SPACES OF PARTIAL MAPS, FIBRED MAPPING SPACES AND THE COMPACT-OPEN TOPOLOGY

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The category Top of topological spaces over B is known to have a product, the fibred product. In order to determine exponential laws for this product, we construct first a compact-open topology on a function space of partial maps with closed domain, and from this construct in Top a fibred mapping space. Basic properties of this space are established, including conditions for exponential laws and for Hausdorffness.

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0. Introduction

Let Top denote the category of topological spaces and TopB the category of spaces and maps over a fixed space B (spaces over B are maps p : X → B, q : Y → B, r : Z → B, etc, and maps p → q are maps f : X → Y such that qf = p). It is well-known that TopB has a product, the so-called fibred product or pullback p ∩ q : X ∩ Y → B (X ∩ Y is often written X ×B Y) of p and q. We consider the question: when does the obvious pullback functor for q

TopB → TopB, p → p ∩ q,

have a right adjoint? In other words when is there a fibred exponential law giving a bijection

ϕ : M(p ∩ q, r) → M(p, (qr)):

for a suitable map (qr) : (YZ) → B, where M(p, q) denotes the set of maps p → q?

From the set-theoretic point of view, it is easy to see that (YZ) should be the disjoint union of the sets of maps M(Yb, Zb) of fibres of q into corresponding fibres
of r. The problem is to provide a topology for \((YZ)\) and to obtain, at least in some circumstances, such a fibred exponential law.

Analogous laws (previously called exponential laws of maps) have already been considered in categories \(C_b\), where \(C\) is either the category \(Q\) of quasi-topological spaces [2], the category \(K\) of \(t\)-spaces [4], or (generalising the previous case) the category of \(A\)-generated spaces [12]; furthermore one of us has applied this construction to problems in homotopy theory [4, 5, 6, 7]. However, the main purpose of this paper is to obtain a topology on \((YZ)\) by a simple modification of the compact-open topology, and so relate this space to the more usual function spaces of General Topology. For this reason, and in order to make clear which results are valid in \(\text{Top}_b\), we avoid working in a convenient category of topological spaces (except in Section 7).

The fibred exponential laws are themselves derived from exponential laws for partial maps (i.e. partial maps with closed domain) using a simple idea, the representability of partial maps. The function space of partial maps is likely to be of independent interest.

The structure of this paper is as follows. Section 1 discusses the space of partial maps (with the compact-open topology) and its exponential laws. Section 2 defines the fibred mapping space \((YZ)\) for spaces \(Y, Z\) over a Hausdorff space \(B\) and Section 3 gives the fibred exponential laws. Section 4 gives some elementary examples of fibred mapping spaces (more interesting ones will be considered in [8]) and Section 5 shows that \((YZ)\) is Hausdorff if \(Z\) is Hausdorff, \(B\) is locally compact and \(q : Y \rightarrow B\) is a submersion. Sufficient conditions for maps to be submersions are discussed in Section 6.

Section 7 contains a few remarks describing the analogues of our arguments and results in convenient categories, in particular in the convenient category \(K\) of \(t\)-spaces.

In a sequel [8] to this paper we relate our construction to the well-known Hom construction for vector bundles and to other constructions, both new and old, in the theories of principal bundles, ex-spaces and relative lifting spaces. It is hoped that these two papers will give some idea of the breadth of applications of fibred mapping spaces.

1. Exponential laws for partial maps

If \(X, Y\) are spaces, then \(M(X, Y)\) will denote the set of all maps (i.e. continuous functions) \(X \rightarrow Y\), and \(M(X, Y)\) will denote the same set with the compact-open topology. (If \(X = \emptyset\), then \(M(X, Y)\) has a unique element, namely the empty map indexed by \(Y\). If \(X \neq \emptyset\), \(Y = \emptyset\), then \(M(X, Y) = \emptyset\).)

A closed domain partial map from \(X\) to \(Y\) is a map \(A \rightarrow Y\) for some closed subspace \(A\) of \(X\)—such a partial map is called a parc map and denoted by \(X \rightarrow Y\). The set of all such maps is written \(P(X, Y)\); note that we include in \(P(X, Y)\) the empty parc map.
Our first simple observation is that $P(X, Y)$ can be given a compact-open topology which has as sub-basis the sets

$$W(K, U) = \{f \in P(X, Y) : f(K) \subseteq U\}$$

for all compact subsets $K$ of $X$ and open subsets $U$ of $Y$, where of course $f(K) = \{f(x) : x \in K, f(x) \text{ is defined}\}$. The resulting space will be denoted by $P(X, Y)$. Of course $P(X, Y)$ is in general non-Hausdorff, since any non-empty open set contains the empty map $\emptyset$.

Our second simple observation is that the functor $X \mapsto P(X, Y)$ is representable. To prove this, let $Y^\sim$ be the set $Y \cup \{\omega\}$ (where $\omega \notin Y$) with the topology in which $C$ is closed in $Y^\sim$ if and only if $C = Y^\sim$ or $C$ is closed in $Y$.

**Proposition 1.1.** The function

$$\lambda : M(X, Y^\sim) \to P(X, Y),$$

$$f \mapsto f \mid f^{-1}(Y)$$

is a homeomorphism.

The proof is simple.

This proposition enables facts on mapping spaces with the compact-open topology to be translated into new results on part maps. We list the ones that we need, first of all discussing functoriality.

**Proposition 1.2.** (i) Maps $f : W \to X$, $g : Y \to Z$ induce by composition maps

$$g_* : P(X, Y) \to P(X, Z),$$

$$f^* : P(X, Y) \to P(W, Y).$$

(ii) If $X_0, Y_0$ are subspaces of $X, Y$ such that $X_0$ is closed, then $P(X_0, Y_0)$ is a subspace of $P(X, Y)$.

We recall that the exponential function

$$\theta : M(X \times Y, Z) \to M(X, M(Y, Z)),$$

$$\theta(f)(x)(y) = f(x, y), \quad x \in X, y \in Y, f \in M(X \times Y, Z),$$

is a well-defined injection.

**Definition 1.3.** The pair $(X, Y)$ is called an exponential pair if for all spaces $Z$, the exponential function $\theta : M(X \times Y, Z) \to M(X, M(Y, Z))$ is surjective.

Note that $(X, Y)$ is an exponential pair if (i) $Y$ is locally compact, i.e. each point of $Y$ has a fundamental system of compact neighbourhoods ([10; p. 156] or [13, 16] for the Hausdorff case), or (ii) $X \times Y$ is a Hausdorff $k$-space [13]. (A $k$-space is one
which has the weak topology with respect to its compact subspaces. For non-
Hausdorff spaces, it is more useful to study \(T\)-spaces as defined in Section 7.)

**Theorem 1.4 (Exponential law for paro maps).** The exponential function

\[
\theta : P(X \times Y, Z) \rightarrow M(X, P(Y, Z)),
\]

\[
\theta(f)(x)(y) = f(x, y), \quad f \in P(X \times Y, Z), \quad x \in X, \quad y \in Y,
\]

is well-defined.

(i) If \((X, Y)\) is an exponential pair, then \(\theta\) is surjective.

(ii) If \(X\) is Hausdorff, then \(\theta\) is continuous.

(iii) If \(X, Y\) are Hausdorff, then \(\theta\) is a homeomorphism into.

**Proof.** These follow from Proposition 1.1 and standard results on exponential laws
for maps [13, 16], on replacing \(Z\) by \(Z^\sim\).

**Remark 1.5.** If \(Z\) is a singleton, then \(P(Y, Z)\) coincides with the space of closed sets
of \(X\) with the topology considered in [1, §7].

**Remark 1.6.** Similar methods give exponential laws for **paro maps** (partial maps
with open domains). The representing space for paro maps from \(X\) to \(Y\) is
\(Y' = Y \cup \{0\}\) with open sets \(U\) such that \(U = Y'\) or \(U\) is open in \(Y\).

2. **Fibred mapping spaces**

Let \(Y, Z, B\) be spaces and \(q : Y \rightarrow B, r : Z \rightarrow B\) be maps. For each \(b \in B\), let
\(Y_b = q^{-1}(b), Z_b = r^{-1}(b)\). Let \((YZ)\) be the set

\[
\bigsqcup_{b \in B} M(Y_b, Z_b)
\]

where \(M(Y_b, Z_b)\) is as in Section 1. Let \((qr) : (YZ) \rightarrow B\) be the function with fibre
over \(b \in B\) the set \(M(Y_b, Z_b)\). The conventional definition for \(M(Y_b, Z_b)\) when \(Y_b\)
or \(Z_b\) is empty implies that

\[
(qr)(YZ) = (B \setminus q(Y)) \cup r(Z).
\]

Assume from now on that \(B\) is \(T_1\); then each fibre \(Y_b\) is closed. (The assumption
that \(B\) is \(T_1\) could be weakened in many places to the assumption that certain maps
have closed fibres.) Let \(i : (YZ) \rightarrow P(Y, Z)\) be the function which sends a map
\(Y_b \rightarrow Z_b\) to the paro map \(Y \rightarrow Z\) with the same domain and same values. Note
that \(i\) need not be injective, since whenever \(Y_b\) is empty, the unique map \(Y_b \rightarrow Z_b\)
is sent by \(i\) to the empty paro map \(Y \rightarrow Z\).
Definition 2.1. The modified compact-open topology on \((YZ)\) is the initial topology with respect to the two functions

\[
\begin{align*}
(YZ) & \xrightarrow{i} \mathcal{P}(Y, Z) \\
& \xrightarrow{(qr)} B.
\end{align*}
\]

This topology has as a sub-basis the sets \(W(U) = (qr)^{-1}(U)\) for all \(U\) open in \(B\) and \(W(A, V) = \{f \in (YZ) : f(A) \subseteq V\}\) for all compact subsets \(A\) of \(Y\) and open subsets \(V\) of \(Z\). (Notice that if \(f : Y_0 \to Z_0\) where \(Y_0\) is empty, then \(f\) belongs to any \(W(A, V)\).) The space \((YZ)\) will be called the fibred mapping space of \(Y\) into \(Z\) over \(B\), and the map \((qr) : (YZ) \to B\) the fibred mapping projection or functional projection of \((YZ)\) to \(B\).

Proposition 2.2. (i) Maps \(f : p \to q, g : r \to s\) induce by composition maps

\[
g^* : (qr) \to (qs). \\
f^* : (qr) \to (pr).
\]

(ii) If \(q : Y \to B, r : Z \to B\) are maps and \(B_0\) is a closed subspace of \(B\), then \((Y_0Z_0)\) is a subspace of \((YZ)\), where \(Y_0\) is \(q^{-1}(B_0)\) and \(Z_0 := r^{-1}(B_0)\).

Proof. This follows easily from properties of paracompact maps and initial topologies.

3. The fibred exponential law

This result will be given in three parts, i.e., Theorems 3.1, 3.3 and 3.5. Let \(p : X \to B, q : Y \to B\) be maps. The fibred product \(pq\) with the two projections \(p \parallel q \to p, p \parallel q \to q\) is, of course, the product in the category \(\text{Top}_B\). In order to apply the methods of paracompact maps, we need to assume \(X \parallel Y\) is closed in \(X \times Y\). So for convenience we assume for the remainder of this section that \(B\) is Hausdorff and leave the reader to see how the results apply under the weaker hypothesis that the pullback spaces which occur are closed.

Theorem 3.1. Let \(p : X \to B, q : Y \to B, r : Z \to B\) be maps. Then \(p \parallel q\) is a fibred exponential function:

\[
\phi : M(p \parallel q, r) \to M(p, (qr)),
\]

which for \(f : X \parallel Y \to Z\) a map over \(B\), satisfies \(\phi(f)(x)(y) = f(x, y)\) all \((x, y) \in X \parallel Y\) and which is injective and natural in the obvious sense.

Proof. Let \(f : X \parallel Y \to Z\) be a map over \(B\) and let \(x \in X\), \(b = p(x)\). Then \(\phi(f)(x)\) is given by \(y \mapsto f(x, y)\) and hence maps \(Y_b \to Z_b\) (if \(Y_b = \emptyset\) we take \(\phi(f)(x)\) to be the
unique empty map \( Y_0 \to Z_0 \). Thus \( \phi(f) : X \to (YZ) \) is a function over \( E \). But \((qr) \circ \phi(f) = p \) which is continuous, and \( i \circ \phi(f) = \theta(f') \), where \( f' : X \times Y \to Z \) is the parc map corresponding to \( f \), and so \( i \circ \phi(f) \) is continuous. Hence \( \phi(f) \) is continuous.

The injectivity of \( \phi \) is immediate, and details of naturality are left to the reader.

Remark 3.2. The above proof does not use the continuity of \( p, q, r \) except to ensure that \( X \cap Y \) and the fibres of \( q \) are closed.

Let \( M(p, q) \) denote \( M(p, q) \) with the compact-open topology.

Theorem 3.3. (i) If \( X \) is Hausdorff, then \( \phi : M(p \cap q, r) \to M(p, (qr)) \) is continuous.

(ii) If \( X \) and \( Y \) are Hausdorff, then \( \phi \) is a homeomorphism into.

Proof. (i) It follows from [13, XII 5.1(a)] that a sub-basis for \( M(p, (qr)) \) consists of the sets \( W(C, W(D, V)) \), \( W(C, W(U)) \) for \( C \) compact in \( X \), \( D \) compact in \( Y \), \( V \) open in \( Z \) and \( U \) open in \( B \). But in this space \( W(C, W(U)) \) is either the whole space or the empty set according as \( p(C) \) is or is not contained in \( U \), and \( \phi^{-1}(W(C, W(D, V))) = W(C \times D, V) \). Hence \( \phi \) is continuous.

(ii) By [13, XII 5.1(b)] a sub-basis for the open sets of \( M(p \cap q, r) \) consists of sets of the form \( W(C \cap D, V) \) for \( C, D \) compact in \( X, Y \) respectively and \( V \) open in \( Z \). The result follows easily.

Definition 3.4. If \( p : X \to B \) is a map and \( A \subseteq B \) then \( X \mid A \) will denote the space \( p^{-1}(A) \) and \( p \mid A \) the restriction of \( p \) to \( X \mid A \). Then \( p \) is said to be locally trivial with fibre \( F \) if each \( b \in B \) has an open neighbourhood \( U \) such that there is an associated homeomorphism \( u_b : p \mid U \to t \), where \( t : F \times U \to U \) is the projection.

Theorem 3.5. The exponential function \( \phi : M(p \cap q, r) \to M(p, (qr)) \) is a surjection (and hence a bijection) if any one of the following list of conditions is satisfied.

(a) \( (X, Y) \) is an exponential pair.

(b) \( X, Y \) are Hausdorff and \( X \cap Y, Y \) are \( k \)-spaces.

(c) \( X = F \times A \) where \( F \) is Hausdorff, \( A \subseteq B \), \( p : X \to B \) is the projection, \( Y \) is a Hausdorff \( k \)-space and \( F \times (Y \mid A) \) is a \( k \)-space.

(d) \( Y = G \times A \) where \( X, G \) are Hausdorff, \( A \subseteq B \), \( q : Y \to B \) is the projection, and \( G \times A \) and \( G \times (X \mid A) \) are \( k \)-spaces.

(e) \( p \) is locally trivial with Hausdorff fibre \( F \), \( Y \) is a Hausdorff \( k \)-space and \( F \times Y \) is a \( k \)-space.

(f) \( B \) is regular, \( X \) is Hausdorff, \( q \) is locally trivial with Hausdorff fibre \( G \) such that \( G \times B \) and \( G \times X \) are \( k \)-spaces.

The proof is removed to Section 8.

Definition 3.6. If \( p : X \to B \) is a map, then a section of \( p \) is a map \( s : E \to X \) such
that \( ps = 1_b \). We write \( \text{sec} \, p \) for the set \( \text{sec} \, p \) of sections to \( \mu \) topologised with the compact-open topology.

**Corollary 3.7.** If \( q : Y \to B \), \( r : Z \to B \) are maps, then there is a natural injection

\[
\phi : M(q, r) \to \text{sec}(qr), \quad \phi(f)(b) = f \upharpoonright Y_b : Y_b \to Z_b, \quad f \in M(q, r), \quad b \in B.
\]

If \( Y \) is a Hausdorff \( k \)-space, then \( \phi : M(q, r) \to \text{sec}(qr) \) is a homeomorphism.

**Proof.** This follows easily from Theorem 3.3 and Theorem 3.5(c), putting \( A = B \), \( p = 1_B \) and noticing that \( 1_B \cap q = q \) and \( \text{sec}(qr) = M(1_B, q) \).

There are two more corollaries covering the case where \( r \) is a trivial fibration; they require the following preliminary result.

**Lemma 3.8.** (i) Given a map \( q : Y \to B \) and a space \( W \); if \( t = t(W) \) denotes the projection \( W \times B \to B \), then

\[
\xi : M(Y, W) \to M(q, t), \quad \xi(f) = (f, q), \quad f \in M(Y, W)
\]

is a bijection. In other words the functor

\[
\text{Top}_B \to \text{Top}, \quad (q : Y \to B) \mapsto Y, \quad (f : p \to q) \mapsto (f : X \to Y)
\]

is left adjoint to the functor

\[
\text{Top} \to \text{Top}_B, \quad W \mapsto t(W), \quad (f : W \to W') \mapsto (f \times 1_B).
\]

(ii) If \( Y \) is Hausdorff, then \( \xi : M(Y, W) \to M(q, t) \) is a homeomorphism.

**Proof.** (i) \( \xi \) is clearly a well defined function. If \( s : W \times B \to W \) denotes the projection, then we define

\[
\xi' : M(q, t) \to M(Y, W), \quad \xi'(g) = s g, \quad g \in M(q, t).
\]

Now \( \xi \xi' \) and \( \xi' \xi \) are identity functions and so \( \xi \) is a bijection.

We notice that \( \xi' \) is continuous, for it is the composite of the inclusion \( M(q, t) \to M(Y, W \times B) \) and the map \( M(Y, W \times B) \to M(Y, W) \) induced by the projection \( W \times B \to W \).

(ii) The composite of \( \xi \) and the inclusion \( M(q, t) \to M(Y, W \times B) \) is also the composite

\[
M(Y, W) \xrightarrow{\delta} M(Y, W) \times M(Y, B) \xrightarrow{\eta} M(Y, W \times B),
\]

where \( \delta \) is \( f \mapsto (f, q) \) and \( \eta \) is the standard bijection. Certainly \( \delta \) is continuous, and \( \eta \) is continuous since \( Y \) is Hausdorff, so \( \xi \) is continuous. Hence \( \xi \) is a homeomorphism.

The next result will be applied elsewhere to the problem of computing the
cohomology of pullback spaces; its $\mathbb{I}$-space version is (implicitly) the central concept in the argument of \cite{7}.

**Corollary 3.9.** If \( p : X \to B \), \( q : Y \to B \) are maps, \( W \) is a space and \( t = t(W) : W \times B \to B \), then there is a natural injection

\[
\psi : M(X \cap Y, W) \to M(p,(qt)), \quad \psi(f)(x)(y) = (f(x,y), q(y)),
\]

\( f \in M(X \cap Y, W) \), \((x,y) \in X \cap Y\).

(i) If \( p,q \) satisfy any one of the conditions (a) \( \cdots \) (f) of Theorem 3.5 then \( \psi \) is a surjection.

(ii) If \( X \) and \( Y \) are Hausdorff then \( \psi : M(X \cap Y, W) \to M(p,(qt)) \) is a homeomorphism into.

**Proof.** \( \psi \) is the composite of \( \phi \) of Theorem 3.3, and \( \xi \) of Lemma 3.8.

The next result is relevant to the problem of computing the cohomology of the total space of a fibration (a relative version of the corresponding result in the category \( K \) has already been used to study \( H^n(E,F; \pi) \) for a fibration \( F \to E \to B \) [5, pp. 334–336] and [6, p. 21]).

**Corollary 3.10.** If \( q : Y \to B \) is a map, \( W \) is a space and \( t = t(W) : W \times B \to B \) then there is a natural injection

\[
\eta : M(Y, W) \to \sec(qt), \quad \eta(f)(b)(y) = (f(y), b),
\]

\( f \in M(Y, W) \), \( y \in Y \), \( b = q(y) \).

If \( Y \) is a Hausdorff \( k \)-space, then \( \eta : M(Y, W) \to \sec(q,t) \) is a homeomorphism.

**Proof.** \( \eta \) is the composite of \( \phi \) of Corollary 3.7 and \( \xi \) of Lemma 3.8. If \( X \) and \( Y \) are spaces then

\[
e : M(X, Y) \times X \to Y, \quad (f,x) \mapsto f(x), \quad f \in M(X, Y), \ x \in X,
\]
is called the evaluation function for \( M(X, Y) \); it is well-known that if \( X \) is locally compact then \( e \) is continuous. We conclude this section by examining the fibred mapping space analogue of \( e \); the result will prove useful in Section 4 below. If \( q : Y \to B \), \( r : Z \to B \) are maps, then

\[
e : (qr) \cap q \to r, \quad e(f,x) = f(x), \quad (f,x) \in (YZ) \cap Y
\]
will be called the evaluation function for \( (qr) \).

**Proposition 3.11.** If \( Y \) is locally compact, then \( e : (qr) \cap q \to r \) is continuous.

**Proof.** This follows by applying Theorem 3.5(a) to the identity map \( (qr) \to (qr) \).
4. Elementary examples of fibred mapping spaces

Example 4.1. If $Y, Z$ are spaces and $q : Y \to \ast, r : Z \to \ast$ are maps to the singleton space $\ast$, then $(YZ)$ may be identified with the usual function space $M(Y, Z)$. From this and Proposition 2.2(ii) we deduce:

Corollary 4.2. If $q : Y \to B, r : Z \to B$ are maps, then for each $b$ in $B$ the fibre of $(qr)$ over $b$ is the space $M(Y_b, Z_b)$.

Notice also that if $B = \ast$, then the fibred exponential laws of Section 3 reduce to the usual exponential laws for topological spaces.

Example 4.3. Suppose now that $q : Y \to B$ is a map and $r : Z \to B$ is the inclusion of the subspace $Z$ of $B$. Then there is a continuous bijection $g : (YZ) \to Z \cup (B \setminus q(Y))$ which is a homeomorphism if $B$ is Hausdorff.

Proof. The projection $(qr) : (YZ) \to B$ is continuous, and (since $r : Z \to B$ is an inclusion) is injective and has range $A = Z \cup (B \setminus q(Y))$. So $(qr)$ restricts to a continuous bijection $g : (YZ) \to A$. However, the projection $A \cap Y \to Z$ is continuous, and hence, by Theorem 3.1 and the Hausdorff condition on $B$, the adjoint map $A \to (YZ)$ is continuous. Since this map is $g^{-1}$, the result follows.

Example 4.4. If $q : Y \to B, r : Z \to B$ are both inclusions of subspaces and $B$ is Hausdorff, then $(YZ)$ is homeomorphic to $(B \setminus Y) \cup (Y \cap Z)$. (This follows from Example 4.3.)

Example 4.5. If $q : Y \to B$ is a map, and $B$ is Hausdorff, then $(q1_B) : (YB) \to B$ is a homeomorphism. (This follows from Example 4.3.)

Example 4.6. If $r : Z \to B$ is a map, then there is a natural bijection $g : Z \to (BZ)$ over $B$. If $B$ is Hausdorff then $g$ is continuous. If also $B$ is locally compact, then $g$ is a homeomorphism.

Proof. The function $g$ is defined by the rule that if $z \in Z_b$, then $g(z) : \{b\} \to Z_b$ has value $z$. Thus $g$ is adjoint to the projection $Z \cap B \to Z$ and so is continuous if $B$ is Hausdorff. Suppose $B$ is locally compact. Then the evaluation function $e : (BZ) \cap B \to Z$ is continuous (Proposition 3.11); since $g^{-1}$ is the composite of $e$ and the canonical identification $(BZ) \to (EZ) \cap B$, it follows that $g^{-1}$ is continuous.

5. Hausdorffness of $(YZ)$

We obtain both sufficient conditions for $(YZ)$ to be Hausdorff and an example of a situation in which $(YZ)$ is non-Hausdorff.
Theorem 5.1. Assume $Z$ is Hausdorff and $q : Y \to B$ has the following property: if $y \in Y$, then $q(y)$ has a neighbourhood $U$ which is the image by $q$ of a compact subset $A$ of $Y$ such that $A \cap q^{-1}q(y) = \{y\}$. Then $(YZ)$ is Hausdorff.

Proof. Let $f : Y_b \to Z_b$, $g : Y_b \to Z_b$, belong to $(YZ)$ and suppose $f \neq g$. If $h \neq h'$, then we can choose disjoint neighbourhoods $V, V'$ of $b, b'$ so that $(qr)^{-1}(V)$, $(qr)^{-1}(V')$ are disjoint neighbourhoods of $f, g$. Suppose $b = b'$. Since $f \neq g$, $Y_b$ is non-empty and so there is a $y$ in $Y_b$ such that $f(y) \neq g(y)$. Let $U, U'$ be disjoint neighbourhoods of $f(y), g(y)$. Let $C, A$ be as in the assumptions of the theorem. Then $f \in W(A, U)$, $g \in W(A, U')$ and so

$$W(A, U) \cap (qr)^{-1}(C),$$

$$W(A, U') \cap (qr)^{-1}(C)$$

are disjoint neighbourhoods of $f, g$ respectively. This completes the proof.

Recall that a map $q : Y \to B$ is a submersion if for each $y$ in $Y$ there is a neighbourhood $V$ of $q(y)$ and map $\lambda : V \to Y$ such that $\lambda q(y) = y$ and $q\lambda = 1_Y$. Various sufficient conditions for maps to be submersions are given in the next section.

Corollary 5.2. If $Z$ is Hausdorff, $B$ is locally compact and $q : Y \to B$ is a submersion, then $(YZ)$ is Hausdorff.

Proposition 5.3. Let $G \times B \to B$ be the projection where $G$ is a space with more than one element and $B$ is not locally compact. Then $(G \times B \ G \times B)$ is not Hausdorff.

Let $b$ be a point of $B$ which has no compact neighbourhood. Let $f, g : G \times \{b\} \to G \times \{b\}$ be two distinct elements of $(G \times B \ G \times B)$ (e.g. constant maps to different points of $G$) and suppose

$$M = W(U) \cap W(C_1, U_1) \cap \cdots \cap W(C_m, U_m)$$

is a basic neighbourhood of $f$ and

$$N = W(V) \cap W(D_1, V_1) \cap \cdots \cap W(D_n, V_n)$$

is a basic neighbourhood of $g$. Then

$$E = C_1 \cup \cdots \cup C_m \cup D_1 \cup \cdots \cup D_n$$

is compact in $G \times B$ and so its projection $p(E)$ in $B$ is not a neighbourhood of $b$. Hence there is an element $x$ in $(U \cap V) \setminus p(E)$. Let $h : G \times \{x\} \to G \times \{x\}$ be the identity function. Then $h(C_1), h(D_1)$ are empty and so $h \in M \cap N$. Thus $(G \times B \ G \times B)$ is not Hausdorff.
6. Submersions

It is clear that any locally trivial map, either in our sense or in the sense of [13, p. 404], is a submersion.

**Theorem 6.1.** If \( q : Y \to B \) is a Hurewicz fibration and \( B \) is a CW-complex then \( q \) is a submersion.

**Proof.** This follows from the result [13, p. 406, Corollary 4.4], that \( q \) is locally trivial in the sense of [13, p. 404].

A stronger result can be obtained using an (as yet) unpublished result of P.R. Heath, that Hurewicz fibrations with LEC base spaces are regular.

**Theorem 6.2.** (R. Brown and P.R. Heath). If \( q : Y \to B \) is a Hurewicz fibration and \( B \) is an LEC space, then \( q \) is a submersion.

This generalises 6.1 because CW-complexes are LEC spaces [14].

**Proof.** LEC spaces are locally contractible [14]. Let \( V \) be a neighbourhood of \( q(y) \) in \( B \) and \( \phi : V \to B \) be a homotopy such that \( \phi_0 \) is an inclusion and \( \phi_1 \) is the constant map relative to \( q(Y) \).

The constant map \( \phi_1 \) lifts to \( V \to \{ y \} \) and \( \phi \) lifts regularly to \( \psi \). Then \( \psi \) is a section to \( q \) over \( V \) taking \( q(y) \) to \( y \), hence \( q \) is a submersion.

If \( X \) is any pointed space then \( q : PX \to X \) will denote the path fibration over \( X \).

**Theorem 6.3.** If \( X \) is a space such that all of the inclusions \( \{ x \} \to X \) of its points are closed cofibrations, i.e. \( X \) is a \( \mathcal{T}_1 \)-space with no degenerate points, then \( q : PX \to X \) is a submersion.

**Proof.** Let \( x_0 \) be the base point of \( X \) and \( \gamma : I \to X \) be a path in \( X \) with \( \gamma(0) = x_0 \), \( \gamma(1) = x_1 \). By [10, 7.3.6(C)] there is a map \( u : X \to [0, \infty] \) and a map \( \phi : X \to X \), where

\[
X_\lambda = \{(x, t) \in X \times I : 0 \leq t \leq \min\{u(x), 1\}\},
\]

such that \( u(x_1) = 0 \), \( \phi(x_0, 0) = x \) for all \( x \in X \), and \( \phi(x, u(x)) = x \) if \( u(x) \leq 1 \). Let \( U = u^{-1}([0, 1]) \) and \( v(y) = (1 + u(y))^{-1}, y \in U \). Define a map \( \psi : U \times I \to X \) by

\[
\psi(y, t) = \begin{cases} 
\lambda(t v(y)), & 0 \leq t \leq v(y), \\
\phi(y, (1-t)(1+u(y))), & v(y) \leq t \leq 1.
\end{cases}
\]

Then \( \psi \) is continuous and its adjoint \( \psi' : U \to PX \) satisfies \( p\psi'(y) - y (\forall y \in U) \). \( \psi'(x_1) = \lambda \).
Theorem 6.4. If $p : X \to B$ is a submersion and $f : A \to B$ is a map, then the induced map $p_\ast : X \cap A \to A$ is a submersion.

Proof. This is easy and left to the reader.

It follows, from Corollary 5.2 and the results of this section, that some of the fibred mapping spaces most likely to arise in applications to homotopy theory are Hausdorff.

7. Convenient categories

Because the usual topological category $\text{Top}$ does not have a function space and product satisfying an exponential law (i.e., $\text{Top}$ is not a cartesian closed category) various other related categories have been proposed as replacements. In this section we discuss their significance for fibred exponential laws.

7.1. The category of Hausdorff $k$-spaces

This was one of the first convenient categories proposed [9], and has since been considered by many writers. The function space is the $k$-ification of the usual function space $M(X, Y)$. However the difficulty which arises here is that many fibred mapping spaces, and presumably also their $k$-ifications, are not Hausdorff.

7.2. The category $\mathcal{K}$ of $t$-spaces

This category is considered in [3, pp. 276-279], and also in [18, p. 554] (where it is denoted by $\mathcal{H}G$). We outline how the results of Sections 1, 2, 3 extend to this category.

If $X$ is any space, then $tX$ is defined to be $X$ retrotopologised with the final topology with respect to all maps of compact Hausdorff spaces into $X$. We say $X$ is a $t$-space if $X = tX$. The category of $t$-spaces and continuous maps is written $\mathcal{K}$; clearly $X \mapsto tX$ defines a reflection $t : \text{Top} \to \mathcal{K}$, so that if $X$ is a $t$-space and $Y$ is any space, then $M(X, Y) = M(tX, tY)$ (c.f. [3, Proposition 4.3]).

The basic constructions in $\mathcal{K}$ are then obtained by applying $t$ to the corresponding constructions in $\text{Top}$. This gives a product space $X \times tY = t(X \times Y)$ [3, p. 278], subspace [3, p. 277], function space $\mathcal{K}(X, Y) = tM(X, Y)$ (see [3, p. 279]).

A $t$-space $X$ is said to be $t$-Hausdorff if the diagonal subset $\Delta$ is closed in $X \times X$. Clearly any Hausdorff $t$-space is $t$-Hausdorff. Also if $X$ is $t$-Hausdorff then $X$ is $T_1$, since if $y \in X$ then $\{y\} = j^{-1}(\Delta)$ where $j : X \to X \times tX$ is the map $x \mapsto (y, x)$.

If $X, Y$ are $t$-spaces, then $\mathcal{PK}(X, Y)$ will denote the set of all parac maps $X \to Y$ topologised in such a way that the bijection

$$\lambda : \mathcal{K}(X, t(Y^\sim)) \to \mathcal{PK}(X, Y),$$

defined as in Proposition 1.1, is a homeomorphism.
Theorem 7.1 (Exponential law for part maps in $\mathbb{K}$). Given $t$-spaces $X, Y, Z$ the exponential function

$$\theta : PK(X \times_t Y, Z) \to K(K(X, PK(Y, Z)))$$

is a well-defined homeomorphism.

Proof. That $\theta$ is a well-defined bijection follows from the exponential law in $\mathbb{K}$ (see [4, Theorem 1.1]). That $\theta$ is a homeomorphism now follows from a standard argument using the associativity of the product and the fact that all the spaces concerned lie in $\mathbb{K}$.

Now let $B$ be a $t$-space and let $\mathbb{K}_B$ denote the category of $t$-spaces over $B$. This category has a product, namely the pull-back space $X \prod_t Y$ [4, p. 312] with projection $p \prod_t q : X \prod_t Y \to B$. If $p : X \to B$, $q : Y \to B$ are $t$-spaces over $B$, then $K(p, q) = tM(p, q)$ is the space of maps in $\mathbb{K}_B$.

Definition 7.2. Let $q : Y \to B$, $r : Z \to B$ be maps in $\mathbb{K}$, where $B$ is $T_1$. Let $(YZ)^i$ have the same underlying set as $(YZ)$ but with the initial topology with respect to the two functions

$$\begin{array}{ccc}
PK(Y, Z) & \xrightarrow{i} & B \\
(YZ) & \xleftarrow{(qr)'} & \\
\end{array}$$

Let $\left(YZ\right)_h = t((YZ)^i)$, and let $(qr)_h$ denote the projection $(YZ)_h \to B$.

Theorem 7.3. (Fibred exponential law for $t$-spaces). If $p : X \to B$, $q : Y \to B$, $r : Z \to B$ are maps in $\mathbb{K}$ and $B$ is $t$-Hausdorff, then the exponential function

$$\phi : K(p \prod_t q, r) \to K(p, (qr)_h)$$

is well-defined and is a homeomorphism.

Proof. The proof that $\phi$ is a bijection is analogous to that of Theorems 3.1, 3.5, but using Theorem 7.1 instead of Theorem 1.4. We also use [4, Lemma 5.1] to ensure that $X \prod_t Y$ is a subspace of $X \times_t Y$, and the $t$-Hausdorff condition to ensure that $X \prod_t Y$ is closed in $X \times_t Y$. That $\phi$ is a homeomorphism now follows by a standard argument.

Remark 7.4. (1) Since right-adjoints are unique, Theorem 7.3 shows that $(qr)_h$ coincides with the fibred mapping space projection of [3, p. 280]. Further, our part map approach explains the requirement for $B$ to be $t$-Hausdorff in [3, p. 280] and in [4, 6, 7].

(2) We know little about conditions for $(YZ)_h$ to be $t$-Hausdorff other than those
which imply it is Hausdorff (as in Section 5) or when \( B \) is a point (for then \( (YZ)_B = K(Y, Z) \) which is \( t \)-Hausdorff if \( Z \) is \( t \)-Hausdorff [11, p. 27]).

(3) The remarks of this section extend to other convenient categories ([18, 19, 12, Theorem 3.4]).

8. Proof of Theorem 3.5

Let \( f : X \to (YZ) \) be a map over \( B \). Then \( g = \phi^{-1}(f) \) is a function \( X \cap Y \to Z \) over \( B \), and the only problem is to prove \( g \) continuous.

(a) Let \( f' = \pi \circ f : X \to P(Y, Z) \). Since \( (X, Y) \) is an exponential pair, \( \theta^{-1}(f') \) is a continuous part map \( X \times Y \to Z \). But \( g = \theta^{-1}(f') \) as functions, so \( g \) is continuous.

(b) Let \( C \) be a compact subset of \( X \cap Y \). We have to prove \( g \mid C \) continuous. Let \( A \) be image of \( C \) under the projection \( X \cap Y \to X \). Then \( A \) is compact. So \( (A, Y) \) is an exponential pair, since \( Y \) and hence \( A \times Y \) is a Hausdorff \( k \)-space. So \( \phi^{-1}(f \mid A) : A \cap Y \to Z \) is continuous. Since \( C \) is contained if \( A \cap Y \), it follows that \( g \mid C \), and hence also \( g \), is continuous.

(c) \( X \cap Y \) is homeomorphic to \( F \times (Y \mid A) \) and the result follows from (b).

(d) \( X \cap Y \) is homeomorphic to \( G \times (X \mid A) \) and the result follows from (b).

(e) It is sufficient to prove that if \( p \mid U \) is trivial then, \( \phi^{-1}(f \mid U) = g \mid U \) is continuous on its domain \( (X \mid U) \cap Y = (X \cap Y) \mid U \). Since \( U \) is open \( F \times (Y \mid U) \) is an open subspace of the \( k \)-space \( F \times Y \) and hence is a \( k \)-space [14, 1.5.3, p. 10]. Also \( p \mid U \) is essentially the projection \( F \times U \to U \) followed by the inclusion \( U \to B \), so the result follows from (c).

(f) The proof of this depends on a result of [3], and is given there (Corollary 1.6).

References