u>$m \times n$-subdivision</u> of a square $\alpha : I^2 \to X$ is a factorisation $\alpha = \alpha' \circ \phi_{m,n}$; its <u>parts</u> are the squares $\alpha_{ij}: I^2 \to X$ defined by $\alpha_{ij}(x,y) = \alpha'(x+i-1, y+j-1)$. We then say that α is the <u>composite</u> of the squares α_{ij} and write $\alpha = [\alpha_{ij}]$. Similar definitions apply to subdivisions of paths and of cubes.

Multiplication in ρ_1 is defined as follows. Let $s,t \in \rho_1$ satisfy $\delta_1 s = \delta_0 t$. Let $\sigma,\tau \in R_1$ be paths representing s,t. Then $\delta_1\sigma \sim \delta_0\tau$ so we may choose a path λ in Z such that the composite path $\psi = [\sigma,\lambda,\tau]$ exists. We define $st = \overline{\psi}$ and show that it is independent of the choices made. Suppose that σ',τ',ζ' are alternative choices. Then $\sigma \sim \sigma'$, and $\tau \sim \tau'$, so there exist homotopies $h : \sigma \sim \sigma'$ and $k : \tau \sim \tau'$ so that we have a diagram

$$\begin{array}{ccc} \sigma & \lambda & \tau \\ \hline \multicolumn{1}{|c|}{h} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{k} \\ \hline \sigma' & \lambda' & \tau' \end{array}$$

in Y with vertical edges in Z. Since λ,λ' are paths in Z and since by hypothesis $\pi_1(Z,\zeta) \to \pi_1(Y,\zeta)$ is trivial for all $\zeta \in Z$, we may choose a square H in Y to fill the middle square of the diagram. The composite square $[h,H,k]$ now gives in Y a homotopy $[\sigma,\lambda,\tau] \sim [\sigma',\lambda',\tau']$, and so st is well-defined. It is easy to see by similar arguments that this multiplication is associative and has the elements e_z as identities. Also the element $\overline{\sigma}$ has as inverse the element $\overline{\sigma^*}$ where $\sigma^*(x) = \sigma(1-x)$; so (ρ_1,ρ_0) is a groupoid.

We next define addition, that is the horizontal composition, on ρ_2. Let $a,b \in \rho_2$ with $\partial_1 a = \partial_0 b$, and choose $\alpha,\beta \in R_2$ representing them. Then $\partial_1\alpha \sim \partial_0\beta$, so there is in Y a square H with boundaries as in the diagram

$$\begin{array}{ccc} & \sigma & \\ \hline \multicolumn{1}{|c|}{\alpha} & \multicolumn{1}{c|}{H} & \multicolumn{1}{c|}{\beta} \\ \hline & \tau & \end{array}$$

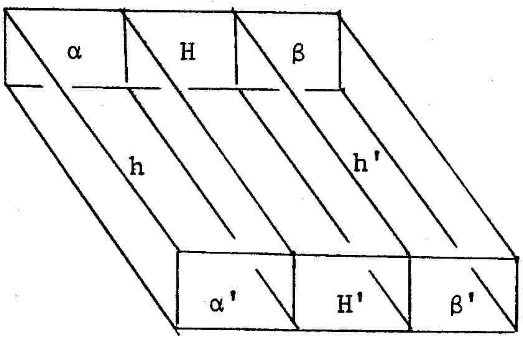
where σ,τ are paths in Z. We define

$$\gamma = [\alpha,\ H,\ \beta]$$

and set $$a + b = \overline{\gamma}\ .$$

Notice that $\varepsilon_0\gamma = [\varepsilon_0\alpha, \sigma, \varepsilon_0\beta]$, and σ is a path in Z. Thus $\overline{\varepsilon_0\gamma} = \overline{\varepsilon_0\alpha} \cdot \overline{\varepsilon_0\beta}$ and similarly $\overline{\varepsilon_1\gamma} = \overline{\varepsilon_1\alpha} \cdot \overline{\varepsilon_1\beta}$.

To see that this addition is well-defined, let $\gamma' = [\alpha', H', \beta']$ be alternative choices. Then there exist homotopies $h : \alpha \sim \alpha'$, $k : \beta \sim \beta'$ where $h,k : (I^2, \dot{I}^2, \ddot{I}^2) \times I \to (X,Y,Z)$. This gives a diagram



in X in which the front and back composite faces are γ and γ', the faces joining front to back are in Y and the edges joining front to back are in Z. Thus the hole in the middle has four faces in Y and all its edges in Z. Since $\pi_1(Z, \zeta) \to \pi_1(Y,\zeta)$ is trivial, we may fill the bottom of this hole with a square in Y. Then by retracting the unit cube onto five faces, we may fill the hole with a cube K in Y. The composite cube $[h,K,h']$ now gives a homotopy from γ to γ' mapping $(\dot{I}^2, \dot{I}^2, \ddot{I}^2) \times I \to (X,Y,Z)$. Thus $\gamma \sim \gamma'$ as required. Again it is easy to see that this addition makes (ρ_2,ρ_1) a groupoid with boundary maps ∂_0,∂_1 and identity elements 0_s for $s \in \rho_1$. A similar procedure gives a vertical groupoid structure with operations $\circ$, ${}^{-1}$, boundary maps $\varepsilon_0,\varepsilon_1$ and identity 1_s.

Also we have the rules $\varepsilon_i(a + b) = \varepsilon_i a \cdot \varepsilon_i b$, $\partial_i(a \circ c) = \partial_i a \cdot \partial_i c$. for $i = 0,1$.

We next verify the interchange law. Let a,b,c,d be elements of ρ_2 satisfying the conditions in our statement of the interchange law in §1. Let $\alpha,\beta,\gamma,\delta$ in R_2 be squares representing them. Then $\partial_1\alpha \sim \partial_0\beta, \partial_1\gamma \sim \partial_0\delta, \varepsilon_1\alpha \sim \varepsilon_0\gamma$ and $\varepsilon_1\beta \sim \varepsilon_0\delta$. So there exist squares f,g,h,k in Y such that the diagram

$$
\begin{array}{c|c|c|c|c}
\multicolumn{5}{c}{x'} \\ \cline{2-4}
 & \alpha & f & \beta & \\ \cline{2-4}
y' & g & \begin{smallmatrix} & x & \\ y & & t \\ & z & \end{smallmatrix} & k & t' \\ \cline{2-4}
 & \gamma & h & \delta & \\ \cline{2-4}
\multicolumn{5}{c}{z'}
\end{array}
$$

exists in X, has all edges in Y, all vertices in Z and further has the marked edges x,y,z,t,x',y',z',t' in Z.

By our basic hypothesis we can fill the centre hole with a square H in Y. Now the square

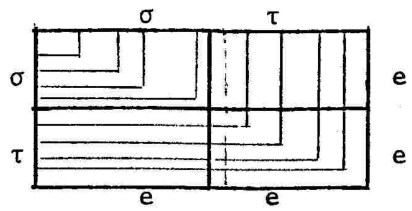
$$
\theta = \begin{bmatrix} \alpha & f & \beta \\ g & H & k \\ \gamma & h & \delta \end{bmatrix}
$$

represents a class $\bar{\theta}$ in ρ_2 which can be computed in two ways. Let $\theta_1 = [\alpha,f,\beta]$, $\theta_2 = [g,H,k]$ and $\theta_3 = [\gamma,h,\delta]$. Then $\bar{\theta}_1 = a+b, \bar{\theta}_3 = c+d$ (by definition) and θ_2 is a square in Y with two edges y',t' in Z. Hence again by definition $\bar{\theta}_1 \circ \bar{\theta}_3 = \bar{\theta}$, that is $\bar{\theta} = (a+b)\circ(c+d)$. A similar argument shows that $\bar{\theta} = (a\circ c) + (b \circ d)$.

Finally we must verify that the transport law holds in ρ. Let $s,t \in \rho_1$ with $\delta_1 s = \delta_0 t$. We may choose σ,τ in R_1 representing s,t such that $\delta_1\sigma = \delta_0\tau$ (for example, if λ is a path in Z from $\delta_1\sigma$ to $\delta_0\tau$, we may replace τ by $[\lambda,\tau]$). Then the composite square

$$\gamma = \begin{bmatrix} \Gamma(\sigma) & 1_\tau \\ 0_\tau & \Gamma(\tau) \end{bmatrix}$$

exists and represents the element $(\Gamma(s) + 1_t) \circ \Gamma(t) = (\Gamma(s) \circ 0_t) + \Gamma(t)$. But γ has the form

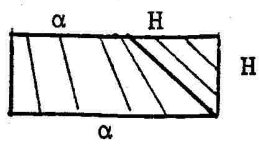


and so represents $\Gamma(st)$.

This completes the proof that $\rho = (\rho_2,\rho_1,\rho_0)$ is a double groupoid.

We need two further observations about composites of squares. Suppose α,β are squares in R_2 such that $\partial_1\alpha = \partial_0\beta = \sigma$ say. Then the square $\gamma = [\alpha, \beta]$ is defined and is clearly equivalent in R_2 to $[\alpha, 0_\sigma, \beta]$, which represents $\bar{\alpha} + \bar{\beta}$. Hence $[\alpha, \beta]$ also reppesents $\bar{\alpha} + \bar{\beta}$. Similar remarks apply to a composite square $\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$. More generally, if we have a subdivision $\alpha = [\alpha_{ij}]$ of a square α in R_2 and if each α_{ij} is in R_2, then $\bar{\alpha}$ is an element of ρ_2 which is also the uniquely defined composite $[\bar{\alpha}_{ij}]$.

On the other hand, suppose $a,b \in \rho_2$ satisfy $\partial_1 a = \partial_0 b$. Then we can always chose α',β' representing a, b respectively such that $[\alpha',\beta']$ is defined and represents $a+b$. This is clear since we can choose α,H,β such that $\bar{\alpha} = a, \bar{\beta} = b$ and $[\alpha, H, \beta]$ is defined and represents $a + b$, and then set $\alpha' = [\alpha, H]$, $\beta' = \beta$; the following diagram illustrates that $[\alpha,H] \sim \alpha$.



Proposition 3. Let (X,Y,Z) be a triple with $\pi_1(Z,\zeta) \to \pi_1(Y,\zeta)$ trivial for all $\zeta \in Z$, and let $\rho = \rho(X,Y,Z)$.

(i) If σ is a path in Z, then $\bar{\sigma}$ is an identity e_z in ρ_1.

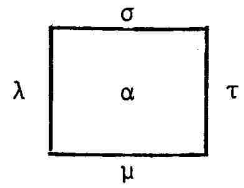
(ii) If α is a square in Z, then $\bar{\alpha} = 0_z$ in ρ_2 for some z.

(iii) If α is a square in Y with vertices in Z, then $\bar{\alpha}$ is a degenerate element of ρ_2.

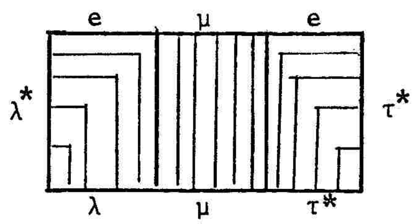
Proof. (i) Clearly $\sigma \sim e_\xi$ where $\zeta = \delta_0\sigma$.

(ii) Again α is homotopic in Z to the constant square at ζ, where $\zeta = \delta_0\partial_0\alpha$, say.

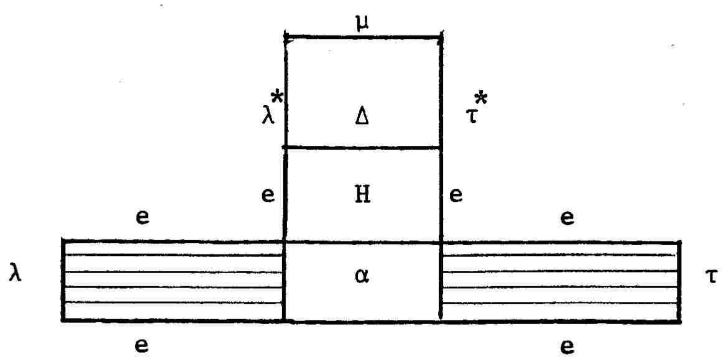
(iii) Let α have edges as follows



Let $\sigma' = [\lambda,\mu,\tau^*]$. Then there is in Y a homotopy $H : \sigma \sim \sigma'$ rel end points. Further there is a square Δ whose class in ρ_2 is degenerate and which is pictured as follows, where the



internal lines denote lines of constancy. So we can form five faces of a cube, the faces folded flat



being as above. All these faces are in Y and so by retraction can be filled in by a cube in Y. This cube gives an equivalence from α to a square whose class in ρ_2 is degenerate.

Our next proposition , which is of the same general type, needs some standard notation for the faces of a cube. We adopt one which is <u>ad hoc</u> but suits our purpose. Each face of $h : I^3 \to X$ is determined by a face $\eta' : I^2 \to I^3$ of the unit cube and is denoted by $\eta h = h \circ \eta'$. The six faces we choose are

$$\alpha_i' : (x,y) \mapsto (x,y,i) \quad (i = 0,1)$$
$$\beta_i' : (x,y) \mapsto (x,i,y) \quad (i = 0,1)$$
$$\gamma_i' : (x,y) \mapsto (i,y,x) \quad (i = 0,1).$$

With this choice, five faces of h fit together to form a plane diagram

$$\begin{array}{|c|c|c|}
\cline{2-2}
\multicolumn{1}{c|}{} & (\gamma_0 h)^{-1} & \multicolumn{1}{c}{} \\
\hline
-\beta_0 h & \alpha_0 h & \beta_1 h \\
\hline
\multicolumn{1}{c|}{} & \gamma_1 h & \multicolumn{1}{c}{} \\
\cline{2-2}
\end{array}$$

where $-$ and $^{-1}$ denote reflections of squares in vertical and horizontal lines.

<u>Proposition 4.</u>- (The Homotopy Addition Lemma) Let X,Y,Z,ρ be as in Proposition 3. Let h be a cube in X with edges in Y and vertices in Z, and let the elements of ρ_2 represented by its faces be

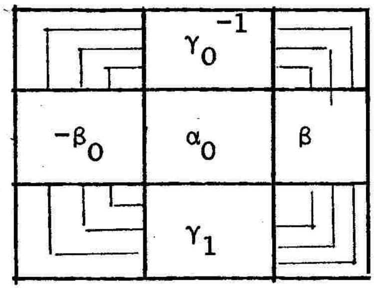
$$a_i = \overline{\alpha_i h}\ , \ b_i = \overline{\beta_i h}\ , \ c_i = \overline{\gamma_i h} \qquad i = 0,1.$$

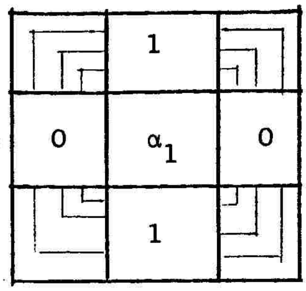
Then in ρ_2

$$a_1 = \begin{bmatrix} -\Gamma^{-1} & c_0^{-1} & \Gamma^{-1} \\ -b_0 & a_0 & b_1 \\ -\Gamma & c_1 & \Gamma \end{bmatrix},$$

where each Γ stands for a square $\Gamma(p)$ for an appropriate edge p.

Proof . The following diagrams define maps of I^2 to I^3 which agree on $\dot{I}^2$, where the first diagram defines a map onto all

$\phi_0 =$ 

$\phi_1 =$ 

faces of I^3 except α_1, and the latter diagram defines a map onto α_1. Since I^3 is convex, these two maps are homotopic rel $\dot{I}^2$. Therefore $h\phi_0$, $h\phi_1$, define the same element of ρ_2. But $\overline{h\phi_0}$ is clearly the composite matrix given above, while $\overline{h\phi_1} = a_1$.

A map $f:(X,Y,Z) \to (X',Y',Z')$ of triples of spaces clearly defines a map $\rho(f): \rho(X,Y,Z) \to \rho(X',Y',Z')$ of cubical complexes, and also a morphism of double groupoids if both double groupoids are defined.

Proposition 5 . If $f:(X,Y,Z) \to (X',Y',Z')$ is a map of triples of spaces such that each of $f_X : X \to X'$, $f_Y : Y \to Y'$, $f_Z : Z \to Z'$ is a homotopy equivalence, then f induces an isomorphism $\rho(f):\rho(X,Y,Z) \to \rho(X',Y',Z')$.

Proof . This is an immediate consequence of (10.11) of [12]. (An alternative proof can be given using the fibrations $X^{I^2} \to X^{\dot{I}^2} \to X^{\ddot{I}^2}$, the subspaces $Y^{\ddot{I}^2}$, $Z^{\ddot{I}^2}$ of $X^{\dot{I}^2}$, $X^{\ddot{I}^2}$ and the cogglueing theorem of [5] as applied to pull-backs over subspaces).

Remark . The conditions of Proposition 5 do not imply that $f:(X,Y,Z) \to (X',Y',Z')$ is a homotopy equivalence of triples except under further cofibration conditions on the two triples (cf. Ch7 of [4]) - these conditions would be inconvenient in the applications.

From the homotopy double groupoid $\rho(X,Y,Z)$ we obtain, according to the procedure of §1, a crossed module $\gamma(\rho(X,Y,Z),\bar{\zeta})$ for each $\bar{\zeta} \in \pi_0 Z$. We also have, for each $\zeta \in Z$, a crossed module $(\pi_2(X,Y,\zeta), \pi_1(Y,\zeta), \partial)$ which we shall always denote by $\mu(X,Y,\zeta)$, or by $\mu(X,Y)$ if the basepoint is clear.

<u>Proposition 6.</u> For any $\zeta \in Z$, the crossed module $\gamma(\rho(X,Y,Z),\bar{\zeta})$ is naturally isomorphic to $\mu(X,Y,\zeta)$.

<u>Proof.</u> Let (A,B,ε) be the crossed module of $\rho(X,Y,Z)$ at $\bar{\zeta}$. It is clear that B is naturally isomorphic to $\pi_1(Y,\zeta)$. (Note that we are assuming that $\pi_1(Z,\zeta) \to \pi_1(Y,\zeta)$ is trivial for all $\zeta \in Z$, whence $\pi_1(Y,\zeta)$ is canonically isomorphic to $\pi_1(Y,\zeta')$ if ζ and ζ' are in the same path component of Z). Now the elements of $\pi_2(X,Y,\zeta)$ are homotopy classes of maps $(I^2,\{0\} \times I,\dot{J}^2) \to (X,Y,\zeta)$, where $\dot{J}^2 = (\{1\} \times I) \cup (I \times \dot{I})$. Clearly each such map determines an element of $\rho(X,Y,Z)$, and in this way we obtain a function $\theta : \pi_2(X,Y,\zeta) \to \rho_2(X,Y,Z)$ which is a morphism from the group structure on $\pi_2(X,Y,\zeta)$ to the horizontal groupoid structure on $\rho(X,Y,Z)$. The rest of the proof consists of proving that θ is an isomorphism onto the group A and that θ commutes with all the operations. We omit the details.

3. The adjunction space theorem

The following theorem is a special case of our main theorem (the Union Theorem) whose statement and proof are postponed until §6.

Theorem B. Let X be a space which is the union of the interiors of two subspaces X_1, X_2 of X, and let $X_0 = X_1 \cap X_2$. Let Y be a subspace of X, let Z be a subspace of Y, and define

$$Y_\nu = Y \cap X_\nu,\ Z_\nu = Z \cap X_\nu, \nu = 0,1,2.$$

Suppose that

(i) the induced morphisms $\pi_1(Z_\nu, \zeta_\nu) \to \pi_1(Y_\nu, \zeta_\nu)$ are trivial for all $\zeta_\nu \in Z_\nu$ and for $\nu = 0,1,2,$ - ;

(ii) the induced maps $\pi_0(Z_\nu) \to \pi_0(X_\nu)$ and $\pi_0(Z_\nu) \to \pi_0(Y_\nu)$ are surjective for $\nu = 0,1,2$;

(iii) the induced morphisms of groupoids $\rho_1(Y_\nu, Z_\nu) \to \rho_1(X_\nu, Z_\nu)$ are surjective for $\nu = 0,1,2$.

Then the induced diagram

$$\begin{tikzcd}
\rho(X_0,Y_0,Y_0) \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & \rho(X_2,Y_2,Z_2) \arrow[d, "\lambda_2"] \\
\rho(X_1,Y_1,Z_1) \arrow[r, "\lambda_1"'] & \rho(X,Y,Z)
\end{tikzcd}$$

is a pushout in the category of double groupoids.

Corollary. Let X, Y, X_ν, Y_ν be as in Theorem B and let $\zeta \in Y_0$. Assume that all the X_ν, Y_ν are path-connected and that $\pi_1(Y_\nu,\zeta) \to \pi_1(X_\nu,\zeta)$ is surjective for $\nu = 0,1,2$. Then the induced diagram

$$\begin{tikzcd}
\mu(X_0,Y_0,\zeta) \arrow[r] \arrow[d] & \mu(X_2,Y_2,\zeta) \arrow[d] \\
\mu(X_1,Y_1,\zeta) \arrow[r] & \mu(X,Y,\zeta)
\end{tikzcd}$$

is a pushout in the category of crossed modules.

Proof. Take $Z = \{\zeta\}$ in Theorem B and apply Theorem A and Proposition 6 to pass from pushouts of double groupoids to pushouts of crossed modules.

The main purpose of this section is to prove Theorem C, which is a double groupoid pushout theorem for adjunction spaces. Thus Theorem B and Theorem C are the 2-dimensional analogues of the results on the fundamental groupoid given in [4] 6.7.2 and 8.4.2.

For Theorem C (The Adjunction Space Theorem) we suppose given an "adjunction space triple", by which is meant a commutative square of maps of triples

$$\begin{array}{ccc} (X_0,Y_0,Z_0) & \xrightarrow{f} & (X_2,Y_2,Z_2) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ (X_1,Y_1,Z_1) & \xrightarrow[\bar{f}]{} & (X,Y,Z) \end{array} \qquad (1')$$

in which (X_0,Y_0,Z_0) are closed subspaces of X_1,Y_1,Z_1 respectively, i is the inclusion of triples, and if f_X,f_Y,f_Z are the restrictions of f mapping $X_0 \to X_2$, $Y_0 \to Y_2$, $Z_0 \to Z_2$ respectively, then the above diagram represents X,Y,Z as the adjunction spaces $X = X_2 \cup_{f_X} X_1$, $Y = Y_2 \cup_{f_Y} Y_1$, $Z = Z_2 \cup_{f_Z} Z_1$.

It should be noted that if we are given (X_ν,Y_ν,Z_ν) for $\nu = 0,1,2$, and the maps i,f, then some conditions are required to ensure that the induced maps $Z \to Y \to X$ of adjunction spaces are homeomorphisms into, so that (X,Y,Z) can be regarded as a triple. Useful conditions are given in the next Proposition.

Proposition 7. Suppose given maps $i : (X_0,Y_0) \to (X_1,Y_1)$, $f : (X_0,Y_0) \to (X_2,Y_2)$ of pairs such that $i_X : X_0 \to X_1$, $i_Y : Y_0 \to Y_1$ are inclusions of closed subspaces. Let $X = X_2 \cup_{f_X} X_1$, $Y = Y_2 \cup_{f_Y} Y_1$, and let $\phi : Y \to X$ be the induced map. Then

(i) if Y_1, Y_2 are closed in X_1, X_2 respectively, then ϕ is a closed map;

(ii) ϕ is injective if and only if $f_X | W$ is injective and $f_X(W) \cap Y_2 = \phi$, where $W = (Y_1 \cap X_0) \backslash Y_0$;

(iii) ϕ is injective if $Y_0 = X_0 \cap Y_1$.

<u>Proof</u>. (i) If F is closed in Y and $F_1 \sqcup F_2$ is the inverse image of F under the quotient map $Y_1 \sqcup Y_2 \to Y$, then $\phi(F)$ has inverse image in $X_1 \sqcup X_2$ by the quotient map $X_1 \sqcup X_2 \to X$ the set $(F_1 \cup f_X^{-1}(F_2)) \sqcup F_2$.

(ii) This follows from writing Y as the set-theoretic sum $(Y_1 \backslash X_0) \sqcup W \sqcup Y_2$.

(iii) This follows from (ii).

In order to state Theorem C we will need the mapping cylinders $M(f_X)$, $M(f_Y)$, $M(f_Z)$ and the natural map, $p = (p_X, p_Y, p_Z) : (M(f_X) \cup X_1, M(f_Y) \cup Y_1, M(f_Z) \cup Z_1) \to (X,Y,Z)$. If each of $X_0 \to X_1$, $Y_0 \to Y_1$, $Z_0 \to Z_1$ is a cofibration, then p_X, p_Y, p_Z are homotopy equivalences ([4]7.5.4).

<u>Theorem C</u> (The Adjunction Space Theorem) Suppose given the adjunction space triple of diagram (1). Suppose also that (i) the induced morphisms $\pi_1(Z_\nu, \zeta_\nu) \to \pi_1(Y_\nu, \zeta_\nu)$ are trivial for all $\zeta_\nu \in Z_\nu$ and $\nu = 0,1,2,-$;

(ii) the induced maps $\pi_0(Z_\nu) \to \pi_0(X_\nu)$ are surjective for $\nu = 0,1,2$;

(iii) the induced morphisms of groupoids $\rho_1(Y_\nu, Z_\nu) \to \rho_1(X_\nu, Z_\nu)$ are surjective for $\nu = 0,1,2$;

(iv) the maps p_X, p_Y, p_Z are homotopy equivalences.

Then the induced diagram

$$\begin{array}{ccc} \rho(X_0, Y_0, Y_0) & \xrightarrow{f} & \rho(X_2, Y_2, Z_2) \\ {\scriptstyle i}\downarrow & & \downarrow \\ \rho(X_1, Y_1, Z_1) & \longrightarrow & \rho(X, Y, Z) \end{array}$$

is a pushout in the category of double groupoids.

Proof Let $X' = M(f_X) \cup X_1$, $X'_1 = X'\backslash X_2$, $X'_2 = X'\backslash X_1$, so that $X'_0 = X'_1 \cap X'_2 = X_0 \times]0, 1[$. Let $Y' = M(f_Y) \cup Y_1$, $Z' = M(f_Z) \cup Z_1$. Then the projections $p_\nu : (X'_\nu, Y'_\nu, Z'_\nu) \to (Y_\nu, Y_\nu, Z_\nu)$ for $\nu = 0,1,2$ are triples of homotopy equivalences, as is the projection $p : (X',Y',Z') \to (X,Y,Z)$ by assumption (iv). It follows that assumptions (i) - (iii) of Theorem B (applied to primed spaces) are satisfied. Also Proposition 5 implies that the morphisms of homotopy double groupoids induced by these maps are isomorphisms.

Consider the following diagram:

$$\begin{tikzcd}
\rho(X_0,Y_0,Z_0) \arrow[rrr] \arrow[ddd] & & & \rho(X_2,Y_2,Z_2) \arrow[ddd] \\
& \rho(X'_0,Y'_0,Z'_0) \arrow[r, "(2)"] \arrow[d] \arrow[ul, "p_0"'] \arrow[l, phantom, "(1)"] & \rho(X'_2,Y'_2,Z'_2) \arrow[d] \arrow[ur, "p_2"'] & \\
& \rho(X'_1,Y'_1,Z'_1) \arrow[r, "(4)"'] \arrow[dl, "p_1"] & \rho(X',Y',Z') \arrow[dr, "p"'] \arrow[r, phantom, "(3)"] & \\
\rho(X_1,Y_1,Z_1) \arrow[rrr] & & & \rho(X,Y,Z).
\end{tikzcd}$$

The cells (1) - (4) are obviously commutative, because they are induced by commutative diagrams of maps. So the diagram defines an isomorphism from its inner square to its outer square. But the inner square is a pushout of double groupoids by Theorem B. So the outer square is a pushout of double groupoids.

Remarks 1) Suppose given an adjunction space $X = X_2 \cup_{f_X} X_1$ where X_1, X_2 are CW-complexes, X_0 is a subcomplex of X_1 and $f_X : X_0 \to X_2$ is cellular Let Y_ν, Z_ν be respectively the 1- and 0- skeletons of X_ν for $\nu = 0,1,2$ - . Then the induced diagram (1) is an adjunction space triple, and also conditions (i) - (v) of Theorem C are satisfied.

2) The assumption (iv) of Theorem C can be weakened to the conditions that $\pi_1(M(f_Z) \cup Z_1,\zeta) \to \pi_1(M(f_Y) \cup Y_1,\zeta)$ is trivial for all ζ in $M(f_Z) \cup Z_1$ and p induces an isomorphism of homotopy double groupoids.

Corollary Suppose given the adjunction space triple of diagram (1) with Z_ν consisting of a base point of Y_ν for $\nu = 0,1,2$ - . Suppose that $X_0 \to X_1$, $Y_0 \to Y_1$ are closed cofibrations, that all the X_ν, Y_ν are path-connected and that $\pi_1(Y_\nu) \to \pi_1(X_\nu)$ is surjective for $\nu = 0,1,2$. Then the induced square

$$\begin{array}{ccc} \mu(X_0,Y_0) & \longrightarrow & \mu(X_2,Y_2) \\ \downarrow & & \downarrow \\ \mu(X_1,Y_1) & \longrightarrow & \mu(X,Y) \end{array}$$

is a pushout in the category of crossed modules.

Proof. This is as for the Corollary to Theorem B.

Remark. A particular adjunction space triple which arises commonly is when X is the union of closed subspaces X_1,X_2 with intersection X_0; Y is a subspace of X which is the union of closed subspaces Y_1,Y_2 with intersection Y_0 and with $Y_1 \subset X_1$, $Y_2 \subset X_2$; and Z is a common base point of all the X_ν, Y_ν. Then the Corollary to Theorem C applies to give a result similar to the Corollary to Theorem B but without the assumptions $Y_\nu = Y \cap X_\nu$, $\nu = 0,1,2$.

4. Pushouts of crossed modules.

The direct application of the adjunction theorem or of the union theorem requires computation of pushouts in the category of double groupoids, about which we know little. However, for path-connected spaces with base-point, double groupoids with one vertex suffice, and the applications require only knowledge of pushouts in the category $\mathbf{C}$ of crossed modules. (See the Corollaries to Theorems B and C.) This section describes the necessary computations in $\mathbf{C}$.

There is no problem about the existence of pushouts in $\mathbf{C}$. Crossed modules form an equational class of 2-sorted algebras and therefore $\mathbf{C}$ has all limits and colimits (see[7,13,14]). It also has various kinds of "free" objects given by left adjoints to various forgetful functors. In particular, if $\mathbf{G}$ denotes the category of groups, let $\mathbf{C}_0$ be the category whose objects are the morphisms of $\mathbf{G}$ and whose morphisms are commutative squares in $\mathbf{G}$. The functor $\mathbf{C} \to \mathbf{C}_0$ which forgets the group action of a crossed module has a left adjoint $F : \mathbf{C}_0 \to \mathbf{C}$ and, for a group homomorphism $\theta : A \to B$, we call $F(\theta) = F(A,B,\theta)$ the <u>free crossed module</u> on θ.

<u>Proposition 8</u>. The free crossed module on the group homomorphism $\theta : A \to B$ has the form $\partial : G \to B$, where G is the group generated by the set $A \times B$ with defining relations

(i) $(a_1,b)(a_2,b) = (a_1a_2,b)$ and

(ii) $(a_1,b_1)^{-1}(a_2,b_2)(a_1,b_1) = (a_2,b_2b_1^{-1}(\theta a_1)b_1)$

for all $a_1,a_2 \in A$ and $b,b_1,b_2 \in B$. The action of $b \in B$ on G is given by $(a,b_1)^b = (a,b_1b)$, and $\partial : G \to B$ is given by $\partial(a,b) = b^{-1}(\theta a)b$.

<u>Proof</u>. Let H be the group with generators $A \times N$ and defining relations (i). Then, for each $b \in B$, the set $\{(a,b);\ a \in A\}$ is a subgroup A^b of H, isomorphic to A, and H is the free product of these copies of A. We may therefore

define a group homomorphism $d : H \to B$ by $d(a,b) = b^{-1}(\theta a)b$. Also B acts naturally on H by permuting the free factors A^b according to the rule $(a,b_1)^b = (a,b_1 b)$. It is easy to see that $d(h^b) = b^{-1}(dh)b$ for all $h \in H, b \in B$, but (H,B,d) fails to be a crossed module because $h_1^{dh} \neq h^{-1}h_1 h$ in general. Let N be the normal subgroup of H generated by all elements $h_1^{dh}(h^{-1}h_1 h)^{-1}$ for $h,h_1 \in H$. Then N is stable under the action of B and is contained in the kernel of d. Hence B acts on $G = H/N$, and d induces a homomorphism $\partial : G \to B$ making (G,B,∂) a crossed module. It is easy to see that N is generated as normal subgroup by the elements $h_1^{dh}(h^{-1}h_1 h)^{-1}$ with h,h_1 running through the generators (a,b) of H; that is G is obtained from H by imposing the extra relations (ii). This argument proves the existence of a crossed module of the form described in the proposition with a canonical group homomorphism $\alpha : A \to G$ given by $a \mapsto (a,1)$. One needs only to check its universal property. But this is clear; for suppose we have a crossed module (M,N,δ) and a commuting diagram

$$\begin{tikzcd}
A \arrow[rr, "\mu"] \arrow[dd, "\theta"'] \arrow[dr, "\alpha"] & & M \arrow[dd] \\
& G \arrow[dd, "\partial"] & \\
B \arrow[rr, "\nu"] \arrow[dr, "1_B"'] & & N \\
& B \arrow[ur, "\nu"'] &
\end{tikzcd}$$

in which μ and ν are group homomorphisms. Then μ induces a group homomorphism $\mu^* : G \to M$ by the rule $(a,b) \mapsto (\mu a)^{\nu b}$, the counterparts of the relations (i) and (ii) holding in M by virtue of the crossed module axioms. This μ^* satisfies $\mu^* \circ \alpha = \mu$ and gives the required morphism of crossed modules $(\mu^*,\nu) : (G,B,\partial) \to (M,N,\delta)$.

Remark. In the special case when A is a free group with free generators $\{x_i\}$, this free crossed module is determined by B and the elements $y_i = \theta(x_i) \in B$, and it coincides with Whitehead's "free crossed B-module" ([24], p.455) as can be seen by comparing the two presentations.

A more general concept is useful in practice. We say that a morphism $(\alpha,\beta) : (A,B,\partial) \to (A^*,B^*,\partial^*)$ of crossed modules is <u>universal</u> if, whenever $(\mu,\nu) : (A,B,\partial) \to (M,N,\delta)$ is a morphism of crossed modules and ν factorizes in the form $\nu = \nu^* \circ \beta$ (where $\nu^* : B^* \to N$ is a group morphism), there is a unique group morphism $\mu^* : A^* \to M$ such that $\mu^* \circ \alpha = \mu$ and $(\mu^*,\nu^*) : (A^*,B^*,\partial^*) \to (M,N,\delta)$ is a morphism of crossed modules

$$\begin{tikzcd}
A \arrow[rr, "\mu"] \arrow[dr] \arrow[dd, "\partial"'] & & M \arrow[dd, "\delta"] \\
& A^* \arrow[ur, dashed] \arrow[dd, "\partial^*"] & \\
B \arrow[rr, "\nu"] \arrow[dr, "\beta"'] & & N \\
& B^* \arrow[ur, "\nu^*"'] &
\end{tikzcd}$$

Clearly, this condition determines A^*,α and ∂^* to within isomorphism if the crossed module (A,B,∂) and the group morphism $\beta : B \to B^*$ are given; we therefore refer to (A^*,B^*,∂^*) as the universal crossed module induced from (A,B,∂) by the group morphism $\beta : B \to B^*$. The existence of A^*, α and ∂^* under these circumstances is contained in the proof of the first of the following descriptions.

<u>Proposition 9</u>. Let $(\alpha,\beta) : (A,B,\partial) \to (A^*,B^*,\partial^*)$ be a universal morphism of crossed modules.

(i) If the free crossed module on the group morphism $\beta \circ \partial : A \to B^*$ is denoted by $d : G \to B^*$, with canonical group morphism $\gamma : A \to G$, then $A^* \simeq G/R$, where R is the B^*-subgroup of G generated by all $\gamma(a^b)(\gamma a)^{-\beta b}$ for $a \in A$, $b \in B$.

(ii) If $\beta : B \to B^*$ is a surjection, then $A^* \cong A/[A,K]$ (with $\alpha : A \to A^*$ the quotient map) where $K = \operatorname{Ker} \beta$ and $[A,K]$ is the subgroup of A generated by all $a^{-1}a^k$ for $a \in A$, $k \in K$.

(iii) If $\beta : B \to B^*$ is an injection and T is a right transversal of $\beta(B)$ in B^*, let H be the free product of groups $A_t (t \in T)$ each isomorphic to A by an isomorphism $a \mapsto a_t (a \in A)$. Let B^* act on H by the rule $(a_t)^{b^*} = (a^b)_u$, where $tb^* = (\beta b)u$ in B^* and $u \in T$. Let $\delta : H \to B^*$ be defined by $a_t \mapsto t^{-1}(\beta\partial a)t$. Then $A^* = H/S$, where S is the normal subgroup of H generated by all $g^{-1}h^{-1}g^{\delta g}$ for $g, h \in H$ (or for all g,h in some generating set of H).

<u>Proof</u> (i) Given a morphism of crossed modules $(\mu,\nu): (A,B,\partial) \to (M,N,\delta)$ and a group morphism $\nu^* : B^* \to N$ as in the diagram above, there is (by the universal property of free crossed modules) a group morphism $\mu' : G \to M$ such that $(\mu',\nu^*) : (G,B^*,d) \to (M,N,\delta)$ is a morphism of crossed modules and $\mu' \circ \gamma = \mu$. Now $d : G \to B^*$ kills R since it is a morphism of B^*-groups and when applied to $\gamma(a^b)(\gamma a)^{-\beta b}$ gives

$$\begin{aligned} d\gamma(a^b)(\beta b)^{-1}(d\gamma a)^{-1}(\beta b) &= \beta\partial(a^b)(\beta b)^{-1}(\beta\partial a)^{-1}(\beta b) \\ &= \beta\{\partial(a^b)b^{-1}(\partial a)^{-1}b\} = 1. \end{aligned}$$

Hence, putting $G^* = G/R$ we obtain an induced crossed module $d^* : G^* \to B^*$ and we now have a morphism of crossed modules $(\gamma^*,\beta) : (A,B,\partial) \to (G^*,B^*,d^*)$ since the action of B on A is carried to that of B^* on $G^* = G/R$. Also, $\mu' : G \to M$ kills R since its kernel is a B^*-subgroup of G and

$$\mu'\{\gamma(a^b)(\gamma a)^{-\beta b}\} = \mu'\gamma(a^b)(\mu'\gamma a)^{-\nu^*\beta b} = \mu(a^b)(\mu a)^{-\nu b} = 1.$$

Thus we have a morphism of crossed modules $(\mu^*,\nu^*) : (G^*,B^*,d^*) \to (M,N,\delta)$ with $\mu^* \circ \gamma^* = \mu$. This shows that $(\gamma^*,\beta) : (A,B,\partial) \to (G^*,B^*,d^*)$ is universal and hence that $A^* \cong G^*$.

(ii) For $b \in B$, $a \in A$ and $k \in K$ we have

$(a^{-1}a^k)^b = (a^b)^{-1} (a^b)^{b^{-1}kb} \in [A,K]$, so that $[A,K]$ is stable under the action of B. Thus B acts on $A/[A,K]$, and K acts trivially, giving an action of $B^* \cong A/K$ on $A/[A,K]$. One easily sees that $(A/[A,K], B^*,\partial^*)$ is a crossed module, where $\partial^* : A/[A,K] \to B^*$ is induced by $\beta \circ \partial : A \to B^*$, and that is has the right universal property.

(iii) The proof is similar to that of Proposition 8. One first verifies that the given action of B^* on H is a group action. With this action, $\delta : H \to B^*$ is a morphism of B^*-groups since, if $tb^* = (\beta b)u$ with $t, u \in T$, $b \in B, b^* \in B^*$, then

$$\begin{aligned}
\delta((a_t)^{b^*}) = \delta((a^b)_u) &= u^{-1}(\beta\partial(a^b))u \\
&= u^{-1}\beta(b^{-1}(\partial a)b)u \\
&= u^{-1}(\beta b)^{-1}(\beta\partial a)(\beta b)u \\
&= (b^*)^{-1}\, t^{-1}(\beta\partial a)tb^* \\
&= (b^*)^{-1}\, \delta(a_t)b^*.
\end{aligned}$$

Hence, as in Proposition 8, we obtain a crossed module $(H/S, B^*, \partial^*)$ which clearly has the required universal property.

Remarks. (1) Composites of universal morphisms of crossed modules are again universal. Hence, by combining (ii) and (iii) of Proposition 9, one can obtain for any universal morphism $(\alpha,\beta) : (A,B,\partial) \to (A^*,B^*,\partial^*)$ a presentation of A^* as a group in terms of A,B,B^*,∂ and β. An alternative presentation can be obtained by combining Proposition 9 (i) with Proposition 8.

(2). For a general pushout of crossed modules

$$\begin{tikzcd}
 & (A_1,B_1,\partial_1) \arrow[dr] & \\
(A_0,B_0,\partial_0) \arrow[ur, "{(\alpha_1,\beta_1)}"] \arrow[dr, "{(\alpha_2,\beta_2)}"'] & & (A,B,\partial) \\
 & (A_2,B_2,\partial_2) \arrow[ur] &
\end{tikzcd}$$

one can obtain explicit presentations of A and B as groups in terms of the $A_i,B_i,\partial_i,\alpha_i,\beta_i$. First, B is the pushout in $\underset{\sim}{G}$ (the category of groups) of the $\beta_i : B_0 \to B_i$ ($i = 1,2$,): this is because the forgetful functor $(A,B,\partial) \mapsto B$ from $\underset{\sim}{C}$ to $\underset{\sim}{G}$ has a right adjoint $B \mapsto (0,B,0)$. This gives a presentation of B and we may approach A in several ways. For example, the group morphisms $\beta_i : B_i \to B$ induce universal morphisms

$$(\alpha_1', \beta_i) : (A_i B_i, \partial_i) \longrightarrow (A_i', B, \partial_i') \qquad (i = 0,1,2)$$

giving rise to a pushout diagram.

$$\begin{tikzcd}[column sep=small] & (A_1', B, \partial_1') \arrow[dr] & \\ (A_0', B, \partial_0') \arrow[ur] \arrow[dr] & & (A, B, \partial) \\ & (A_2', B, \partial_2') \arrow[ur] & \end{tikzcd}$$

of crossed B-modules (for fixed B). We can obtain presentations of A_0', A_1', A_2' from Proposition 9 and hence of the pushout A' of the induced morphisms $A_0' \to A_1'$ and $A_0' \to A_2'$ in $\underset{\sim}{G}$. The group A' is equipped with a morphism $\partial: A' \to B$ and an action of B on A'. The group A is now obtained as a quotient A'/N, where N is the normal closure of the elements $x^{-1}y^{-1}x\,y^{\partial x}$ for $x,y \in A'$.

<u>Proposition 10</u> (i) Given a group morphism $\theta : A \to B$, the crossed module (G,B,∂) is the free crossed module on θ, with canonical morphism $\alpha : A \to G$, if and only if the morphism of crossed modules $(\alpha,\beta) : (A, A, 1_A) \to (G,B,\partial)$ is universal.

(ii) A morphism of crossed modules

$$(\alpha,\beta) : (A,B,\partial) \longrightarrow (A^*,B^*,\partial^*)$$

is universal if and only if the diagram

$$\begin{tikzcd}[column sep=small] & (0, B^*, 0) \arrow[dr] & \\ (0, B, 0) \arrow[ur] \arrow[dr] & & (A^*, B^*, \partial^*) \\ & (A, B, \partial) \arrow[ur] & \end{tikzcd}$$

is a pushout of crossed modules.

(iii) A diagram

$$\begin{tikzcd}[column sep=small] & (0,B_1,0) \arrow[dr] & \\ (0,B_0,0) \arrow[ur] \arrow[dr] & & (A,B,\partial) \\ & (A_2,B_2,\partial_2) \arrow[ur] & \end{tikzcd}$$

is a pushout of crossed modules if and only if $(A_2,B_2,\partial_2) \to (A,B,\partial)$ is universal and the induced diagram

$$\begin{tikzcd}[column sep=small] & B_1 \arrow[dr] & \\ B_0 \arrow[ur] \arrow[dr] & & B \\ & B_2 \arrow[ur] & \end{tikzcd}$$

of morphisms of groups is a pushout of groups.

<u>Proof</u>. This is entirely a matter of checking the appropriate universal properties.

5. Applications to second relative homotopy groups

The basic result for our applications is the following special case of the corollaries to Theorems B and C.

Theorem D. Suppose that the commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X_1 & \xrightarrow[\bar{f}]{} & X \end{array}$$

of pointed spaces satisfies one of the two following hypotheses.

Hypothesis A : the maps $i, f, \bar{i}, \bar{f}$, are inclusions of subspaces, $X_0 = X_1 \cap X_2$, and X is the union of the interiors of X_1 and X_2.

Hypothesis B: the map i is a closed cofibration and X is the adjunction space $X_2 \cup_f X_1$.

Suppose also that X_0, X_1, X_2 are path-connected and that $i_* : \pi_1(X_0) \to \pi_1(X_1)$ is surjective. Then $\bar{f}_* : \mu(X_1, X_0) \to \mu(X, X_2)$ is a universal morphism of crossed modules.

Proof. Consider the diagram of triples

$$\begin{array}{ccc} (X_0, X_0, *) & \xrightarrow{f} & (X_2, X_2, *) \\ {\scriptstyle (i,1,*)}\downarrow & & \downarrow \\ (X_1, X_0, *) & \xrightarrow{\bar{f}} & (X, X_2, *) \end{array} .$$

When Hypothesis A (resp. Hypothesis B) is satisfied we may apply Theorem B, Corollary (resp. Theorem C, Corollary) to obtain a pushout of crossed modules

$$\begin{array}{ccc} \mu(X_0, X_0) & \xrightarrow{f_*} & \mu(X_2, X_2) \\ \downarrow & & \downarrow \\ \mu(X_1, X_0) & \xrightarrow[\bar{f}_*]{} & \mu(X, X_2) \end{array}$$

But $\mu(X_0,X_0)$ and $\mu(X_2,X_2)$ are the trivial crossed modules $0 \to \pi_1(X_0)$ and $0 \to \pi_1(X_2)$, so $\bar{f}_* : \mu(X_1,X_0) \to \mu(X,X_2)$ is universal by Proposition 10(ii).

<u>Corollary 1</u>. (Homotopy excision theorem in dimension 2).
Let X be the union of subspaces X_1,X_2 satisfying either Hypothesis $\underset{\sim}{A}$ or the following Hypothesis $\underset{\sim}{B}'$: X_1 and X_2 are closed, $X_0 = X_1 \cap X_2$ and $i : X_0 \to X_1$ is a cofibration. Assume that X_0,X_1,X_2 are non-empty and path-connected and choose a base-point in X_0. If $\pi_1(X_1,X_0) = \pi_1(X_0,X_0) = 0$ then the excision map $\varepsilon : \pi_2(X_1,X_0) \to \pi_2(X,X_2)$ is surjective. If, in addition, $\pi_2(X_2,X_0) = 0$ then ε is an isomorphism.

<u>Proof</u>. Since $\pi_1(X_1,X_0) = 0$, $\pi_1(X_0) \to \pi_1(X_1)$ is surjective and the theorem therefore gives a universal morphism

$$\begin{array}{ccc} \pi_2(X_1,X_0) & \xrightarrow{\varepsilon} & \pi_2(X,X_2) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ \pi_1(X_0) & \xrightarrow{j} & \pi_1(X_2) \end{array} ,$$

where j is the inclusion of X_0 in X_2.
Since $\pi_1(X_2,X_0) = 0$, j_* is surjective, and therefore so is ε by Proposition 9(ii). If also $\pi_2(X_2,X_0) = 0$ then j_* is an isomorphism, and therefore so is ε.

We observe that Theorem D is stronger than the 2-dimensional homotopy excision theorem ([12],p.211) in several respects:

(i) It gives an algebraic description of $\pi_2(X,X_2)$ without the assumption that $\pi_1(X_2,X_0) = 0$.

(ii) In the case $\pi_1(X_1,X_0) = \pi_1(X_2,X_0) = 0$ it not only gives the surjectivity of $\varepsilon : \pi_2(X_1,X_0) \to \pi_2(X,X_2)$ but also determines its kernel. In fact, if K is the kernel of $j_* : \pi_1(X_0) \to \pi_1(X_2)$ then K acts on $G = \pi_2(X_1,X_0)$ and the kernel of ε is $[G,K]$, by Proposition 9(ii).

(iii) The extra condition $\pi_2(X_2,X_0) = 0$ needed to show that the excision map is an isomorphism can be weakened to the assumption that the boundary map $\pi_2(X_2,X_0) \to \pi_1(X_0)$ is trivial.

(iv) The theorem applies to a general adjunction space situation. We illustrate these points with some further applications.

Corollary 2. Let $f : Z \to Y$ be a map of path-connected, pointed space and let $X = Y \cup_f CZ$, where CZ is the non-reduced cone on Z. Then the crossed module $\mu(X,Y)$ is the free crossed module on the group morphism $f_* : \pi_1(Z) \to \pi_1(Y)$.

Proof. We take $X_0 = Z$, $X_1 = CZ$, $X_2 = Y$ in Theorem D. This gives a universal morphism $\mu(CZ,Z) \to \mu(X,Y)$. But CZ is contractible so the boundary map $\partial : \pi_2(CZ,Z) \to \pi_1(Z)$ is an isomorphism. Hence, by Proposition 10(i) $\mu(X,Y)$ is the free crossed module on $f_* : \pi_1(Z) \to \pi_1(Y)$.

Remark. Any space $\bar{Y}$ obtained from Y by attaching a family of 2-cells is homotopy equivalent, rel Y, to a space $X = Y \cup_f CZ$, where Z is a bunch of circles. In this case $\pi_1(Z)$ is a free group and we recover Whitehead's theorem that $\pi_2(\bar{Y},Y)$ is a free crossed $\pi_1(Y)$-module, with one generator $y_i \in \pi_1(Y)$ for each cell attached (See[24], p.493).

Corollary 3 Let $X = Y \cup_f CZ$ as in Corollary 2, and suppose that the attaching map $f : Z \to Y$ induces a surjection $f_* : \pi_1(Z) \to \pi_1(Y)$. Then $\pi_2(X,Y) \cong \pi_1(Z)/[\pi_1(Z),K]$, where $K = \operatorname{Ker} f_*$ and $[\ ,\ ]$ denotes a commutator subgroup. Hence there is an exact sequence

$$\pi_2(Y) \to \pi_2(X) \to K/[\pi_1(Z), K] \to 0.$$

Proof. As in Corollary 2, we have a universal morphism of crossed modules

$$\begin{array}{ccc} \pi_1(Z) & \longrightarrow & \pi_1(X,Y) \\ \Big\downarrow 1 & & \Big\downarrow \partial \\ \pi_1(Z) & \xrightarrow{f_*} & \pi_1(Y) \end{array}$$

and since f_* is surjective, Proposition 9(ii) gives $\pi_2(X,Y) \cong \pi_1(Z)/[\pi_1(Z),K]$. We therefore have a homotopy exact sequence

$$\pi_2(Y) \to \pi_2(X) \to \pi_1(Z)/[\pi_1(Z),K] \xrightarrow{\partial} \pi_1(Y),$$

and since ∂ is induced by $f_* : \pi_1(Z) \to \pi_1(Y)$ its kernel is $K/[\pi_1(Z),K]$, as asserted.

Remarks. (i) If we take Y to be a point in Corollary 3 we can deduce as a special case that $\pi_2(SZ) \cong \pi_1(Z)^{Ab}$ for any path-connected space Z; this also follows from the absolute Hurewicz theorem.

(ii) Under the hypotheses of Corollary 3, the van Kampen theorem gives $\pi_1(X) = \pi_1(Y \cup_f CZ) = 0$. Hence, by the Hurewicz theorem, $\pi_2(X) \cong H_2(X)$. If we further assume that $\pi_2(Y) = 0$, we obtain $H_2(X) \cong K/[\pi_1(Z),K]$ and so the homology sequence of (X,Y) becomes

$$\ldots \to H_3(X,Y) \to H_2(Y) \to K/[\pi_1(Z),K] \to H_2(X,Y) \to H_1(Y) \to 0.$$

Since $H_n(X,Y) \cong H_n(Y \cup_f CZ, Y) \cong H_n(SZ) \cong H_{n-1}(Z)$, the sequence can be rewritten

$$\ldots \to H_2(Z) \to H_2(Y) \to K/[\pi_1(Z),K] \to H_1(Z) \to H_1(Y) \to 0.$$

Applying this to Eilenberg-MacLane spaces $Z = K(G,1)$, $Y = K(Q,1)$, we deduce that any short exact sequence of groups

$$0 \to K \to G \to Q \to 0$$

gives rise to an exact sequence

$$H_2(G) \to H_2(Q) \to K/[G,K] \to H_1(G) \to H_1(Q) \to 0,$$

a result which appears in [20] and [21] .

The next corollary covers a more general situation including the attaching of handles.

Corollary 4. Let A,B,Y be the path-connected, pointed spaces, and suppose we are given a map $f : A \times B \to Y$. Let $X = Y \cup_f (CA \times B)$. Then there is a universal morphism of crossed modules

$$\begin{array}{ccc}\pi_1(A) & \longrightarrow & \pi_2(X,Y) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle \partial} \\ \pi_1(A) \times \pi_1(B) & \xrightarrow[f_*]{} & \pi_1(Y)\end{array}$$

where $\pi_1(A)$ acts on itself by inner automorphisms and $\pi_1(B)$ acts trivially on $\pi_1(A)$. Hence $\pi_2(X,Y) \cong G/R$, where $G \to \pi_1(Y)$ is the free crossed module on the map $f_*i_1 : \pi_1(A) \to \pi_1(Y)$ and R is the subgroup of G generated by all $\gamma(a)^y\,\gamma(a)^{-b^*}$ for $a \in \pi_1(A)$, $b \in \pi_1(B)$, $y \in \pi_1(Y)$, γ being the canonical map $\pi_1(A) \to G$ and b^* the image of b under $f_*i_2 : \pi_1(B) \to \pi_1(Y)$.

Proof. Theorem D gives a universal morphism $\mu(CA \times B,\ A \times B) \to \mu(X,Y)$, and it is easy to identify $\pi_2(CA \times B,\ A \times B)$ as $\pi_1(A)$, with boundary map the canonical injection $\pi_1(A) \to \pi_1(A) \times \pi_1(B)$. This gives the first statement. According to the description of universal morphisms given in Proposition 9(i), $\pi_2(X,Y)$ is now obtained as the quotient G/R, where R is the $\pi_1(Y)$-subgroup generated by all $\gamma(a_1^{(a_2,b)})\,\gamma(a_1)^{-f_*(a_2,b)}$. Since $\gamma(a_1^{(a_2,b)}) = \gamma(a_1^{a_2}) = \gamma(a_1)^{a_2^*}$ when $a_2^* = f_*i_1(a_2) \in \pi_1(Y)$ and $f_*(a_2,b) = a_2^*\,b^* = b^*\,a_2^*$, R is generated as a subgroup by all $\gamma(a_1)^y\,\gamma(a_1)^{-b^*y}$.

Corollary 5. If in Corollary 4 we take $A = S^p$, $B = S^q$ $(p,q \geq 1)$ so that $X = Y \cup_f (E^{p+1} \times S^q)$ where $f : S^p \times S^q \to Y$, then

(i) $\pi_2(X,Y) = 0$ if $p \geq 2$;

(ii) for $p = 1$ and $q \geq 2$, $\pi_2(X,Y)$ is the free crossed module over $\pi_1(Y)$ with one generator g whose image in $\pi_1(Y)$ is the class of the loop $fi_1 : A \to Y$;

(iii) for $p = q = 1$, $\pi_2(X,Y)$ is the free crossed module over $\pi_1(Y)$ with one generator g as in (ii) and one defining relation $g^y = g$, where $y \in \pi_1(Y)$ is the class of the loop $fi_2 : B \to Y$.

Proof. (i) When $p \geq 2$, we have $\pi_1(A) = \pi_1(S^p) = 0$ and hence $\pi_2(X,Y) = 0$ by Corollary 4.

(ii) When $p = 1$ and $q \geq 2$, Corollary 4 gives a universal morphism

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "id"'] & \pi_2(X,Y) \arrow[d] \\ Z \arrow[r] & \pi_1(Y) \end{tikzcd}$$

and by Proposition 10(i), this makes $\pi_2(X,Y)$ a free $\pi_1(Y)$-module as described.

(iii) When $p = q = 1$ we have a universal morphism

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i_1"'] & \pi_2(X,Y) \arrow[d] \\ Z \times Z \arrow[r] & \pi_1(Y) \end{tikzcd}$$

and $\pi_2(X,Y) = G/R$ where G is the free crossed $\pi_1(Y)$-module on g as in (ii) and R is generated as $\pi_1(Y)$-submodule by g^y, by Corollary 4.

Remarks. 1) The assumption in Theorem D that $i_* : \pi_1(X_0) \to \pi_1(X_1)$ is surjective cannot be dropped, as is shown by the following example. Let $X_0 = *$, $X_1 = S^1$, $X_2 = S^2$ so that $X = S^1 \vee S^2$. Then $\mu(X_1,X_0) = (0,0,0)$, $\mu(X,X_2) = (\pi_2(S^1 \vee S^2,S^2),0,0)$ and so the induced morphism $\mu(X_1,X_0) \to \mu(X, X_2)$ is not universal since in the homotopy exact sequence of the pair $(S^1 \vee S^2,S^2)$, $\pi_2(S^2) \to \pi_2(S^1 \vee S^2)$ is not surjective (since $\pi_2(X) \cong H_2(\tilde{X})$ is an infinite sum of copies of Z).

2) We can however obtain universal morphisms of crossed modules under more general assumptions than those of Theorem D. Consider for example the situation of the Corollary to Theorem C. Then by Proposition 10(iii) the morphism $\mu(X_1,Y_1) \to \mu(X,Y)$ of crossed modules is universal if in addition to the assumption of this Corollary we have $\pi_2(X_2,Y_2) = 0$, $\pi_2(X_0,Y_0) = 0$.

6. The Union Theorem

Let $\underset{\sim}{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of X whose interiors cover X. For each $\nu = (\nu_1, \ldots, \nu_n) \in \Lambda^n$, let X_ν be the intersection of the sets X_{ν_i}. Let Y be a subspace of X, let Z be a subspace of Y and for each ν in Λ^n define

$$Y_\nu = Y \cap X_\nu \quad , \; Z_\nu = Z \cap X_\nu .$$

We shall suppose that for each ν in Λ^2 and $\zeta \in Z_\nu$ the induced morphism $\pi_1(Z_\nu,\zeta) \to \pi_1(Y_\nu,\zeta)$ is trivial: this will imply that $\rho(X_\nu,Y_\nu,Z_\nu)$ is defined for each $\nu \in \Lambda^2$ and also for each $\nu \in \Lambda$.

The ρ-sequence of the cover $\underset{\sim}{X}$ is defined to be the diagram

$$\bigsqcup_{\nu\in\Lambda^2} \rho(X_\nu,Y_\nu,Z_\nu) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \rho(X_\lambda,Y_\lambda,Z_\lambda) \xrightarrow{c} \rho(X,Y,Z)$$

in which a,b are determined by the inclusions $a_\nu : X_\lambda \cap X_\mu \to X_\lambda$ and $b_\nu : X_\lambda \cap X_\mu \to X_\mu$ for each $\nu = (\lambda,\mu) \in \Lambda^2$ and c is determined by the inclusions $c_\lambda : X_\lambda \to X$, for $\lambda \in \Lambda$.

Theorem E (The Union Theorem). In the above ρ-sequence of the cover $\underset{\sim}{X}$ c is th coequaliser of a,b in the category of double groupoids if

(i) the induced maps $\pi_0(Z_\nu) \to \pi_0(X_\nu)$, $\pi_0(Z_\nu) \to \pi_0(Y_\nu)$ are surjective for all $\nu \in \Lambda^8$,

(ii) the induced maps $\rho_1(Y_\nu,Z_\nu) \to \rho_1(X_\nu,Z_\nu)$ are surjective for all $\nu \in \Lambda^8$.

Remarks 1. Conditions (i),(ii) of the theorem imply conditions on r-fold intersections for $1 \leq r < 8$ since for such r any $\nu \in \Lambda^r$ defines $\nu' \in \Lambda^8$ by allowing suitable repetitions.

2. The conditions on 8-fold intersections can probably be reduced to conditions on 4-fold intersections, but not to 2-fold intersections only. An example for this is given later.

Proof. Suppose that we are given a morphism $f' : \bigsqcup_{\lambda\in\Lambda} \rho(X_\lambda, Y_\lambda, Z_\lambda) \to G$ of double groupoids such that $f' \circ a = f' \circ b$. We have to show that there is a unique morphism $f : \rho(X,Y,Z) \to G$ of double groupoids such that $f \circ c = f'$. We first define f on the 2-dimensional part of $\rho(X,Y,Z)$ and to this end we construct for certain special elements of $R_2(X,Y,Z)$ corresponding elements of G_2.

Suppose that θ in $R_2(X,Y,Z)$ is such that θ lies in some set X_λ of $\underset{\sim}{X}$ (by which we mean that $\theta(I^2)$ is contained in X_λ). Then θ defines uniquely an element $\theta^{(\lambda)}$ of $R_2(X_\lambda, Y_\lambda, Z_\lambda)$ and we obtain an element $F(\theta)$ of G_2 by the rule $F(\theta) = f'(\bar{\theta}^{(\lambda)})$. It is a straightforward consequence of the condition $f' \circ a = f' \circ b$ that $F(\theta)$ is independent of the choice of λ such that θ lies in X_λ.

Suppose now that ϕ, ψ in $R_2(X,Y,Z)$ are such that ϕ lies in X_λ, ψ lies in X_μ and also $\theta = \phi + \psi$ is defined. Then $F(\phi)$, $F(\psi)$ are uniquely defined in G_2. Further $\partial_1\phi = \partial_0\psi$ lies in $X_\lambda \cap X_\mu$ and so the condition $f' \circ a = f' \circ b$ implies that

$$\partial_1 F(\phi) = f'(\overline{(\partial_1\phi)}^{(\lambda)}) = f'(\overline{(\partial_0\psi)}^{(\mu)}) = \partial_0 F(\psi).$$

Hence we obtain in G_2 an element $g = F(\phi) + F(\psi)$ which a priori depends on ϕ, ψ and not just on $\theta = \phi + \psi$.

Similar remarks apply to composites $\phi \circ \xi$.

More generally, suppose we are given a subdivision $\theta = (\theta_{ij})$ of θ in $R_2(X,Y,Z)$ such that each θ_{ij} is in $R_2(X,Y,Z)$ and also lies in some $X_\nu, \nu \in \Lambda^n$, for some n. Then θ_{ij} also lies in some $X_\lambda, \lambda \in \Lambda$, and so $g_{ij} = F(\theta_{ij})$ is well-defined. By a similar argument to the above so also is $g = [g_{ij}]$. We now show in Lemma 1 that this element g is not changed if θ is varied by certain special homotopies. This Lemma is the key place where we use the connections and also the cancellations which the double groupoid structure allows.

A convention we adopt from now on is that given a subdivision $\theta = [\theta_{ij}]$ and given for each (i,j) a ν in Λ^n such that θ_{ij} lies in X_ν, we write X_{ij} for X_ν, Y_{ij} for $Y \cap X_{ij}$ and Z_{ij} for $Z \cap X_{ij}$.

<u>Lemma 1.</u> Let $\theta, \theta^* \in R_2(X,Y,Z)$ and let $H : (I^2, \dot{I}^2, \ddot{I}^2) \times I \to (X,Y,Z)$ be a homotopy from θ to θ^*. Let $H = [H_{ij}]$ be an $(m \times n \times 1)$- subdivision of H such that each H_{ij} lies in some two-fold intersection X_{ij} of elements of $\underset{\sim}{X}$. Suppose also that all the edges of all the H_{ij} lie in Y and all their vertices lie in Z. Let $\theta = [\theta_{ij}]$, $\theta^* = [\theta^*_{ij}]$ be the subdivisions of θ, θ^* induced by that of H, so that θ_{ij}, θ^*_{ij} are in $R_2(X,Y,Z)$. If $g_{ij} = F(\theta_{ij})$, $g^*_{ij} = F(\theta^*_{ij})$, then $[g_{ij}] = [g^*_{ij}]$ in G.

<u>Proof.</u> Let the faces of H_{ij} other than θ_{ij}, θ^*_{ij} be $\beta_{i,j-1}$, $\beta_{i,j}$, $\gamma_{i-1,j}$, γ_{ij}, oriented according to the conventions adopted in the homotopy addition lemma (Proposition 4).

The edges of H_{ij} lie in Y_{ij} and its vertices in Z_{ij}, so Proposition 4 gives a relation in $\rho_2(X_{ij},Y_{ij},Z_{ij})$ between the classes represented by the six faces of H_{ij}. This relation is carried by the morphism f' to a relation in G_2 of the form

$$g^*_{ij} = \begin{bmatrix} -\Gamma^{-1} & c^{-1}_{i-1,j} & \Gamma^{-1} \\ -b_{i,j-1} & g_{ij} & b_{i,j} \\ -\Gamma & c_{ij} & \Gamma \end{bmatrix} \qquad (1)$$

where $b_{ij} = F(\beta_{ij})$, $c_{ij} = F(\gamma_{ij})$.

The interchange law for G allows us to refine the subdivision $g^* = [g^*_{ij}]$ by the substitution (1) and compose the new parts in any convenient fashion. By cancellation of pairs b_{ij}, $-b_{ij}$ and c_{ij}, c^{-1}_{ij}, and by composing degenerate elements and absorbing 0's and 1's, we can obtain a new subdivision of g^* of the form

$$\begin{bmatrix} -\Gamma^{-1} & c^{-1}_{01} & \cdots\cdots & c^{-1}_{0n} & \Gamma^{-1} \\ -b_{10} & g_{11} & \cdots\cdots & g_{1n} & b_{1n} \\ -b_{20} & g_{21} & \cdots\cdots & g_{2n} & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{m0} & g_{m1} & \cdots\cdots & g_{mn} & b_{mn} \\ -\Gamma & c_{m1} & \cdots\cdots & c_{mn} & \Gamma \end{bmatrix} \qquad (2)$$

Now H maps $\ddot{I}^2 \times I$ into Z, so each of the four edges joining θ to θ^* is a path in Z_λ for some λ. By Proposition 3(i) these edges represent identity elements in the corresponding $\rho_1(X_\lambda, Y_\lambda, Z_\lambda)$ and therefore map to identity elements in G_1. Hence each of the degenerate elements at the four corners of (2) is of the form $\Gamma(e) = \circledcirc$. Similarly, H maps $\dot{I}^2 \times I$ into Y, so each of the border squares $b_{i0}, b_{in}, c_{0j}, c_{mj}$ lies in Y and therefore, by Proposition 3(iii), represents a degenerate element of some $\rho_2(X_\lambda, Y_\lambda, Z_\lambda)$. The border elements of (2) are therefore all degenerate in G_2. So we can easily write

$$g^* = \begin{bmatrix} \odot & c_0 & \odot \\ b_0 & [g_{ij}] & b_1 \\ \odot & c_1 & \odot \end{bmatrix}$$

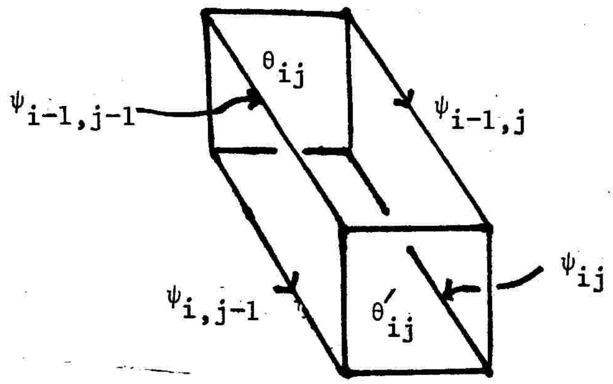
where b_0, c_0, b_1, c_1 are degenerate. Also the vertical edges of c_0, c_1 are identities in G_1, so $c_0 = 1_p, c_1 = 1_q$ where $p = \varepsilon_0 g^*, q = \varepsilon_1 g^*$. Similarly b_0, b_1 are zero squares. So $g^* = [g_{ij}]$, which proves the lemma.

We now show that some more general homotopies H from θ to θ^* can be reduced to those of the type of Lemma 1.

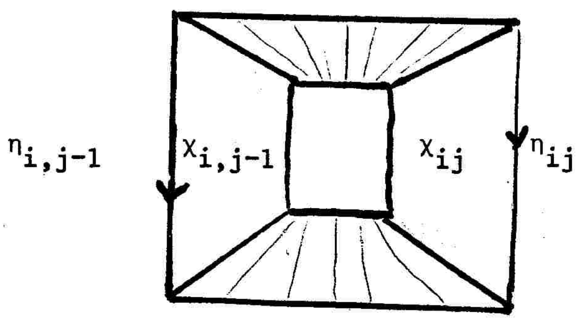
<u>Lemma 2.</u> Let θ, $\theta^* \in R_2(X,Y,Z)$ and let $H : (I^2, \dot{I}^2, \ddot{I}^2) \times I \to (X,Y,Z)$ be a homotopy from θ to θ^*. Let $H = [H_{ij}]$ be an $(m \times n \times 1)$-subdivision of H such that each H_{ij} lies in some two-fold intersection X_{ij} of elements of $\underset{\sim}{X}$. Let $\theta = [\theta_{ij}]$, $\theta^* = [\theta^*_{ij}]$ be the subdivision of θ, θ^* induced by that of H, and suppose that θ_{ij}, θ^*_{ij} are in $R_2(X,Y,Z)$ for all i,j. Then there is a homotopy $\hat{H} = [\hat{H}_{ij}]$ from $[\theta_{ij}]$ to $[\theta^*_{ij}]$ with $\hat{H}_{ij}$ lying in X_{ij} and such that the edges of $\hat{H}_{ij}$ lie in Y. Further $[F(\theta_{ij})] = [F(\theta^*_{ij})]$ in G_2.

<u>Proof.</u> The final conclusion follows from Lemma 1, and so it is enough to construct the homotopy $\hat{H}$.

The vertices of H_{ij} all lie in Z_{ij}, and those edges of H_{ij} which lie in θ_{ij} or θ^*_{ij} also lie in Y_{ij}. We label the other four edges as follows



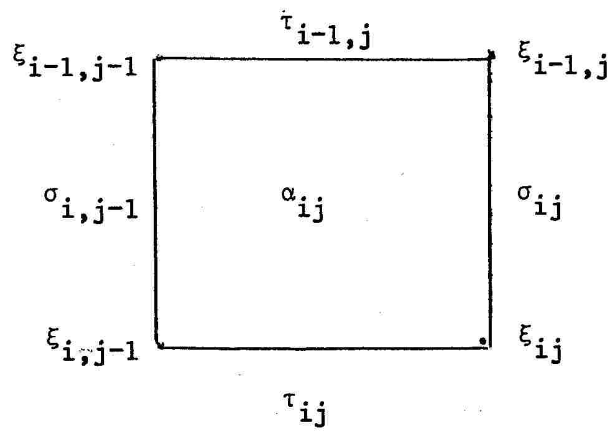
We now invoke hypothesis (ii) of the theorem to choose, for each ψ_{ij} a path η_{ij} in Y with the same end points as ψ_{ij} and a homotopy χ_{ij} in X which is a homotopy rel end points from ψ_{ij} to η_{ij}. We make these choices in such a way that χ_{ij} lies in the intersection $X_{ij} \cap X_{i+1,j} \cap X_{i,j+1} \cap X_{i+1,j+1}$ (this is where we use the assumption about 8-fold intersections) if ψ_{ij} does not lie on the boundary of H, while for each of the edges $\psi_{0,j}$, $\psi_{i,0}$, $\psi_{m,j}$, $\psi_{i,n}$ which do lie on the boundary of H, and are therefore in Y, we choose $\eta_{ij} = \psi_{ij}$ and χ_{ij} to be the trivial homotopy. We now replace each cube H_{ij} by a cube $\hat{H}_{ij}$ having θ_{ij} and θ^*_{ij} as opposite faces and having $\eta_{i-1,j-1}$, $\eta_{i-1,j}$, $\eta_{i,j-1}$, η_{ij} as edges joining them. This can be done by a canonical construction (of which we omit details) so that the faces of $\hat{H}_{ij}$ apart from θ_{ij}, θ^*_{ij} are typically of the following type:



So the cubes $\hat{H}_{ij}$ determine a homotopy H from θ to θ^* as required.

We now turn to the construction of $F(\alpha) \in G_2$ for a general element α of $R_2(X,Y,Z)$. Since X is covered by the interiors of the sets X_λ, there is an $(m \times n)$-subdivision $\alpha = [\alpha_{ij}]$ (in the strict sense defined in § 2) such that α_{ij} lies in some set X_{ij}, say. However α_{ij} need not be an element of $R_2(X,Y,Z)$, and so we shall construct a homotopy $h = [h_{ij}]$ from $\alpha = [\alpha_{ij}]$ to an element $\theta = [\theta_{ij}]$ such that each θ_{ij} lies in some X_λ and is in $R_2(X,Y,Z)$, and then define $F(\alpha) = [F(\theta_{ij})]$.

To this end, we start by labelling the edges and vertices of α_{ij} as follows:



and construct h_{ij} from α_{ij} to θ_{ij}.

In what follows it is convenient to set $X_{0,j} = X_{1,j}$, $X_{m+1,j} = X_{m,j}$, $X_{i,0} = X_{i,1}$, $X_{i,n+1} = X_{i,n}$.

(1) The point ξ_{ij} will lie in $X_{ij} \cap X_{i+1,j} \cap X_{i,j+1} \cap X_{i+1,j+1}$, and so we choose a point ζ_{ij} in Z and a path ω_{ij} from ξ_{ij} to ζ_{ij} such that ω_{ij} is a path in this intersection. This choice is possible by hypothesis (i). We restrict the choices still further by insisting that if ξ_{ij} belongs to Y then ω_{ij} is a path in Y (again invoking hypothesis (i)) and if ξ_{ij} belongs to Z then ω_{ij} is a path in Z (for example a constant path).

(2) Having chosen the ω_{ij} for $0 \le i \le m$, $0 \le j \le n$ we next choose for each horizontal edge τ_{ij} a square $T_{ij} : I^2 \to X$ whose

restrictions to $I \times \{0\}$, $\{0\} \times I$ and $\{1\} \times I$ are respectively τ_{ij}, $\omega_{i,j-1}$ and ω_{ij} and whose restriction to $I \times \{1\}$ is a path τ'_{ij} in Y and such that T_{ij} is a square in $X_{ij} \cap X_{i+1,j}$. This choice is possible by hypothesis (ii). We further insist that if τ_{ij} is a path in Y (whence by previous choices $\omega_{i,j-1}$ and ω_{ij} are paths in Y) then T_{ij} is a square in Y; for example we may take T_{ij} to be degenerate with $\tau'_{ij} = [\omega^{-1}_{i,j-1}, \tau_{ij}, \omega_{ij}]$.

(3) For each vertical edge σ_{ij} we choose in like manner a square S_{ij} with edges σ_{ij}, $\omega_{i-1,j}$, ω_{ij} and σ'_{ij} subject to similar conditions.

(4) We map $B = (I^2 \times \{0\}) \cup (\dot{I}^2 \times I)$ to X by combining the following maps:

$$(x,y,0) \mapsto \alpha_{ij}(x,y,0),$$

$$(x,0,z) \mapsto S_{i-1,j}(x,z)\ ,\ (x,1,z) \mapsto S_{i,j}(x,z),$$

$$(0,y,z) \mapsto T_{i-1,j}(y,z)\ ,\ (1,y,z) \mapsto T_{i,j}(y,z)\ .$$

This map is now composed with a fixed retraction $I^2 \times I \to B$ to define a homotopy h_{ij} from α_{ij} to the opposite face θ_{ij}. The cubes h_{ij} clearly form an $(m \times n \times 1)$ - subdivision of a cube h, which is a homotopy from $\alpha = [\alpha_{ij}]$ to $\theta = [\theta_{ij}]$, and since α has edges in Y and vertices in Z, our construction ensures that h maps $(I^2, \dot{I}^2, \ddot{I}^2) \times I$ into (X,Y,Z). Hence α and θ represent the same element $\bar{\alpha}$ of $\rho_2(X,Y,Z)$.

If a morphism $f' : \rho(X,Y,Z) \to G$ exists satisfying $f \circ c = f'$, then

$$f(\bar{\alpha}) = f(\bar{\theta}) = [f(\bar{\theta}_{ij})] = [F(\theta_{ij})].$$

This proves the uniqueness of f. To prove the existence of f, we show that (a) $F(\alpha) = [F(\theta_{ij})]$ depends only on the class $\bar{\alpha}$ of α in $\rho(X,Y,Z)$ (b) the map f defined by F can be extended to $\rho_1(X,Y,Z)$ and $\rho_0(X,Y,Z)$ so as to define a morphism f of double groupoids satisfying $f \circ c = f'$.

To emphasise the way that $F(\alpha)$ depends on the choices involved, namely on the choice of the subdivision $\alpha = [\alpha_{ij}]$, on the choice of X_{ij} such that α_{ij} lies in X_{ij}, and on the choice of the squares S_{ij} , T_{ij} or, equivalently, on the choice of homotopies h_{ij}, we write $F(\alpha) = F(\alpha,(X_{ij}), (h_{ij}))$.

<u>Lemma 3.</u> $F(\alpha,(X_{ij}), (h_{ij}))$ depends only on α.

<u>Proof.</u> The homotopy h from α to θ maps $(I^2,\dot{I}^2,\ddot{I}^2) \times I$ into (X,Y,Z), and in the induced subdivision $\theta = [\theta_{ij}]$, each θ_{ij} maps $(I^2,\dot{I}^2, \ddot{I}^2)$ into (X_{ij},Y_{ij},Z_{ij}). Let $h' = [h'_{k\ell}]$ be an alternative homotopy constructed under the same rules with $h'_{k\ell}$ lying in $X'_{k\ell}$, inducing subdivisions $\alpha = [\alpha'_{k\ell}]$, $\theta = [\theta'_{k\ell}]$. Then $(\alpha'_{k\ell})$ and (α_{ij}) have a common refinement (α''_{pq}), in the sense that each α_{ij} is subdivided into a block of the α''_{pq} (all blocks being the same size) and the same is true for each $\alpha'_{k\ell}$. Each of the α''_{pq} is necessarily in some X''_{pq} of $\underset{\sim}{X}$ and we can construct corresponding homotopies h''_{pq}. It is enough, therefore, to prove that $F(\alpha,(X_{ij}), (h_{ij})) = F(\alpha,(X'_{k\ell}), (h'_{k\ell}))$ in the case when $(\alpha'_{k\ell})$ is a refinement of (α_{ij}).

Each square $\alpha'_{k\ell}$ is part of a unique square α_{ij}, and we set $X^!_{k\ell} = X'_{k\ell} \cap X_{ij}$. We now apply the hypotheses (i) and (ii) of our theorem, with $\nu \in \Lambda^8$, to construct according to our recipe a homotopy $h^! = [h^!_{k\ell}]$ from α to $\theta^!$ such that each $h^!_{k\ell}$ lies in $X^!_{k\ell}$. The homotopies $h'_{k\ell}: \alpha'_{k\ell} \sim \theta'_{k\ell}$, $h^!_{k\ell} : \alpha'_{k\ell} \sim \theta^!_{k\ell}$ together define homotopies $H'_{k\ell} : \theta'_{k\ell} \sim \theta^!_{k\ell}$ which form a subdivision of a homotopy $H' : \theta' \sim \theta^!$ such that each $H'_{k\ell}$ lies in $X'_{k\ell}$. We may then apply Lemma 2 to conclude that in G_2,

$$F(\alpha,(X'_{k\ell}) , (h'_{k\ell})) = F(\alpha, (X'_{k\ell}), (h^!_{k\ell})). \quad (1)$$

The latter element is in fact $g^{!} = [g^{!}_{k\ell}]$ where $g^{!}_{k\ell} = F(\theta^{!}_{k\ell})$. However if $\alpha'_{k\ell}$ is a part of α_{ij}, then $\theta^{!}_{k\ell}$ is also a square in X_{ij}. Let θ^*_{ij} be the composite of those squares $\theta^{!}_{k\ell}$ such that $\alpha'_{k\ell}$ is a part of α_{ij}. Then we have also a subdivision $h^{!} = [h^*_{ij}]$ where h^*_{ij} is a homotopy from α_{ij} to θ^*_{ij} lying in X_{ij}. Let $g^*_{ij} = F(\theta^*_{ij})$. Then by composing in G_2 we find that $g^{!} = [g^*_{ij}]$, i.e. that

$$F(\alpha, (X'_{k\ell}), (h^{!}_{k\ell})) = F(\alpha, (X_{ij}), (h^*_{ij})). \qquad (2)$$

But now the same argument as for equation (1) shows that

$$F(\alpha, (X_{ij}), (h^*_{ij})) = F(\alpha, (X_{ij}), (h_{ij})). \qquad (3)$$

The result follows from (1), (2) and (3).

Lemma 4. $F(\alpha, (X_{ij}), (h_{ij}))$ depends only on the class of α in $\rho(X,Y,Z)$.

Proof. In view of Lemma 3 it is enough to show that if $\alpha \sim \alpha'$ in $R_2(X,Y,Z)$, then $F(\alpha, (X_{ij}), (h_{ij})) = F(\alpha', (X'_{ij}), (h'_{ij}))$ for some convenient choice of $(X_{ij}), (X'_{ij})$, (h_{ij}) and (h'_{ij}). So let $K : (I^2, \dot{I}^2, \ddot{I}^2) \times I \to (X,Y,Z)$ be a homotopy from α to α'. Since the interiors of the X_λ cover X, there is an $(m \times n \times p)$ - subdivision $K = [K_{ijk}]$ such that each K_{ijk} lies in some X_λ, say X_{ijk}. Let $\alpha = [\alpha_{ij}]$, $\alpha' = [\alpha'_{ij}]$ be the induced subdivisions of α, α'.
We show that if $(h_{ij}), (h'_{ij})$ are any homotopies based on these subdivisions and constructed to our recipe, then $F(\alpha, (X_{ij0}), (h_{ij})) = F(\alpha', (X_{ijp}), (h'_{ij}))$. A simple induction on p (the number of layers between α and α') reduces us to the case $p = 1$, so we may assume that the subdivision of K has a single layer, $K = [K_{ij}]$, each K_{ij} being a homotopy from α_{ij} to α'_{ij}, lying in X_{ij}. Then the homotopies $h_{ij} : \alpha_{ij} \sim \theta_{ij}$, $h'_{ij} : \alpha'_{ij} \sim \theta'_{ij}$ also can be chosen to lie in X_{ij}, and these homotopies combine with the K_{ij} to give homotopies

$H_{ij} : \theta_{ij} \sim \theta'_{ij}$, lying entirely in X_{ij}, and forming a subdivision of a homotopy H from $\theta = [\theta_{ij}]$ to $\theta' = [\theta'_{ij}]$. By Lemma 2, we deduce that $F(\alpha,(X_{ij}),(h_{ij})) = F(\alpha,(X_{ij}),(h'_{ij}))$.

We have now proved that there is a well-defined map $f : \rho_2(X,Y,Z) \to G_2$, given by $f(\bar{\alpha}) = F(\alpha,(X_{ij}),(h_{ij}))$, and which satisfies $f \circ c = f'$ at least on 2-dimensional elements of ρ. We show next that f preserves addition.

Let $a,a',a'' \in \rho_2(X,Y,Z)$ satisfy $a = a' + a''$. Then we can choose α,α',α'' representing a,a',a'' such that $\alpha = [\alpha'_{,}\alpha'']$. We now compute $f(\alpha)$ by means of a subdivision of α which is a refinement $[\alpha'_{ij}|\alpha''_{ij}]$ of $[\alpha'_{,}\alpha'']$. Since $\alpha',\alpha'' \in R_2(X,Y,Z)$ our prescription will ensure that over the common edge of α' and α'' all choices of paths and squares will be made in Y. We then obtain elements g'_{ij},g''_{ij} of G_2 such that $f(a') = [g'_{ij}]$, $f(a'') = [g''_{ij}]$ and $f(a) = [g'_{ij}|g''_{ij}]$. It follows from the interchange law that $f(a) = f(a') + f(a'')$. A similar argument shows that f preserves vertical compositions of squares.

To see that f satisfies condition (iii) of Proposition 1, let $s \in \rho_1(X,Y,Z)$ be represented by $\sigma \in R_1(X,Y,Z)$. Then $\Gamma(s)$ is represented by $\Gamma(\sigma) = \alpha$, say. Choose a subdivision $\alpha = [\alpha_{ij}]$ such that each α_{ij} is in some X_{ij} of the cover $\underset{\sim}{X}$. Since α is a square in Y, each α_{ij} is in Y_{ij}, so we may choose the squares θ_{ij} and the homotopies h_{ij} entirely within Y. Then θ_{ij} represents a degenerate element of $\rho_2(X_{ij},Y_{ij},Z_{ij})$ by Proposition 3. Hence $g_{ij} = f(\bar{\theta}_{ij})$ is a degenerate element of G_2 and therefore $f(\Gamma(s)) = f(\bar{\alpha}) = [g_{ij}] = g$ is a degenerate element of G_2 by Proposition 2.

Now since $\alpha = \Gamma(\sigma)$, the edges $\partial_1\alpha$ and $\varepsilon_1\alpha$ are constant paths at $\zeta = \sigma(1)$. We may therefore choose the h_{ij} so that the corresponding edges of $\theta = [\theta_{ij}]$ are also constant at ζ. It follows that g is of the form $\Delta(t \overset{t}{\underset{e}{}} e)$, and is therefore $\Gamma(t)$ by Proposition 2.

Proposition 1 now implies that f can be extended uniquely to a morphism of double groupoids $f : \rho(X,Y,Z) \to G$ and it only remains to show that $f \circ c = f'$. However we already know this holds in dimension 2, and since $f \circ c$ and f' are morphisms of double groupoids, this also holds in dimensions 1 and 0.

This completes the proof of the Union Theorem.

Corollary. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of X whose interiors cover X. Let Z be a subspace of X, and let $Z_\nu = Z \cap X_\nu$. Then the induced diagram of groupoids

$$\bigsqcup_{\nu\in\Lambda^2} \rho_1(X_\nu, Z_\nu) \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \rho_1(X_\lambda, Z_\lambda) \xrightarrow{c} \rho_1(X,Z)$$

(where a,b,c are induced as in Theorem E) is a coequaliser diagram in the category of groupoids if the induced maps $\pi_0(Z_\nu) \to \pi_0(X_\nu)$ are surjective for all $\nu \in \Lambda^8$.

Proof. In Theorem E we take $Y = X$, and obtain a coequaliser diagram of homotopy double groupoids from which we obtain the above diagram of groupoids by applying the forgetful functor $(G_2,G_1,G_0) \mapsto (G_1,G_0)$ from double groupoids to groupoids. However this forgetful functor has a right-adjoint $(A_1,A_0) \mapsto (A_2,A_1,A_0)$ where A_2 consists of all quadruples $(p \; {}^{r}_{q} \; s)$ in A_1 such that $p^{-1}rsq^{-1}p^{-1}$ is defined, and so the forgetful functor preserves colimits.

Remark. The above Corollary seems to be new even for the case $Z = \{\zeta\}$. The conditions on 8-fold intersection can probably be reduced to conditions on 3-fold intersections. In [22] (p.763 and p.775) it is stated that conditions on 2-fold intersections are sufficient, but this is false as is shown by the following example.

Example. Let $X = R^2$ and let $X_1 = \{(x,y) : x^2 + y^2 = 4\}$, $X_2 = \{(x,y) : x > -1\}$, $X_3 = \{(x,y) : x < 1\}$, $Z = \{(0,2)\}$. Then $X, X_2, X_3, X_1 \cap X_2, X_2 \cap X_3, X_3 \cap X_1$ are all 1-connected, but $\pi_1(X_1,(0,2)) = Z$. Therefore the diagram of the Corollary is not a co-equaliser diagram. Of course we do obtain a coequaliser diagram of groupoids if we take $Z = \{(0,2),(0,-2)\}$.

This example does not contradict results of [11], since there the cover $\{X_\lambda\}_{\lambda\in\Lambda}$ is assumed closed under finite intersection.

R E F E R E N C E S

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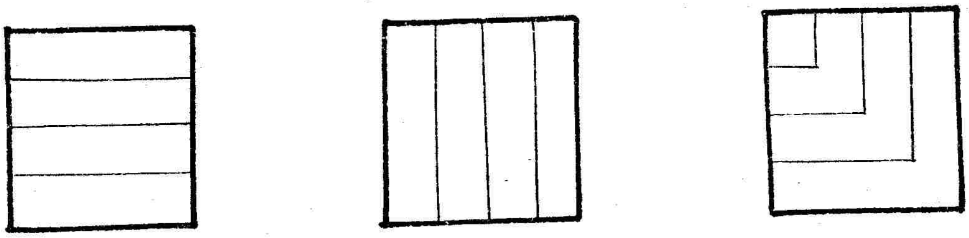
Figures p.11



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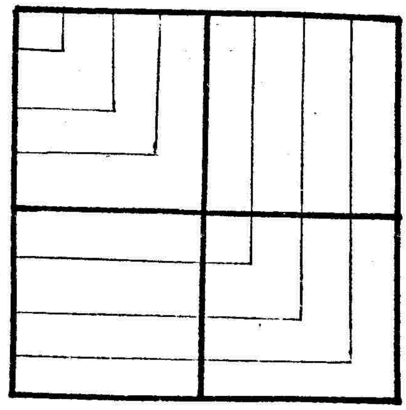

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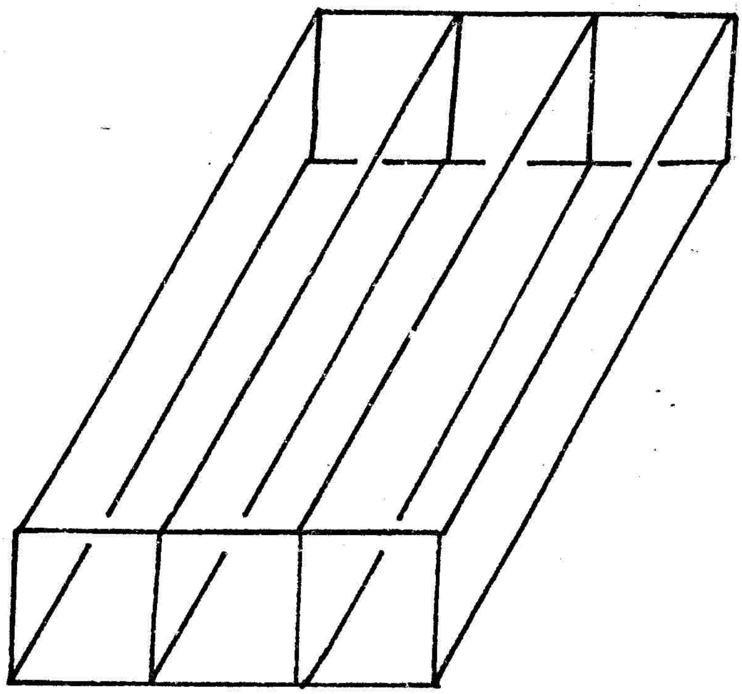

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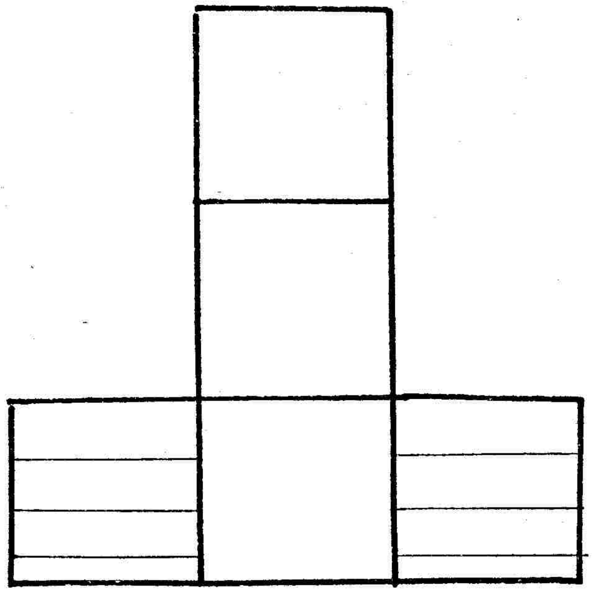

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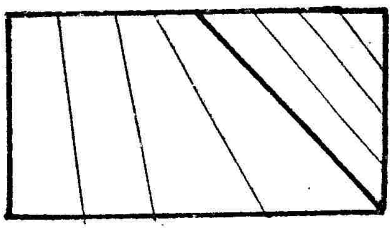

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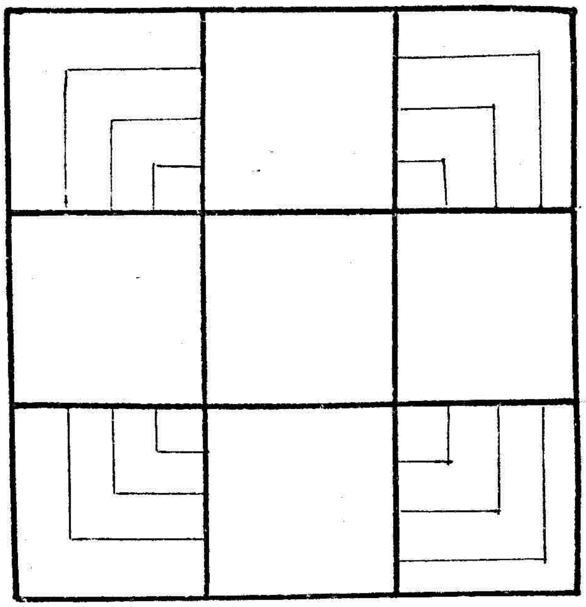

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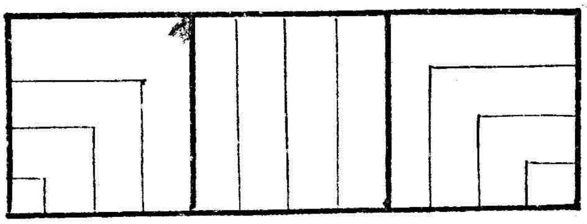

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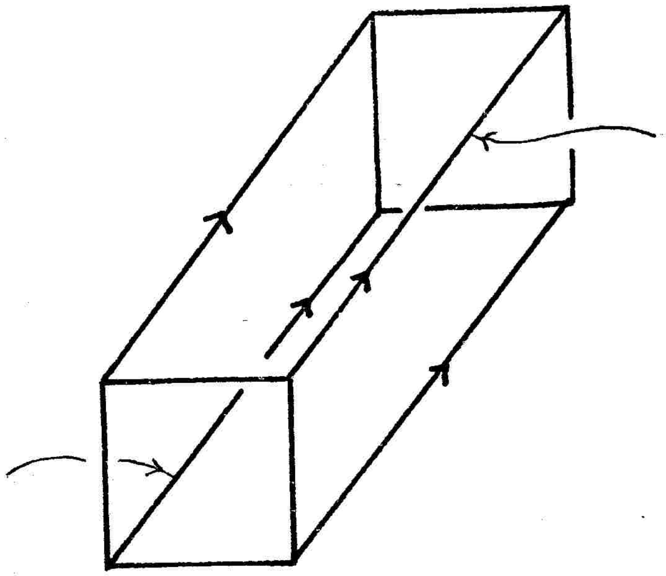

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