GROUPOIDS AND VAN KAMPEN'S THEOREM

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[Received 5 January 1966]

Introduction

The fundamental groupoid $\pi(X)$ of a topological space X has been known for a long time but has been regarded, usually, as of little import in comparison with the fundamental group—for example, the groupoid is described in ((**3**) 155) as an 'interesting curiosity'. In this paper we shall generalize the fundamental group at a point a of X, namely $\pi(X, a)$, to the fundamental groupoid on a set A, written $\pi(X, A)$, which consists of the homotopy classes of paths in X joining points of $A \cap X$. If $a \in A$, the fundamental group $\pi(X, a)$ can be recovered from knowledge of $\pi(X, A)$, but the latter groupoid is often easier to describe because the category of groupoids has exactly the right properties to model successfully the geometric constructions in building up spaces.

As an example of this, consider an adjunction space $B \cup_f Z$, where $f\colon Y \to B$ is continuous and Y is a closed subspace of Z (so that $B \cup_f Z$ is obtained by glueing Z to B by means of f). The following is a basic step in computing the fundamental group of a space: *Compute the fundamental group of $B \cup_f Z$ in terms of the fundamental groups of B, Y, Z and the maps induced by $i\colon Y \to Z$, $f\colon Y \to B$.* This is insoluble without some local conditions on Y in Z, as an example of H. B. Griffiths (**6**) shows. With suitable local conditions (and other topological conditions which may be regarded as inessential) a special case of this problem was solved by van Kampen in (**10**). His answer was a formula describing the fundamental group of $B \cup_f Z$ in terms of generators and relations.

We shall use groupoids to give in Theorem 4.2 a general and natural solution to this problem. With this theorem one can derive by a uniform method the fundamental groups of spaces from a large class which includes all CW-complexes, and, *a fortiori*, all simplicial complexes. Even for the latter spaces the methods here are simpler than the classical combinatorial methods.

Theorem 4.2 is a deduction from our main result, Theorem 3.4, which determines a groupoid $\pi(X, A_0)$ when X is the union of the interiors of two sets X_1, X_2. Other work on the fundamental group of a union has been done by P. Olum (**11**), the author (**1**), R. H. Crowell (**2**), and A. I. Weinzweig (**12**). The results of (**11**) and (**1**) are not as powerful as the groupoid theorems here, since the exact sequence considered in these papers does not determine the fundamental group completely when $X_1 \cap X_2$ is not path-connected. The papers (**2**) and (**12**) are concerned with the fundamental group of a union $\bigcup X_\alpha$: (**2**) deals only with the case in which each X_α is path-connected; the general case is considered in (**12**), but the formulation is in terms of groups only and is hence most complicated. We shall in § 6 indicate theorems which certainly contain those of (**2**) and probably also those of (**12**).

The use of groupoids in the present context was suggested by a reading of (**7**). The fact that groupoids *are* useful suggests the principle that one should take notice of all the structure there is in a given situation. Now one generalization of a path in a space is a mapping of an n-dimensional cube. For such mappings there is an addition, sometimes defined, in each of n directions; so one is led to consider n-dimensional groupoids. The advantage of these is that, unlike the homotopy groups, they permit subdivisions, and so one can hope for a van-Kampen-type theorem. At the moment I can prove the n-dimensional version of the case $A = X$ of Theorem 3.4, but the formulation of the other cases, and so the use of these objects for computing homotopy groups, presents difficulties. I hope to say more on this another time.

I would like to thank W. H. Cockcroft, P. J. Higgins, and R. M. F. Moss for helpful comments on an earlier draft of this paper.

1. Groupoids

A *groupoid* G is a category whose objects form a set and in which each map is an isomorphism. The objects of G are called *places*, the maps of G are called *roads*; if $\alpha: x \to y$, $\beta: y \to z$ are maps in G, their composite is written $\alpha+\beta: x \to z$, while the inverse of α is $-\alpha: y \to x$.

A *subgroupoid* H of a groupoid G is a subcategory of G such that H is also a groupoid. This condition is simply that if α is a road of H, then so also is $-\alpha$. The subgroupoid H is *full* if for any two places x, y of H the two sets $H(x,y)$, $G(x,y)$ of roads from x to y in H, G respectively, are equal. Any set of places of G determines in an obvious way a full subgroupoid of G. In particular, for a single place x of G this full subgroupoid is a group, the *place group* (or *vertex group*) at x.

A groupoid G is *connected* if for any places x, y of G there is a road in G from x to y. The *components* of any groupoid G are the maximal connected subgroupoids of G; it is easily seen that these exist and are full subgroupoids of G. If the components of G are all place groups, then G is called *totally disconnected.*

A subgroupoid H of G is *representative* if for each place x of G there is a road from x to a place of H; thus H is representative if H meets each component of G.

Let G, H be groupoids. A *morphism* $f\colon G \to H$ is a (covariant) functor. Thus f assigns to each place x of G a place $f(x)$ of H, and to each road $\alpha\colon x \to y$ in G a road $f(\alpha)\colon f(x) \to f(y)$ in H, and f satisfies

$$f(\alpha+\beta) = f(\alpha)+f(\beta)$$

when $\alpha+\beta$ is defined.

In addition to the obvious notion of isomorphism of groupoids there is also the notion of equivalence. First let f, $g\colon G \to H$ be morphisms of groupoids. A *homotopy* $\theta\colon f \simeq g$ is simply a natural equivalence of f, g when these are considered as functors. Thus θ assigns to each place x of G a road $\theta(x)\colon f(x) \to g(x)$ in H such that if $\alpha\colon x \to y$ in G then the following diagram commutes:

$$\begin{array}{ccc} f(x) & \xrightarrow{\theta(x)} & g(x) \\ {\scriptstyle f(\alpha)}\downarrow & & \downarrow{\scriptstyle g(\alpha)} \\ f(y) & \xrightarrow[\theta(y)]{} & g(y) \end{array}$$

that is $f(\alpha)+\theta(y) = \theta(x)+g(\alpha)$. Two groupoids G, H are *equivalent* if there are morphisms $f\colon G \to H$, $g\colon H \to G$ such that $gf \simeq 1$, $fg \simeq 1$.

In fact any groupoid is equivalent to a totally disconnected groupoid consisting of one place group in each component. This is a special case of Lemma 1.1 below (cf. (8)).

Let H be a subgroupoid of G, and let $i\colon H \to G$ be the inclusion. A *retraction* of G onto H is a morphism $r\colon G \to H$ such that $ri = 1$; and such a retraction is a *deformation retraction* if $ri = 1$, $ir \simeq 1$. If such r exists, we call H a *retract*, respectively *deformation retract*, of G.

1.1. LEMMA. *A full, representative subgroupoid of G is a deformation retract of G.*

Proof. Let H be the subgroupoid and $i\colon H \to G$ the inclusion. If x is a place of H let $r(x) = x$ and let $\theta(x) = 0\colon x \to x$.

If x is a place of G but not of H let $r(x)$ be a place of H in the same component of G as x, and let $\theta(x)$ be a road from $r(x)$ to x.

Finally, if $\alpha: x \to y$ is a road in G then $r(\alpha)$ is uniquely defined by the condition that it make the following diagram commutative:

$$\begin{tikzcd} x \arrow[r, "\alpha"] & y \\ r(x) \arrow[u, "\theta(x)"] \arrow[r, "r(\alpha)"'] & r(y) \arrow[u, "\theta(y)"'] \end{tikzcd}$$

It is easy to check that r is a morphism $G \to H$. Clearly $ri = 1$, $\theta: ir \simeq 1$.

We note a fact used later, that, for any x in G,

$$r(\theta(x)) = 0: \; r(x) \to r(x). \tag{1.2}$$

In order to model the processes of identification occurring in the topology, we need two kinds of identification of groupoids.

First we need the *quotient groupoid* ((7) 9). A subgroupoid N of G is *normal* if N and G have the same places and (i) $N(x,y) = \emptyset$, $x \neq y$, (ii) $\alpha + N(y,y) - \alpha \subseteq N(x,x)$ for all α in $G(x,y)$, all places x, y. Such a normal subgroupoid determines a quotient groupoid G/N.

Any family of sets R_x such that $R_x \subseteq G(x,x)$ has a *normal closure* N. The elements of N are all *consequences* of R, that is $N(x,x)$ consists of all finite sums

$$\sum_{i=1}^{n} (\alpha_i + r_i - \alpha_i)$$

for α_i in $G(x,x_i)$ and r_i an element, or the inverse of an element, of R_{x_i}. Any morphism $f: G \to H$ such that $f(\alpha)$ is a zero of H for each road α of R factors uniquely through the projection $G \to G/N$, and the latter groupoid is called the *groupoid G with the relations $r = 0$, $r \in R$.*

Secondly we need the universal σ-groupoid of (7). Let G_p be the set of places of a groupoid G, let S be a set, and let $\sigma: G_p \to S$ be a function. A groupoid U with set of places S is defined as follows. The set $U(x,y)$ of roads from x to y is non-trivial if and only if both x and y belong to the image of σ. In this case a set $W(x,y)$ of words is defined by stipulating that $\alpha_1 \ldots \alpha_n$ is a word if for some x_i, y_i in G_p we have (i) $\alpha_i \in G(x_i, y_i)$, (ii) $\sigma x_1 = x$, $\sigma y_n = y$, (iii) $\sigma y_i = \sigma x_{i+1}$, $i = 1, \ldots, n-1$. An equivalence relation between words is generated by the basic relations

$$u0v \sim uv, \quad u'(\alpha+\beta)v' \sim u'\alpha\beta v'$$

for words u, v, u', v'. Intuitively, two words are equivalent if one can be obtained from the other by a sequence of computations in G. The set $U(x,y)$ is the set of equivalence classes of words. The obvious multiplication of words induces an addition of equivalence classes of words, and it is easy to see that U is a groupoid. To show the dependence of U on G and on σ, we write U as $U_\sigma(G)$.

A morphism $f\colon G \to H$ of groupoids induces a function $f_p\colon G_p \to H_p$ of the sets of places. We say that f is a *σ-morphism* if $f_p = \sigma$. For example the morphism $\sigma^*\colon G \to U_\sigma(G)$ which sends a road α to the equivalence class of the word α, is a σ-morphism.

We say that $f\colon G \to H$ is a *universal* σ-morphism if it satisfies the following conditions: (i) f is a σ-morphism, (ii) if $g\colon G \to K$ is any $\tau\sigma$-morphism then there is a unique τ-morphism $g^*\colon H \to K$ such that $g^*f = g$. It is easy to prove that $\sigma^*\colon G \to U_\sigma(G)$ is a universal σ-morphism. (The definition of universal σ-morphism is phrased slightly more strongly than that of (7), which is restricted to σ-morphisms. The advantage of this definition is that it implies directly that if $f\colon G \to H$ is a universal σ-morphism, and $g\colon H \to K$ is a universal τ-morphism, then $gf\colon G \to K$ is a universal $\tau\sigma$-morphism. It follows easily that $U_{\tau\sigma}(G)$ is isomorphic to $U_\tau(U_\sigma(G))$, and this enables one to shorten some of the proofs of (7).)

Let S consist of one element, and let $\sigma\colon G_p \to S$ be the unique function. Then $U_\sigma(G)$ is a groupoid with one place, that is $U_\sigma(G)$ is a group. This is called the *universal group* of G and is written $U(G)$.

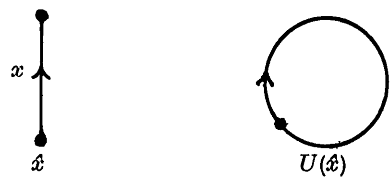
Because a groupoid may have many zeros, the concept of cokernel of a morphism is not useful. What is useful is the difference cokernel (5) of two morphisms. Let $f, g\colon G \to H$ be two morphisms. A *difference cokernel* of f, g is a morphism $c\colon H \to C$ to some groupoid C such that (i) $cf = cg$, (ii) if $d\colon H \to D$ is any morphism such that $df = dg$ then there is a unique morphism $d'\colon C \to D$ such that $d'c = d$.

1.2. LEMMA. *Any morphisms f, $g\colon G \to H$ admit a difference cokernel.*

Proof. Let $\sigma\colon H_p \to S$ be the difference cokernel in the category of sets of the two functions $f_p, g_p\colon G_p \to H_p$ (thus S is obtained by factoring H_p by the equivalence relation generated by $f_p(x) \sim g_p(x)$ for all x in G_p). If α is a road in G then $\sigma^*f(\alpha)$, $\sigma^*g(\alpha)$ are roads in $U_\sigma(H)$ with the same end-points. Let C be $U_\sigma(H)$ with the relations $\sigma^*f(\alpha) - \sigma^*g(\alpha) = 0$ for all roads α in G, and let $c\colon H \to C$ be the composite $H \to U_\sigma(H) \to C$. It is easy to see that c is a difference cokernel of f, g.

Let $\{G^\lambda\}_{\lambda \in \Lambda}$ be a family of groupoids. In the category of groupoids the sum $\coprod_\lambda G^\lambda$ is simply the disjoint union of the family. This rather trivial construction, and the less trivial construction of $U_\sigma(G)$, give the category of groupoids a great deal of power. For example let X be a set. For each x in X let $\hat{x}$ be the *elementary groupoid* with two places 0, 1 and one road x from 0 to 1 and $-x$ from 1 to 0. Let $\hat{F}(X)$ be the sum of these elementary groupoids $\hat{x}$ for each x in X. Then $U(\hat{F}(X))$, the universal group of $\hat{F}(X)$, is $F(X)$, the free group on X. A similar construction gives the free groupoids of (7).

We note by the way that $\hat{x}$ and $U(\hat{x})$ can be represented by the diagrams



and of course $U(\hat{x})$ is isomorphic to Z. This, in essence, is why the fundamental group of the circle is Z.

An *I-groupoid* is a groupoid G with set of places I. The *free product* $G = G_1 * G_2$ of two I-groupoids is $U_\sigma(G_1 \sqcup G_2)$ where σ is the obvious projection $(G_1 \sqcup G_2)_p \to I$. Let $i_r \colon G_r \to G$ $(r = 1, 2)$ be the obvious morphisms. It follows from (7) that i_1, i_2 are injections, and so we shall assume that G_1, G_2 are subgroupoids of G.

The existence of arbitrary sums and of difference cokernels implies that the category of groupoids admits arbitrary right roots (5). (As in (5), we restrict the term 'direct limit' to right root on a directed set.) We leave the reader to prove the existence of direct products, difference kernels, and left roots.

2. The fundamental groupoid

The fundamental groupoid $\pi(X)$ of a topological space X is a quotient of the category, or semi-groupoid, $P(X)$ of paths in X defined as follows. The places (or objects) of $P(X)$ are the points of X. *A path* from x to y is a map $a \colon [0, r] \to X$ for some $r \geqslant 0$ such that $a(0) = x$, $a(r) = y$. If $a \colon [0, r] \to X$, $b \colon [0, s] \to X$ are paths from x to y, y to z respectively, then $a + b$ is the path from x to z defined by

$$(a+b)(t) = \begin{cases} a(t), & 0 \leqslant t \leqslant r, \\ b(t-r), & r \leqslant t \leqslant r+s. \end{cases}$$

Two paths a, b are *equivalent* $(a \sim b)$ if there are constant paths a', b' such that $a + a'$, $b + b'$ are homotopic rel end-points. (This definition was suggested to me by Dr I. M. James.) This is an equivalence relation.

The places of the groupoid $\pi(X)$ are the points of X, and the roads from x to y are the equivalence classes of paths from x to y. The projection $P(X) \to \pi(X)$ is written p_X.

A map $f \colon X \to Y$ of spaces induces a morphism $\pi(f) \colon \pi(X) \to \pi(Y)$ of groupoids. A homotopy $F \colon f \simeq g$ of maps $X \to Y$ induces a homotopy

$\theta: \pi(f) \simeq \pi(g)$, where if $x \in X$ then $\theta(x)$ is the class of the path $t \rightsquigarrow F(x,t)$. Thus the equivalence class of $\pi(X)$ (but not the isomorphism class) is a homotopy invariant of X. From now on we shall abbreviate $\pi(f)$ to f.

We shall be interested in full subgroupoids of $\pi(X)$. Let A be any set. By $\pi(X, A)$ we mean the full subgroupoid of $\pi(X)$ whose places are $A \cap X$. For example if A consists of a single point a of X then $\pi(X, A) = \pi(X, a)$, the fundamental group of X at the point a. Notice that an inclusion $X \to Y$ induces a morphism $\pi(X, A) \to \pi(Y, A)$. We say that A is *representative* in X if A meets each path-component of X.

3. The main theorem

Let $\mathscr{C}$ be a category, and let $i_1: C_0 \to C_1$, $i_2: C_0 \to C_2$ be maps in $\mathscr{C}$. A *union* (see, for example, (**9**)) of i_1, i_2 is a pair (u_1, u_2) of maps $u_1: C_1 \to C$, $u_2: C_2 \to C$ with the following properties:

(i) $u_1 i_1 = u_2 i_2$,

(ii) if $u_1': C_1 \to C'$, $u_2': C_2 \to C'$ are maps such that $u_1' i_1 = u_2' i_2$ then there is a *unique* map $u: C \to C'$ such that $uu_1 = u_1'$, $uu_2 = u_2'$.

It is easily seen that a union, if it exists, is unique up to isomorphism.

Such a union is known in the literature also as a 'pushout' of i_1, i_2 (**5**), and the square

$$\begin{array}{ccc} C_0 & \xrightarrow{i_1} & C_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle u_1} \\ C_2 & \xrightarrow[u_2]{} & C \end{array}$$

is known as a pushout square. The union is a special case of the concept of right root (**5**). In the category of groups, or of groupoids, the union is known also as the free product with amalgamation.

Throughout this section, let X be a topological space, and let X_0, X_1, X_2 be subspaces of X such that the interiors of X_1, X_2 cover X, and $X_0 = X_1 \cap X_2$. Let $\mathbf{X}$ be the square of inclusions

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle u_1} \\ X_2 & \xrightarrow[u_2]{} & X \end{array} \tag{3.1}$$

Then $\mathbf{X}$ is a pushout square.

We suppose that X is path-connected (since if not we may apply the following results to each path-component of X at a time). Then each path-component of X_1 meets X_2, and conversely.

Our object is to obtain direct information on full subgroupoids of $\pi(X, A)$. We assume that A is representative in X_0 (and hence, since X is path-connected, A is representative also in X_1, X_2); in many examples A would be taken to be exactly one point in each path component of X_0. We assume also the following conditions.

(3.2) (i) $A = A_1 \cup A_2$, $A_0 = A_1 \cap A_2$.

(ii) *For each point a of A_2 there is a road $\theta_1(a)$ in $\pi(X_1, A)$ from some point of A_0 to a.*

(iii) *For each point a of A_1 there is a road $\theta_2(a)$ in $\pi(X_2, A)$ from some point of A_0 to a.*

(iv) *If $a \in A_0$ then $\theta_1(a) = \theta_2(a) = 0\colon a \to a$.*

For each a in A, let $\theta(a)$ be $u_1\theta_1(a)$ or $u_2\theta_2(a)$ as $a \in A_2$ or $a \in A_1$. These choices θ_1, θ_2 determine, as in Lemma 1.1, retractions r_1, r_2, and $r = r_1 r_2$, in the following commutative diagram (strictly, $r_1\colon \pi(X_1, A) \to \pi(X_1, A_1)$ is determined by θ_1, and $r_1\colon \pi(X, A) \to \pi(X, A_1)$ is determined by $u_1\theta_1$; similarly for r_2):

$$\begin{tikzcd}
\pi(X_0, A) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \pi(X_1, A) \arrow[r, "r_1"] \arrow[d, "u_1"'] & \pi(X_1, A_1) \arrow[d, "u_1"] \\
\pi(X_2, A) \arrow[r, "u_2"] \arrow[d, "r_2"'] & \pi(X, A) \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, "r"] & \pi(X, A_1) \arrow[d, "r_2"] \\
\pi(X_2, A_2) \arrow[r, "u_2"'] & \pi(X, A_2) \arrow[r, "r_1"'] & \pi(X, A_0)
\end{tikzcd} \tag{3.3}$$

3.4. THEOREM. *In the diagram* (3.3), $(r_2 u_1, r_1 u_2)$ *is the union of* $r_1 i_1$, $r_2 i_2$.

The proof is carried out in later sections in three steps: step 1, the case $A = X$, is in § 5, and is the only step involving topology; step 2, the case $A = A_0$, is in § 6; step 3, the general case, is in § 7.

It should be noted that if X_0, X_1, X_2 are path-connected, and if A consists of a single point $*$ of X_0 (so that $r_2 = r_1 = 1$), then Theorem 3.4 is the theorem of van Kampen as formulated in (**3**), (**11**).

We now try to interpret Theorem 3.4 so as to give more direct information. Let $\sigma_1\colon A_1 \to A_0$, $\sigma_2\colon A_2 \to A_0$ be the functions determined by the retractions r_1, r_2. Let

$$G_1 = U_{\sigma_1}\pi(X_1, A_1), \quad G_2 = U_{\sigma_2}\pi(X_2, A_2),$$

$$t_1 = (r_2 u_1)^*\colon G_1 \to \pi(X, A_0), \quad t_2 = (r_1 u_2)^*\colon G_2 \to \pi(X, A_0),$$

$$s_1 = \sigma_1^* r_1 i_1, \quad s_2 = \sigma_2^* r_2 i_2.$$

So we have a diagram

$$\begin{tikzcd} \pi(X_0, A) \arrow[r, "s_1"] \arrow[d, "s_2"'] & G_1 \arrow[d, "t_1"] \\ G_2 \arrow[r, "t_2"'] & \pi(X, A_0) \end{tikzcd} \tag{3.5}$$

It is a simple deduction from Theorem 3.4, and from the various universal properties involved, that in this diagram (t_1, t_2) is the union of s_1, s_2.

Let A consist of exactly one point in each path-component of X_0. Then in (3.5), G_1, G_2, and $\pi(X, A_0)$ are A_0-groupoids. Further, $\pi(X_0, A)$ is totally disconnected. So we have easily:

3.6. COROLLARY. *$\pi(X, A_0)$ is isomorphic to the free product of the A_0-groupoids $G_1 = U_{\sigma_1}\pi(X_1, A_1)$, $G_2 = U_{\sigma_2}\pi(X_2, A_2)$ with the relations $s_1\alpha = s_2\alpha$ for all α in $\pi(X_0, a)$, a in A.*

3.7. COROLLARY. *Let each path-component of X_1, X_2 be simply connected. Then the fundamental group of X is a free group.*

Proof. We take $A = A_0$. Then $\pi(X, A)$ is a free product of free A-groupoids, with trivial relations. Therefore $\pi(X, A)$ is a free groupoid. The result follows from the corollary to Theorem 7 of (7).

The following is a more elaborate example. Suppose that X_1 is path-connected and that $*$ is a point of X_0. Let A consist of exactly one point in each path-component of X_0 (including $*$), and let

$$A_0 = A_1 = \{*\}, \quad A_2 = A.$$

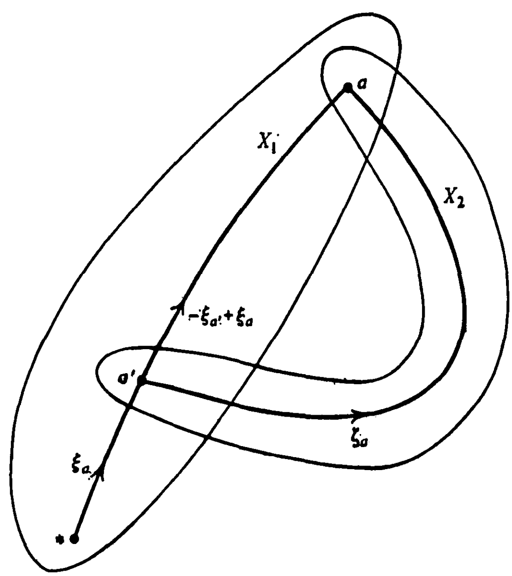
Let A' be a subset of A such that $* \in A'$ and A' contains exactly one point in each path-component of X_2. For each a in A, let a' be the unique point of A' in the same path-component of X_2 as a. Let $T^{a'}$ be a maximal tree groupoid in the component $C^{a'}$ of $\pi(X_2, A)$ containing a'. Let ζ_a be the unique road in $T^{a'}$ from a' to a. Then $F^{a'} = U(T^{a'})$ is a free group on generators μ_a ($a \neq a'$), the images in $F^{a'}$ of the roads ζ_a. Let $\mu_{a'} = 0$.

3.8. Corollary. *With the above conditions, $\pi(X,*)$ is isomorphic to the free product of the groups $\pi(X_1,*)$, $\pi(X_2,a')$, $F^{a'}$, all a' in A', together with the relations*

$$r_1 i_1 \alpha = -\mu_a + (\zeta_a + i_2\alpha - \zeta_a) + \mu_a$$

for each a in A and α in $\pi(X_0, A)$.

Proof. We apply Corollary 3.6: $G_1 = \pi(X_1,*)$, $G_2 = U\pi(X_2,A)$. But $\pi(X_2,A)$ is the sum of its components $C^{a'}$, and so G_2 is the free product of the groups $U(C^{a'})$. By results of (7), $U(C^{a'})$ is isomorphic to the free



product of the groups $\pi(X_2,a')$ and $F^{a'}$; an explicit isomorphism is determined by

$$\beta \rightsquigarrow -\mu_{a_1} + (\zeta_{a_1} + \beta - \zeta_{a_2}) + \mu_{a_2}$$

for β a road in $C^{a'}$ from a_1 to a_2. This proves the formula for the relations.

This result becomes more comprehensible in terms of $\pi(X,A)$. For each a in A, let ξ_a be a road in $\pi(X_1,A)$ from $*$ to a ($\xi_* = 0$), and let $\lambda_a = u_1\xi_a$. Let the retraction r_1 be determined by these roads. Then, for α in $\pi(X_0,A)$,

$$u_1 r_1 i_1 \alpha = \lambda_a + u_1 i_1 \alpha - \lambda_a.$$

In the morphism $\pi(X_2,a') * F^{a'} \to \pi(X,*)$, each element γ of $\pi(X_2,a')$ is sent to the conjugate $\lambda_{a'} + u_2\gamma - \lambda_{a'}$, while a generator μ_a of $F^{a'}$ is sent to

$$\lambda_{a'} + u_2\zeta_a - \lambda_a.$$

Hence the element $-\mu_a + (\zeta_a + i_2\alpha - \zeta_a) + \mu_a$ is sent to $\lambda_a + u_2 i_2 \alpha - \lambda_a$. So the given relations are only consequences (in the groupoid sense) of the relations $u_1 i_1 \alpha = u_2 i_2 \alpha$.

It is an easy consequence of Corollary 3.8 that F, the free product of the groups $F^{a'}$ for a' in A', is a retract of $\pi(X, *)$. Hence F is embedded in $\pi(X, *)$, and the invariant $\tau(\pi(X, *))$ considered by S. Eilenberg in (**4**) is greater than or equal to the number of generators of F.

4. The groupoid of an adjunction space

We consider the problem of determining the fundamental group of an adjunction space $W = B \cup_f Z$, where $f\colon Y \to B$ and Y is closed in Z. The special case of this considered by van Kampen in (**10**) is that in which B is path-connected and Y is the union of a family of disjoint subspaces B_i, $i \in I$, such that $f|B_i\colon B_i \to B$ is a homeomorphism. He also needs a metric on W, and makes additional assumptions on Y, Z, and B to ensure that this exists.†

By definition of adjunction spaces, there is a diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "\bar{f}"] \\ B \arrow[r, "\bar{i}"'] & W \end{tikzcd} \tag{4.1}$$

in which $(\bar{i}, \bar{f})$ is the union of f, i in the category of topological spaces. Let $X = M_f \cup Z$ be obtained from the mapping cylinder M_f by attaching Z by means of the map $y \rightsquigarrow (y, 0)$ on Y. Thus Z is a subspace of X. Let $p\colon X \to W$ be the natural projection.

We assume given a set A which is contained in Y, is representative in Y and in Z, and is such that the set $Q = f(A)$ is representative in B. We assume also that X is path-connected.

4.2. THEOREM. *In the diagram*

$$\begin{tikzcd} \pi(Y, A) \arrow[r, "i"] \arrow[d, "f"'] & \pi(Z, A) \arrow[d, "\bar{f}"] \\ \pi(B, Q) \arrow[r, "\bar{i}"'] & \pi(B \cup_f Z, Q) \end{tikzcd} \tag{4.3}$$

† An earlier theorem is that of H. Seifert in 'Konstruction dreidimensionaler geschlossener Raume', *Ber. Verb. Sächs. Akad. Leipzig, Math. Phys. Kl.* 83 (1931) 26–66. This gives essentially the special case of 4.2 in which W is a simplicial complex and Z, B are subcomplexes of W such that Z, B and $Y = Z \cap B$ are connected.

$(\bar{i},\bar{f})$ is the (groupoid) union of f, i if and only if $p: M_f \cup Z \to B \cup_f Z$ induces an isomorphism of fundamental groups.

Proof. Let $X_1 = X\backslash B$, $X_2 = M_f \cup Y$, and $X_0 = X_1 \cap X_2$. The interiors of X_1, X_2 cover X, and so we can apply Theorem 3.4.

Let A_0 consist of exactly one point in $f^{-1}(q)$ for each q in Q; we shall use conditions (3.2) with $A_2 = A_0$, $A_1 = A$, and θ_2 defined as below.

For each a in A let α_a be the road in $\pi(X_2)$ which is the class of the path from a down the mapping cylinder to $f(a)$, and let

$$\theta_2(a) = \alpha_{a_0} - \alpha_a,$$

where a_0 is the element of A_0 such that $f(a) = f(a_0)$. Note that in $\pi(B)$, $p(\theta_2(a)) = 0: f(a) \to f(a)$.

Consider the following diagram, whose outer part is

$$\begin{tikzcd}
\pi(Y,A) \arrow[rrr] \arrow[ddd] \arrow[dr, "\approx"'] & & & \pi(Z,A) \arrow[ddd] \arrow[dl, "\approx"] \\
& \pi(X_0,A) \arrow[r, "i_1"'] \arrow[d, "r_2 i_2"] \arrow[r, phantom, shift left=6ex, "\text{(i)}"] \arrow[d, phantom, shift right=8ex, "\text{(ii)}"] & \pi(X_1,A) \arrow[d, "r_2 u_1"'] \arrow[d, phantom, shift left=8ex, "\text{(iii)}"] & \\
& \pi(X_2,A) \arrow[r, "u_2"] \arrow[dl, "p"', "\approx"] \arrow[r, phantom, shift right=6ex, "\text{(iv)}"] & \pi(X,A_0) \arrow[dr, "p"] & \\
\pi(B,Q) \arrow[rrr] & & & \pi(W,Q)
\end{tikzcd} \tag{4.4}$$

(4.3), and whose centre square is induced by θ_2 as in Theorem 3.4. The cells (i)–(iv) are commutative; (i) is induced by inclusions, and (iv) is induced by p and its restriction, so these are obviously commutative. The commutativity of (ii) and (iii) is a consequence of $p(\theta_2(a)) = 0$.

Each morphism marked $\approx$ is induced by a homotopy equivalence and is bijective on places; therefore these morphisms are isomorphisms. Therefore (4.4) is an isomorphism of its centre cell to its outer cell if and only if $p: \pi(X, A_0) \to \pi(W,Q)$ is an isomorphism. This morphism is bijective on places. Also, $\pi(X, A_0)$ is connected. Therefore p is an isomorphism if and only if p maps the place groups isomorphically. Thus the theorem follows from Theorem 3.4.

4.5. *Remark.* The map $p: X \to W$ induces an isomorphism of fundamental groups in quite a wide range of cases. If (Z, Y) has the homotopy extension property then p is even a homotopy equivalence.

4.6. *Remark.* It is an easy consequence of Corollary 3.8 that the fundamental group of a wedge of n circles (with the weak topology if n is infinite) is a free group on n generators, the generators being the classes of the loops round one of the circles. From this and Theorem 4.2 one obtains the well-known formulae for the fundamental group of a CW-complex.†

4.7. *Remark.* van Kampen's theorems in (**10**) are of course stated in terms of generators and relations. This form of the theorem can be deduced from Theorem 4.2 and the results of (**7**) in a similar fashion to the argument for Corollary 3.8. In fact the formula of Corollary 3.8 is closely related to van Kampen's.

5. Proof of Theorem 3.4—the case $A = X$

$$\begin{tikzcd} \pi(X_0) \arrow[r, "\dot{\imath}_1"] \arrow[d, "\dot{\imath}_2"'] & \pi(X_1) \arrow[d, "v_1"] \\ \pi(X_2) \arrow[r, "v_2"'] & G \end{tikzcd} \qquad \begin{tikzcd} P(X_0) \arrow[r, "\dot{\imath}_1"] \arrow[d, "\dot{\imath}_2"'] & P(X_1) \arrow[d, "w_1"] \\ P(X_2) \arrow[r, "w_2"'] & G \end{tikzcd}$$

(5.1) (5.2)

Suppose that (5.1) is a commutative square of morphisms of groupoids. Let $w_\alpha = v_\alpha p_{X_\alpha}\colon P(X_\alpha) \to G$ $(\alpha = 1, 2)$. Then (5.2) is commutative. We first prove that there is a unique morphism $w\colon P(X) \to G$ such that $wu_\alpha = w_\alpha\colon P(X_\alpha) \to G$.

Let $a \in P(X)$. By the Lebesgue covering lemma we can make a subdivision

$$a = a_1 + \ldots + a_n$$

such that each a_i belongs to one or other of $P(X_1)$, $P(X_2)$; so we may define $w(a_i)$ to be $w_1(a_i)$ or $w_2(a_i)$, and this makes sense because if $a_i \in P(X_1) \cap P(X_2) = P(X_0)$ then $w_1(a_i) = w_2(a_i)$. Let

$$w(a) = w(a_1) + \ldots + w(a_n).$$

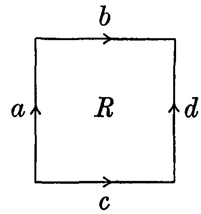
The usual argument of superimposing subdivisions shows that $w(a)$ is independent of the subdivision and that $w(a+b) = w(a) + w(b)$. Clearly w is the unique morphism $P(X) \to G$ such that $wu_\alpha = w_\alpha$, $\alpha = 1, 2$.

We now prove that w, like w_1, w_2, takes the same value on equivalent paths.

Step 1. Since w_1, w_2 map constant paths to zeros, so also does w.

† See the author's forthcoming book *Elements of modern topology* (McGraw-Hill).

Step 2. Let $F: R \to X_\alpha$ be a map, where R is a rectangle. The sides of R determine paths a, b, c, d in X_α (see figure), and clearly $a+b \sim c+d$.



It follows that

$$w_\alpha(a)+w_\alpha(b) = w_\alpha(c)+w_\alpha(d).$$

Step 3. Let $a, b\colon [0,r] \to X$ be paths in X, and let $F\colon [0,r]\times I \to X$ be a homotopy $a \simeq b$ rel end-points. We may by a grid subdivide $[0,r]\times I$ into rectangles so small that each is contained in $F^{-1}(X_1)$ or $F^{-1}(X_2)$. It follows from Step 2 and a simple cancellation argument that

$$w(a) = w(b).$$

Step 4. Let a, b be equivalent paths in $P(X)$. Then there are constant paths a', b' such that $a+a' \simeq b+b'$ rel end-points. So

$$w(a) = w(a+a') = w(b+b') = w(b).$$

It follows now that w determines a morphism $v\colon \pi(X) \to G$ such that $vp_X = w$, $vu_\alpha = v_\alpha$, $\alpha = 1, 2$.

Suppose that $v'\colon \pi(X) \to G$ is a morphism such that

$$v'u_\alpha = v_\alpha\colon\ \pi(X_\alpha) \to G, \quad \alpha = 1, 2.$$

Then $w' = v'p_X\colon P(X) \to G$ is a morphism such that

$$w'u_\alpha = w_\alpha\colon\ P(X_\alpha) \to G, \quad \alpha = 1, 2.$$

Therefore $w' = w$, and so $v' = v$ since p_X is surjective.

6. Proof of Theorem 3.4—the case $A = A_0$

Let $\mathbf{\iota}\colon \pi(\mathbf{X}, A) \to \pi(\mathbf{X})$ be the inclusion. We construct a retraction $\mathbf{r}\colon \pi(\mathbf{X}) \to \pi(\mathbf{X}, A)$ as follows.

Since A is representative in X_0, we may construct as in Lemma 1.1 a retraction $r_0\colon \pi(X_0) \to \pi(X_0, A)$.

In constructing r_0, certain choices are made. We construct retractions $r_1\colon \pi(X_1) \to \pi(X_1, A)$, $r_2\colon \pi(X_2) \to \pi(X_2, A)$ as in Lemma 1.1 but using the choices already made for r_0. This ensures that

$$r_1 i_1 = i_1 r_0, \quad r_2 i_2 = i_2 r_0,$$

and also that the choices of roads which determine r_1, r_2 together determine a retraction $r: \pi(X) \to \pi(X, A)$ such that

$$ru_1 = u_1 r_1, \quad ru_2 = u_2 r_2.$$

Thus we have a map $\mathbf{r}: \pi(\mathbf{X}) \to \pi(\mathbf{X}, A)$ of squares such that

$$\mathbf{r}\iota = 1.$$

Suppose now that $u_1': \pi(X_1, A) \to G$, $u_2': \pi(X_2, A) \to G$ are morphisms of groupoids such that $u_1' i_1 = u_2' i_2$. Let $v_1 = u_1' r_1$, $v_2 = u_2' r_2$. Then

$$v_1 i_1 = u_1' r_1 i_1 = u_1' i_1 r_0 = u_2' i_2 r_0 = u_2' r_2 i_2 = v_2 i_2.$$

By § 5 there is a unique morphism $v: \pi(X) \to G$ such that $vu_1 = v_1$, $vu_2 = v_2$. Let

$$u = v\iota: \ \pi(X, A) \to G.$$

Then, for $\alpha = 1, 2,$

$$uu_\alpha = v\iota u_\alpha = vu_\alpha \iota_\alpha = v_\alpha \iota_\alpha = u_\alpha' r_\alpha \iota_\alpha = u_\alpha'.$$

Suppose that $u': \pi(X, A) \to G$ is a morphism such that $u'u_1 = u_1'$, $u'u_2 = u_2'$. Then, as is easily verified, $v' = ur$ satisfies $v'u_1 = v_1$, $v'u_2 = v_2$. Therefore $v' = v$, and so

$$u' = u' r\iota = v'\iota = v\iota = u.$$

This completes the proof.

6.1. *Remark.* The above proof is a special case of the proof that a retract of a right root is a right root. Precisely, let Γ, Γ', Δ, $\Delta': \mathscr{C} \to \mathscr{D}$ be functors such that Γ', Δ' are constant. Let $\gamma: \Gamma \to \Gamma'$, $\delta: \Delta \to \Delta'$ be natural transformations. Let $\iota: \delta \to \gamma$, $r: \gamma \to \delta$ be maps of natural transformations such that $r\iota = 1$. Then δ *is a right root if* γ *is a right root.*

6.2. *Remark.* Let $\mathscr{U} = \{U\}$ be a set of subsets of X such that the interiors of the U in $\mathscr{U}$ cover X and $\mathscr{U}$ is closed under finite intersection. We regard $\mathscr{U}$ as a category with its objects the elements and its maps the inclusion of one element in another. Let $\mathscr{G}d$ be the category of groupoids. Let Γ, $\Gamma': \mathscr{U} \to \mathscr{G}d$ be the functors $U \rightsquigarrow \pi(U)$, $U \rightsquigarrow \pi(X)$ respectively, and let $\gamma: \Gamma \to \Gamma'$ be the natural transformation induced by the inclusions $U \to X$. Our proof in § 5 generalizes easily to prove that γ is a right root.

Now suppose that A is a set representative in U for each U in $\mathscr{U}$. Let Δ, $\Delta': \mathscr{U} \to \mathscr{G}d$ be the functors $U \rightsquigarrow \pi(U, A)$, $U \rightsquigarrow \pi(X, A)$ respectively. Let $\delta: \Delta \to \Delta'$ be the natural transformation induced by the inclusion $U \to X$. By Remark 6.1, δ is a right root if there is a retraction $r: \gamma \to \delta$.

We say that $\mathscr{U}$ is *stratified* if there is function f from $\mathscr{U}$ to the ordinals such that $f(U) < f(U')$ whenever U, U' are elements of $\mathscr{U}$ such that U is a proper subset of U'. Such a stratification exists if for example $\mathscr{U}$ is finite, or if $\mathscr{U}$ is well ordered. Given such a stratification f, we can construct a retraction $r: \gamma \to \delta$ by transfinite induction as follows.

We may suppose that $\mathscr{U}$ contains the empty set $\emptyset$, that $f(\emptyset) = 0$, and that the image of f is an interval in the set of ordinals. Let $\omega > 0$ be an ordinal in the image of f. Suppose that for each U in $\mathscr{U}$ with $f(U) < \omega$, and for each x in U, we have chosen a road $\theta_U(x)$ in $\pi(U)$ from a point of $A \cap X$ to x so that if $i: U \to U'$ is an inclusion in $\mathscr{U}$, and $f(U') < \omega$, then $i_* \theta_U(x) = \theta_{U'}(x)$. For any V in U such that $f(V) = \omega$, let V° be the union of all proper subsets U of V such that $U \in \mathscr{U}$. If $x \in V^\circ$, we choose U in $\mathscr{U}$ such that $x \in U$ and U is a proper subset of V; we then define $\theta_V(x) = i_* \theta_U(x)$, where $i: U \to V$ is the inclusion. By our inductive assumption, $\theta_V(x)$ is independent of the choice of U. If $x \in V \backslash V^\circ$, we define $\theta_V(x)$ to be any road in $\pi(V)$ from a point of $A \cap V$ to x. The inductive assumption is now easily verified for ordinals $\leqslant \omega$; so we have completed the inductive construction of $r: \gamma \to \delta$.

The case when $\mathscr{U}$ is stratified seems to be the useful one. It can be proved in general that $\delta: \Delta \to \Delta'$ is a right root in two ways: (a) by imitating from the start the proof in (**2**), or (b) by assuming that $\mathscr{U}$ is an open cover and using two subsidiary covers $\mathscr{U}'$, $\mathscr{U}''$. Here $\mathscr{U}'$ is obtained from $\mathscr{U}$ by adding in the unions of all stratified subsets of $\mathscr{U}$, and $\mathscr{U}''$ is a cofinal, stratified subcover of $\mathscr{U}'$.

The condition that $\mathscr{U}$ be closed under finite intersection can in some cases be weakened to the following: there is a subset $\mathscr{V}$ of $\mathscr{U}$ such that the interiors of the V in $\mathscr{V}$ cover X and $\mathscr{U}$ consists of all intersections $V \cap V'$ for V, V' in $\mathscr{V}$. The known cases are when $A = X$ (the proof is easy) and ((**12**) 775) when each set of $\mathscr{U}$ is path-connected and A consists of a single point p in the intersection of all the sets of $\mathscr{U}$.

7. Proof of Theorem 3.4—the general case

Let G be a groupoid, and suppose given a commutative diagram

$$\begin{tikzcd} \pi(X_0, A) \arrow[r, "r_1 i_1"] \arrow[d, "r_2 i_2"'] & \pi(X_1, A_1) \arrow[d, "v_1"] \\ \pi(X_2, A_2) \arrow[r, "v_2"'] & G \end{tikzcd}$$

By § 6 there is a unique morphism $w: \pi(X, A) \to G$ such that

$$wu_1 = v_1 r_1, \quad wu_2 = v_2 r_2.$$

Let $\iota\colon \pi(X, A_0) \to \pi(X, A)$ be the inclusion, and let

$$v = w\iota\colon\ \pi(X, A_0) \to G.$$

We shall prove that

$$vr_2u_1 = v_1, \quad vr_1u_2 = v_2,$$

and that v is the only such morphism.

Let α, in $\pi(X_1, A_1)$, be a road from a to b. Then $\theta(a) = u_2\theta_2(a)$, $\theta(b) = u_2\theta_2(b)$. Therefore

$$\begin{aligned} vr_2u_1(\alpha) &= w(\theta(a) + u_1(\alpha) - \theta(b)) \\ &= wu_2\theta_2(a) + wu_1(\alpha) - wu_2\theta_2(b) \\ &= v_2r_2\theta_2(a) + v_1r_1(\alpha) - v_2r_2\theta_2(b) \\ &= v_1r_1(\alpha) \quad \text{by (1.2)} \\ &= v_1(\alpha) \quad \text{since } \alpha \text{ is a retraction.} \end{aligned}$$

Similarly, $vr_1u_2 = v_2$.

Finally, suppose that $v'\colon \pi(X, A_0) \to G$ is a morphism such that

$$v'r_2u_1 = v_1, \quad v'r_1u_2 = v_2.$$

Then

$$v'ru_1 = v'r_2u_1r_1 = v_1r_1,$$
$$v'ru_2 = v'r_1u_2r_2 = v_2r_2.$$

Hence $v'r = w$, and so $v' = v'r\iota = w\iota = v$.

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