CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 37, n° 1 (1996), p. 61-71.

<http://www.numdam.org/item?id=CTGDC_1996__37_1_61_0>

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Volume XXXVII-1 (1996)

FOLIATIONS, LOCALLY LIE GROUPOIDS AND HOLONOMY

by Ronald BROWN and Osman MUCUK

Résumé: On montre qu'une variété paracompacte feuilletée détermine un groupoïde localement de Lie (ou morceau de groupoïde différentiable, au sens de Pradines). Ceci permet la construction des groupoïdes d'holonomie et de monodromie d'un feuilletage comme cas particulier de constructions générales sur les groupoïdes localement de Lie.

Introduction

We prove that a paracompact foliated manifold determines a locally Lie groupoid. In this way, we explain the relation between classical descriptions of the holonomy groupoid of a foliation, and the Lie version of the construction of the holonomy groupoid of a locally topological groupoid, given in Aof-Brown [1]. This relationship was known to Pradines, and is part of the motivation for his Théorème 1 of Pradines [8]. However, as far we now the result is not in the literature. Thus this paper, with Aof-Brown [1] and the companion paper Brown-Mucuk [2], completes the major part of the exposition of the full details of the constructions and proofs of Théorèmes 1 and 2 of Pradines [8]. Formulations and proofs of these theorems in the locally trivial case have been given in Mackenzie [4]. The part still lacking is an extension of these results to germs of locally Lie groupoids.

There is a related result in Rosenthal [10] which deals with the more general case of a locally topological groupoid arising from a local equivalence relation. However, he essentially puts on the local equivalence relation enough conditions to ensure that it gives rise to a pair (G, W) satisfying all the conditions for a locally topological groupoid. We need to verify that these conditions are satisfied for the local equivalence relation determined by a foliation on a paracompact manifold. We do not know if the paracompactness condition is necessary.

We give full details of the application to the well known Möbius Band example, since the relation between extendibility and non-trivial holonomy is beautifully clear in this example, and we have found the geometry not written down in enough detail for those new to this field.

For a discussion of how locally Lie groupoids arise from Lie algebroids, see Mackenzie [4, 5].

The results of this paper formed part of Mucuk [6]. We are grateful to Kirill Mackenzie for comments and to Jean Pradines for sharing his ideas in this area.

1 Foliations and locally Lie groupoids

We consider C^r manifolds for $r \geq -1$. The convention here is that by a C^{-1} manifold we mean simply a topological space, and a C^{-1} map is just a continuous map. This allows us to state results which include the topological case. However, a C^0 map is simply a continuous map. For ease of writing, we refer simply to a manifold and smooth map, with the value of r to be mentioned as necessary. We denote the space of real numbers by $\mathbb{R}$.

We start with the definition of a foliated manifold, as given in Tamura [11, p.81].

Definition 1.1 *Let X be a connected n-manifold where $n = p + q$. Let $F = \{L_\alpha : \alpha \in A\}$ be a family of path connected subsets of X. We say F is a p-*dimensional foliation* of X if the following are satisfied:*
(i) $L_\alpha \cap L_\beta = \emptyset$ for $\alpha, \beta \in A$ with $\alpha \neq \beta$.
(ii) $\bigcup_{\alpha \in A} L_\alpha = X$.
(iii) Given a point $x \in X$ there exists a chart $(U_\lambda, \phi_\lambda)$ for X about x such that for $\alpha \in A$ with $U_\lambda \cap L_\alpha \neq \emptyset$, each path component of $\phi_\lambda(U_\lambda \cap L_\alpha) \subseteq \mathbb{R}^p \times \mathbb{R}^q$ is of the form

$$(\mathbb{R}^p \times \{c\}) \cap \phi(U_\lambda)$$

for a $c \in \mathbb{R}^q$ determined by the path component.

We call L_α a *leaf* of the foliation, and if U is a subset of X, then a path component of the intersection of U with a leaf is called a *plaque* of U (for the foliation F).

A chart $(U_\lambda, \phi_\lambda)$ satisfying (iii) above is called a *foliated chart*, with domain U_λ. A family of charts whose domains cover X is called a *foliated atlas* for F.

The equivalence relation on X determined by the leaves is written R_F. Then R_F is a subspace of $X \times X$ and becomes a topological groupoid on X with the usual multiplication $(y,z)(x,y) = (x,z)$, for $(x,y),(y,z) \in R_F$. Later, we give examples which show that R_F need not be a manifold.

For any subset U of X we write $R_F(U)$ for the equivalence relation on U whose classes are the plaques of U. If $\Lambda = \{(U_\lambda, \phi_\lambda)\}$ is a foliated atlas for X, we write $W(\Lambda)$ for the union of the sets $R_F(U_\lambda)$ for all domains U_λ of charts of Λ. Let $W(\Lambda)$ have its topology as a subspace of R_F and so of $X \times X$. Later, we give examples to show that $W(\Lambda)$ need not be open in R_F.

However, $W(\Lambda)$ does obtain the structure of manifold from the foliated atlas for F. The reason is that the equivalence relation on each $\phi_\lambda(U_\lambda)$ determined by the connected components of its intersections with the subspaces $\mathbb{R}^q \times \{c\}$ of $\mathbb{R}^p \times \mathbb{R}^q$ for $c \in \mathbb{R}^q$, has a smooth structure, and this induces via ϕ_λ a smooth structure on $W(\Lambda)$.

We recall the definition of locally Lie groupoid, as given in the companion paper [2], and which is due originally to Pradines [8].

Definition 1.2 *A* locally Lie groupoid *is a pair (G, W) consisting of a groupoid G and a manifold W such that:*

G_1) $O_G \subseteq W \subseteq G$;

G_2) $W = W^{-1}$;

G_3) the set $W_\delta = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction of δ to W_δ is smooth;

G_4) the restrictions to W of the source and target maps α and β are smooth and the triple (α, β, W) is locally sectionable;

G_5) W generates G as a groupoid.

Note that, in this definition, G is a groupoid but does not need to have a topology. The locally Lie groupoid (G, W) is said to be *extendible* if there can be found a topology on G making it a Lie groupoid and for which W is an open submanifold.

We will show that the pair (R_F, W) defined as above by a foliation F is a locally Lie groupoid. Its holonomy groupoid $Hol(R, W)$ [1] is a Lie groupoid containing $W(\Lambda)$ as an open subspace. This holonomy groupoid may thus be regarded as a minimal 'resolution' of the singularities of R_F. This point of view is fairly standard, and is explained for example in Phillips [7].

Theorem 1.3 *If X is a paracompact foliated manifold, then a foliated atlas Λ may be chosen so that the pair $(R_F, W(\Lambda))$ is a locally Lie groupoid.*

Proof: We use the notion of *distinguished chart* as stated in Tamura [11, p.93]. For our purposes, we need only the property of these charts stated as Lemma 1.4 below, and which is given in Lemma 4.4, *ibid*, p.93, together with the fact that any open cover of X can be refined by a cover consisting of domains of distinguished charts.

We first choose a cover $\mathcal{V}$ of X by the domains of distinguished charts $(V_\xi, \psi_\xi), \xi \in \Xi$. Since X is paracompact, the cover $\mathcal{V}$ has a star refinement and this again we may refine by a cover $\mathcal{U}$ consisting of the domains of distinguished charts $(U_i, \phi_i), i \in I$. These charts are to form the atlas Λ. The domains of charts of Λ will be called simply *domains* of Λ. Again, $\mathcal{U}$ is a star refinement of $\mathcal{V}$. This property is used in the proof of (iv) below, which is the most tricky part of the proof. Let $W = W(\Lambda)$.

It is immediate from the definition of W that
(i) $X \subseteq W \subseteq R_F$, and
(ii) $W = W^{-1}$.

We now prove
(iii) W generates R_F.

Suppose $(x, y) \in R_F$. Then x, y belong to the same leaf. Hence there is a path a from x to y in the same leaf. By the Lebesgue Covering Lemma there is a subdivision

$$a = a_n + \cdots + a_1,$$

such that each a_i is a path from x_i to x_{i+1} in some domain U_i of Λ, where $x_0 = x$, $x_{n+1} = y$. Hence we have the decomposition in R_F

$$(x, y) = (x_n, y) \dots (x, x_1)$$

where each (x_i, x_{i+1}) is in W. Hence W generates R_F.

Since R_F with its topology as a subspace of $X \times X$ is a topological groupoid, the restriction $\delta_W : W \times_\alpha W \to R_F$ of the difference map on R_F is continuous. We now prove that

(iv) $W_\delta = \delta^{-1}(W) \cap (W \times_\alpha W)$ is open in $W \times_\alpha W$.

Let $((u,v),(u,w)) \in W_\delta$. Since $(u,v),(u,w)$, and

$$(v,w) = \delta((u,w),(u,v))$$

all belong to W, there are $i,j,k \in I$ such that u,v belong to the same plaque of U_i, u,w belong to the same plaque of U_j, and v,w belong to the same plaque Q_k of U_k. Since U_j and U_k meet U_i, there is a distinguished chart $V_\xi, \xi \in \Xi$, containing U_i, U_j, U_k.

We now state Lemma 4.4 from [11].

Lemma 1.4 *Let $(U_k, \phi_k), (V_\xi, \psi_\xi)$ be distinguished charts such that $U_k \subseteq V_\xi$. If Q_k is a plaque of U_k, then there exists a distinguished chart $(U_{k'}, \phi_{k'})$ which satisfies the following:*
(i) $Q_k \subseteq U_{k'} \subseteq U_k$ and Q_k is a plaque of $U_{k'}$.
(ii) If Q_ξ is a plaque of V_ξ which meets $U_{k'}$, then it meets $U_{k'}$ in a plaque of $U_{k'}$.

We now choose $U_{k'}$ according to Lemma 1.4, and prove that if $x \in U_i \cap U_j, y \in U_i \cap U_{k'}, z \in U_j \cap U_{k'}$ and $((x,z),(x,y)) \in W \times_\alpha W$ then $(y,z) \in W$.

Since $(x,y) \in W, x$ and y belong to a plaque Q_m of some U_m. Since U_m meets U_i, then U_m is contained in V_ξ (by the star refinement property). Hence x and y belong to a plaque of V_ξ. Similarly, x and z belong to a plaque of V_ξ. Hence y and z belong to a plaque of V_ξ. By (ii) of Lemma 1.4, y and z belong to a plaque of $U_{k'}$. Hence they also belong to a plaque of U_k. Hence $(y,z) \in W$.

This completes the proof that W_δ is open in $W \times_\alpha W$.

v) The fact that α and β are smooth on W follows by transferring the smoothness of the corresponding functions on $\mathbb{R}^p \times \mathbb{R}^q$.

We now prove

(vi) (α, β, W) has enough smooth admissible sections.

For this let $w = (x,y) \in W$. Then xR_iy for some $i \in I$, so that x and y are in the same path-component of some coordinate neighbourhood U_i in the leaf topology.

Consider a chart $\phi_i : U_i \to \mathbb{R}^{p+q}$. Choose open neighbourhoods U' of $x' = \phi_i(x)$ and V' of $y' = \phi_i(y)$ which are contained in $\phi_i(U_i)$ and are such that $V' \doteq U' + y' - x'$. Let $U = \phi_i^{-1}(U'), V = \phi_i^{-1}(V')$, and for $z \in U$ let $s(z) = (z, \phi_i^{-1}(\phi_i z + y' - x'))$. Then $s \in \Gamma^r(W)$, and $s(x) = (x, y)$. Thus s is a smooth admissible section as required. This can be pictured as in Fig. 1[1].

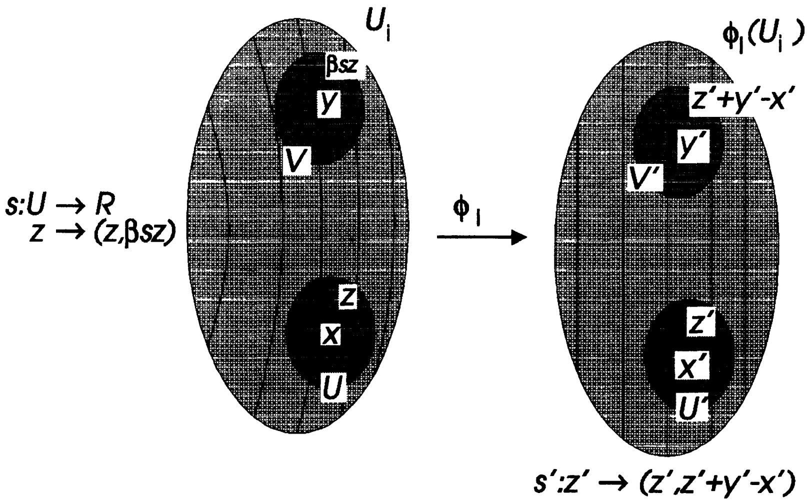

Fig. 1

This completes the proof that (R_F, W) is a locally Lie groupoid. □

We now illustrate the non-extendibility of the locally Lie groupoid arising from a foliation with two simple examples. These illustrate how the description of the holonomy groupoid is related to the well known fact that, under our general range of assumptions, the non-extendibility is precisely equivalent to the existence of non-trivial holonomy.

Example 1.5 This example is a standard foliation of the Möbius Band M. We parameterize M in a way which corresponds roughly to the reverse of cutting the Band along the middle circle.

[1]**The figures in this paper were drawn with CorelDraw and were inserted using Hippocrates Sendouk's programs dviwin 2.5 and clipmeta (sendouk@scf.usc.edu).**

Let $Y = \mathbb{R} \times [0,1]$. Let E be the equivalence relation on Y generated by

$$(r,s) \sim (r+2,s), \qquad (r,0) \sim (r+1,s),$$

$r \in \mathbb{R}, s \in [0,1]$. Let $M = Y/E$, and let $p : Y \to M$ be the projection.

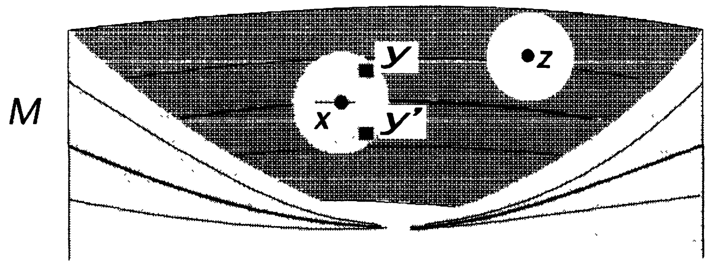

Fig.2

We write $< r,s >$ for $p(r,s) \in M$. The *leaves* of the foliation will be the sets $L_{<s>} = \{< r,s >: r \in \mathbb{R}\}$, for all $s \in [0,1]$. In particular, $L_{<0>}$ is the *centre leaf.* Thus if U is a neighbourhood of a point x on $L_{<0>}$, then $U \cap L_{<s>}$ has at least two path components for $s > 0$ and near to 0.

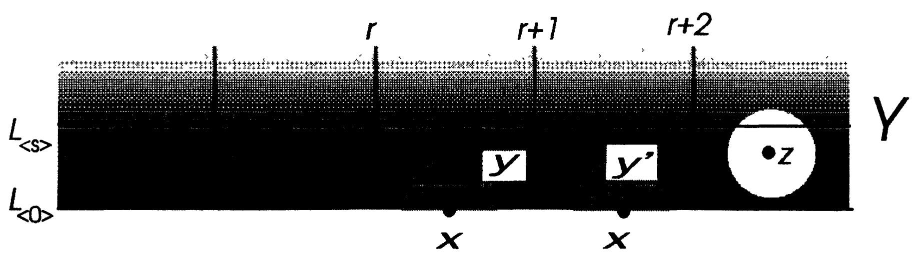

Fig 3

If $s \in (0,1]$, and $r \in \mathbb{R}$, then a chart $\phi_{(r,s)}$ about $< r,s >$ is of the form $< x,y > \mapsto (x,y)$ for (x,y) of distance $s/2$ from (r,s). A chart

$\phi_{(r,0)}$ about $< r, 0 >$ is determined by $(x, y) \mapsto (x, y)$ for (x, y) within distance 1/4 of $(r, 0)$ and $(x, y) \mapsto (x, -y)$ for (x, y) within distance 1/4 of $(r + 1, 0)$. This can be pictured as in Fig. 3. Clearly these leaves and charts determine a foliation of the Möbius Band M.

Let R be the equivalence relation determined by the leaves. Then R is not a manifold, since $\phi_{(0,0)} \times \phi_{(0,0)}$ maps a neighbourhood of $(< 0, 0 >, < 0, 0 >)$ in R to the intersection of a neighbourhood of 0 in $\mathbb{R}^4$ with the union of the two hyperplanes consisting respectively of points (x, s, y, s) and $(x, s, y, -s)$ for all $x, y, s \in \mathbb{R}$.

Let W be defined as in Theorem 1.2. Then W is not open in R, since for any sufficiently small neighbourhood U of $< r, 0 > \in L_{<0>}$, $U \times U$ contains pairs $(< r, s >, < r + 1, s >)$ which lie in R but not in W. In effect, $\phi_{(0,0)} \times \phi_{(0,0)}$ maps an open set of W to one of the two hyperplanes referred to above.

We now show that the locally Lie groupoid (R, W) defined by this foliation is not extendible.

Proof: There is a homeomorphism f of M defined by

$$f < r, s > = < r + \epsilon, s >,$$

where ϵ is chosen sufficiently small so that any point is moved within one of the distinguished charts. Then σ defined by $\sigma(x) = (x, fx)$ is a continuous admissible section of $\alpha : R \to M$. Suitable restrictions of σ give continuous local admissible sections of α_W. There is a product of these restrictions which is a local section τ of α such that for some r,

$$\tau < r, 0 > = (< r, 0 >, < r, 0 >),$$

but for any small $s \neq 0$

$$\tau < r, s > = (< r, s >, < r + 1, s >).$$

Hence $Im\tau$ is not in W, and so the condition (1) of Proposition 4.5 of [2] is not satisfied. Hence the locally topological groupoid (R, W) is not extendible. □

Note that the projection *Hol* $(R, W) \to R$ of the holonomy groupoid is a double cover over each star R_x of a point $x = < r, 0 >$ of the central leaf, and on other stars is a homeomorphism. Essentially, *Hol*(R, W) separates out the self intersections which make R not a manifold.

Example 1.6 Consider the parameterized family of curves shown in Fig. 4 and given in polar coordinates by

$$r_\lambda = 1 + e^{t+\lambda}, \quad 0 \leq \lambda < 2\pi$$

and by

$$r_\mu = \mu, \quad 0 < \mu \leq 1,$$

where the circle $\mid z \mid = 1$ is emphasized. There is a foliation of $\mathbb{R}^2 \setminus \{0\}$ in which these curves are the leaves. Then the locally Lie groupoid (R, W) defined by this foliation is not extendible.

Proof: The argument is similar to the previous one and we do not give it in detail. For a point x on the circle $\mid z \mid = 1$, there is non trivial holonomy as you go once round the leaf. It is this holonomy which gives rise to the non-triviality of the holonomy groupoid, i.e. the non-extendibility of (R, W).

□

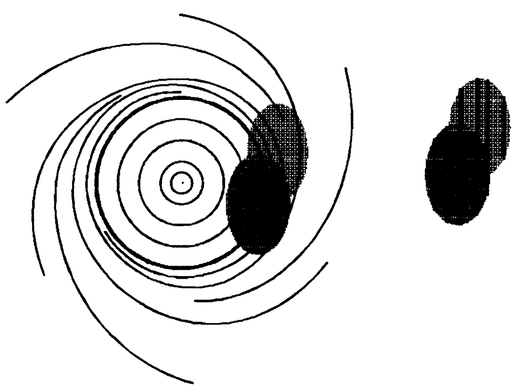

Fig. 4

If (R, W) is the locally topological groupoid arising from a foliation $\mathcal{F}$ on X, the globalisability theorem (Theorem 5.2 of [1], Theorem 4.3 of [2]), gives a new topological groupoid $Hol(R, W)$ and an embedding $W \to Hol(R, W)$ with image $\tilde{W}$, say, such that the topology of $Hol(R, W)$ is determined by $\tilde{W}$ and the continuous local admissible sections of $Hol(R, W)$. Further, there is a canonical morphism $Hol(R, W) \to R$ which is in general not an isomorphism. The kernel of this morphism is a disjoint union of groups, called the *holonomy groups* of the foliation.

By Theorem 5.1 of Brown and Mucuk [2] we can construct the monodromy groupoid $M(X, \mathcal{F}) = M(Hol(R, W), \tilde{W})$ of the holonomy groupoid $(Hol(R, W), \tilde{W})$, together with a surjective morphism

$$p : M(X, \mathcal{F}) \to Hol(R, W)$$

of topological groupoids. As a groupoid, $M(X, \mathcal{F})$ is isomorphic to the fundamental groupoid $\pi_1 X_L$ of the space X with the leaf topology, in which the leaves are the open path components. Hence $M(X, \mathcal{F})$ is isomorphic to the 'homotopy groupoid' $\mathrm{Hom}(X, \mathcal{F})$ defined in Phillips [7].

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