Crossed complexes and chain complexes with operators*

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Introduction

Chain complexes with a group of operators are a well known tool in algebraic topology, where they arise naturally as the chain complex $C_*\tilde{X}$ of cellular chains of the universal cover $\tilde{X}$ of a reduced CW-complex X. The group of operators here is the fundamental group of X.

J.H.C. Whitehead in his classical but little read paper [31] showed that the chain complex $C_*(\tilde{X})$ is useful for the homotopy classification of maps between non-simply connected spaces (see below). His methods must have seemed at the time to be circuitous. In modern parlance, he introduced the categories CW of CW-complexes, HS of homotopy systems, and FCC of free chain complexes with operators, together with functors[1]

$$\mathsf{CW} \xrightarrow{\rho} \mathsf{HS} \xrightarrow{C} \mathsf{FCC}.$$

In each of these categories he introduced notions of homotopy and he proved that C induces an equivalence of the homotopy category of HS with a subcategory of the homotopy category of FCC. He also showed that if X and Y are reduced CW-complexes such that $\dim X \leqslant n$ and $\pi_i Y = 0$ for $2 \leqslant i \leqslant n-1$, then ρ induces a bijection of homotopy classes $[X,Y] \to [\rho X, \rho Y]$. Further, $C\rho X$ is isomorphic to the chain complex $C_*\tilde{X}$ of cellular chains of the universal cover of X, so that under these circumstances there is a bijection of sets of homotopy classes

$$[X,Y] \to [C_*\tilde{X}, C_*\tilde{Y}].$$

This result can be interpreted as an operator version of the Hopf classification theorem. It is surprisingly little known. It includes results of [26, 29] published later, and it enables quite useful calculations to be done easily, such as the homotopy classification of maps from a surface to the projective plane [13].

*This is an edited version of this paper with some changes in notation, and some corrections, 03/04/08.

[1] In this edited version we will later write Π for Whitehead's ρ in order to be consistent with current versions of [6].

A recent account of Whitehead's work is in [1]. However, our aim is somewhat different. In the first place we eliminate from Whitehead's account both the freeness assumptions, and the assumptions of only one vertex. So we consider categories FTop of filtered spaces, Crs of crossed complexes, and Chn of chain complexes with a groupoid as operators. We construct functors

$$\mathsf{FTop} \xrightarrow{\Pi} \mathsf{Crs} \xrightarrow{\nabla} \mathsf{Chn}$$

and show that for a class of filtered spaces X, $\nabla\Pi X$ does give the chain complex of universal covers of X. We also show that ∇ has a right adjoint Θ (which it does not have in the case of only one vertex); these functors are related to some well known tools in the homology of groups, such as relation modules, Alexander modules, and derived modules.

Notions of homotopy and higher homotopy have been analysed for the categories FTop and Crs in [8]. We give a similar analysis for the category Chn and discuss the homotopy-preserving properties of ∇. This enables us to give a more general version of Whitehead's results.

Thus our results and methods give a clearer picture of the relation between Whitehead's work and standard methods of homological algebra. One advantage of our approach is that, because crossed complexes appear in a variety of algebraic situations (cf. [21], and the survey article [3]) one can expect analogues of these methods to form part of the general machinery of non-Abelian homological and homotopical algebra. In fact, applications of our methods, based on an earlier draft, have been found by Porter in commutative algebra [28] and the theory of profinite groups [19].

1 Chain complexes over groupoids

The symmetric monoidal closed structure on the category Crs of crossed complexes, constructed in [8] from tensor products and homotopies, relies crucially on the consideration of crossed complexes over groupoids as well as over groups. The same is true for chain complexes with operators. There are well known definitions of tensor product and internal hom functor for chain complexes of Abelian groups (without operators). If one allows operators from arbitrary groups the tensor product is easily generalized (the tensor product of a G-module and an H-module being a $(G \times H)$-module) but the adjoint construction of internal hom functor does not exist, basically because the group morphisms from G to H do not form a group. To rectify this situation we allow operators from arbitrary groupoids and we start with a discussion of modules over groupoids.

Let Mod denote the category of modules over arbitrary groupoids. An object of Mod is a pair (M, H) where H is a groupoid, M is a family of Abelian groups $M(p), p \in \operatorname{Ob} H$ and H acts on M by $(x, a) \mapsto x^a$ with the usual axioms. (Here x^a is defined when $x \in M(p), a \in H(p, q)$, and then $x^a \in M(q)$.) We normally use additive notation for M and multiplicative notation for H. A morphism in Mod is a pair $(\theta, \phi) : (M, H) \to (N, K)$ where $\phi : H \to K$ is a morphism of groupoids and θ is a family of morphisms of Abelian groups $\theta(p) : M(p) \to N(\phi(p))$ preserving the actions, that is

$$\theta(q)(x^a) = (\theta(p)(x))^{\phi(a)} \quad \text{when} \quad x \in M(p),\, a \in H(p, q).$$

As is customary, we write M for the H-module (M, H) when the operating groupoid H is clear from the context. For a fixed groupoid H, we have a subcategory H-Mod consisting of *all* H-modules and all morphisms of type (θ, id_H); this is just the functor category Ab^H. However, to simplify notation, we will assume throughout this paper that the Abelian groups $M(p)$ for $p \in \mathrm{Ob}\, H$ are all disjoint; any H-module is isomorphic to one of this type.

The *tensor product* in Mod of modules $(M,H),(N,K)$ is the module $(T, H \times K)$ where, for $p \in \mathrm{Ob}\, H$, $q \in \mathrm{Ob}\, K$, $T(p,q) = M(p) \otimes_{\mathbb{Z}} N(q)$ and the action is given by

$$(x \otimes y)^{(a,b)} = x^a \otimes y^b.$$

We write $M \otimes N$ for the $(H \times K)$-module T. This tensor product clearly gives a symmetric monoidal structure to the category Mod, with unit object the module $(\mathbb{Z}, 1)$, where 1 denotes the trivial group.

The internal hom functor in Mod is equally natural. Let $(M,H),(N,K)$ be modules. The morphisms $(\theta,\phi) : (M,H) \to (N,K)$ for fixed $\phi : H \to K$ form an Abelian group under element-wise addition, so all morphisms $M \to N$ form a family of Abelian groups indexed by the set of morphisms $H \to K$. This indexing set is the set of objects of the functor category K^H which is, in fact, a groupoid and it is clear that this groupoid, which we denote $\mathsf{GPD}(H,K)$, acts on the morphisms $M \to N$ giving a module $\mathsf{MOD}(M,N) = \mathsf{MOD}((M,H),(N,K))$. It is straightforward to verify the natural bijection

$$\mathsf{Mod}(L \otimes M, N) \simeq \mathsf{Mod}(L, \mathsf{MOD}(M,N)),$$

where L is a G-module. These families of groups are modules over $\mathsf{GPD}(G \times H, K) \cong \mathsf{GPD}(G, \mathsf{GPD}(H,K))$ and the actions agree, giving a natural isomorphism of modules

$$\mathsf{MOD}(L \otimes M, N) \cong \mathsf{MOD}(L, \mathsf{MOD}(M,N)).$$

Proposition 1.1 *The functors* $\otimes$ *and* MOD *give* Mod *the structure of symmetric monoidal closed category.*

These ideas extend immediately to chain complexes over groupoids. A chain complex M over H is a sequence

$$\cdots \xrightarrow{\partial} M_n \xrightarrow{\partial} M_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} M_1 \xrightarrow{\partial} M_0$$

of H-modules and H-morphisms satisfying $\partial\partial = 0$. A morphism $f : (M,H) \to (N,K)$ is a family of morphisms $f_n : (M_n, H) \to (N_n, K)$ (over some $\phi : H \to K$, independent of n) satisfying $\partial f_n = f_{n-1}\partial$. These form a category Chn and, for a fixed groupoid H, we have a subcategory H-Chn of chain complexes over H.

The *tensor product* of chain complexes M, N over groupoids H, K respectively is the chain complex $M \otimes N = T$ over $H \times K$ where $T_n = \oplus_{i+j=n}(M_i \otimes N_j)$. Here, the direct sum of modules over a groupoid G is defined by taking the direct sum of the Abelian groups at each object of G. The boundary map $\partial : T_n \to T_{n-1}$ is defined on the generators $a \otimes b$ of T_n by

$$\partial(a \otimes b) = \partial a \otimes b + (-)^i a \otimes \partial b,$$

where $a \in M_i$, $b \in N_j$, $i + j = n$.

The internal hom functor $\mathsf{CHN}(-,-)$ is defined as follows. Let M, N be chain complexes over the groupoids H, K respectively and let $F = \mathsf{GPD}(H,K)$. Then the morphisms of chain complexes $M \to N$ form an F-module (as in the case of morphisms of modules). We write S_0 for this module and take it as the 0-dimensional part of the chain complex $S = \mathsf{CHN}(M,N)$. The higher-dimensional elements of S are chain homotopies of various degrees. An *i-fold chain homotopy* ($i \geqslant 1$) from M to N is a pair (s, ϕ) where $s : M \to N$ is a map of degree i (that is, a family of maps $s : M_n \to N_{n+i}$, $n \geqslant 0$) which in each dimension is a morphism of modules over $\phi : H \to K$. Again the i-fold homotopies have the structure of an F-module S_i and we define the boundary map $\partial : S_i \to S_{i-1}$ ($i \geqslant 1$) by

$$(\partial s)(x) = \partial(s(x)) + (-)^{i+1} s(\partial x),$$

the morphism $\phi : H \to K$ being the same for ∂s as for s. We observe that ∂s is of degree $i-1$ and preserves the module structure. Also ∂s commutes or anticommutes with ∂, namely

$$\partial((\partial s)(x)) = (-)^{i+1}(\partial s)(\partial x).$$

It follows firstly that when $i = 1$, ∂s is a morphism of chain complexes, so lies in S_0, and secondly that $\partial\partial : S_i \to S_{i-2}$ is 0 for $i \geqslant 2$. We define $\mathsf{CHN}(M, N)$ to be the chain complex

$$\cdots \longrightarrow S_i \xrightarrow{\partial} S_{i-1} \longrightarrow \cdots \longrightarrow S_0 \text{ over } F.$$

Again, if L is a chain complex over G, there is a natural bijection

$$\mathsf{Chn}(L \otimes M, N) \cong \mathsf{Chn}(L, \mathsf{CHN}(M, N))$$

which extends to a natural isomorphism of chain complexes

$$\mathsf{CHN}(L \otimes M, N) \cong \mathsf{CHN}(L, \mathsf{CHN}(M, N))$$

over $\mathsf{GPD}(G \times H, K) \cong \mathsf{GPD}(G, \mathsf{GPD}(H, K))$.

Proposition 1.2 *The functors $\otimes$ and CHN give Chn the structure of symmetric monoidal closed category. The unit object is the complex*

$$\cdots \to 0 \to \cdots \to 0 \to \mathbb{Z}$$

over the trivial group. The symmetry map $M \otimes N \to N \otimes M$ is given by

$$x \otimes y \mapsto (-)^{ij} y \otimes x$$

for $x \in M_i, y \in N_j$.

2 Derived modules and the functor ∇

A crossed complex is a type of non-Abelian chain complex with operators, the non-Abelian features being confined to dimensions $\leqslant 2$. We recall the definition from [5].

Let C_1 be a groupoid with object set C_0. By a 'non-Abelian' C_1-module we mean a family of groups $A = \{A(p); p \in C_0\}$ on which C_1 operates, obeying all the laws for a C_1-module except commutativity. We write A additively, C_1 multiplicatively. Such a module is a *crossed module* over C_1 if it equipped with a morphism $\delta : A \to C_1$ of groupoids sending $A(p)$ to $C_1(p,p)$ (for $p \in C_0$) which satisfies the two laws

$$\begin{aligned} \delta(a^c) &= c^{-1}(\delta a)c, \quad (a \in A(p), c \in C_1(p,q)), \\ a_1^{\delta a} &= -a + a_1 + a \quad (a, a_1 \in A(p)). \end{aligned}$$

Examples are (i) any Abelian C_1-module, with $\delta = 0$ and (ii) any totally disconnected normal subgroupoid of C_1, with δ the inclusion map. The kernel of δ is always in the centre of A.

A *crossed complex* C is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_0$$

such that

(i) $C_1 \rightrightarrows C_0$ is a groupoid;

(ii) $C_2 \to C_1$ is a crossed module over C_1;

(iii) C_n is a C_1-module (Abelian) for $n \geqslant 3$;

(iv) $\delta : C_n \to C_{n-1}$ is an operator morphism for $n \geqslant 3$;

(v) $\delta\delta : C_n \to C_{n-2}$ is trivial for $n \geqslant 3$;

(vi) δC_2 acts trivially on C_n for $n \geqslant 3$.

We note that all the groupoids C_n $(n \geqslant 1)$ have the same object set C_0 and all the morphisms $\delta : C_n \to C_{n-1}$ map objects identically.

A *morphism* $f : C \to D$ of crossed complexes is a family of groupoid morphisms $f_n : C_n \to D_n$ $(n \geqslant 0)$ which preserves all the structure. This defines the category Crs of crossed complexes.

Details of the functors $\otimes$ and CRS defined on Crs can be found in [8]. For a general understanding of the present paper it is enough to know that they give a symmetric monoidal closed structure on Crs and that they satisfy certain formulae which will be quoted. Our aim now is to construct $\nabla : \mathsf{Crs} \to \mathsf{Chn}$ which relates the two monoidal closed structures.

The basic constructions used to linearise the theory of groups in homological algebra are the group ring $\mathbb{Z}G$ and augmentation module IG of a group G, and the derived module D_ϕ of a group morphism $\phi : H \to G$ (usually appearing in the form $D_\phi = IH \otimes_H \mathbb{Z}G$). These are the ingredients of ∇ and one advantage of working with the category Mod (which includes modules over all groups) is that one can exploit the formal properties of these functorial constructions. We first extend them to the case of groupoids.

Let G be a groupoid. For $q \in \operatorname{Ob} G$, let $\overrightarrow{\mathbb{Z}}G(q)$ be the free Abelian group on the elements of G with target q. Then $\overrightarrow{\mathbb{Z}}G$ is a (right) G-module under the action $(a,g) \mapsto ag$ of G on basis elements. Let $\overrightarrow{\mathbb{Z}}$ be the (right) G-module consisting of the constant family $\overrightarrow{\mathbb{Z}}(p) = \mathbb{Z}$ for $p \in \operatorname{Ob} G$, with trivial action of G (that is, if $g \in G(p,q)$ then g acts as $id_{\mathbb{Z}} : \overrightarrow{\mathbb{Z}}(p) \to \overrightarrow{\mathbb{Z}}(q)$). The augmentation map

$$\varepsilon : \overrightarrow{\mathbb{Z}}G \to \overrightarrow{\mathbb{Z}}, \quad \Sigma n_i g_i \mapsto \Sigma n_i$$

is a morphism of G-modules and its kernel $\overrightarrow{I}G$ is the (right) *augmentation module* of G. The group $\overrightarrow{I}G(q)$ has $\mathbb{Z}$-basis consisting of all $g - 1_q$, where g is a non-identity element of G with target q. Any morphism of groupoids $\phi : H \to G$ induces a module morphism $\overrightarrow{\mathbb{Z}}H \to \overrightarrow{\mathbb{Z}}G$ over ϕ which maps $\overrightarrow{I}H$ to $\overrightarrow{I}G$. So we have functors $\overrightarrow{\mathbb{Z}}$ and $\overrightarrow{I}$ from Gpd to Mod, and we now construct a right adjoint to each of them.

Given a module (M, H), the semidirect product $H \ltimes M$ of H and M is the groupoid with $\mathrm{Ob}\,(H \ltimes M) = \mathrm{Ob}\,H$, $(H \ltimes M)(p, q) = H(p, q) \times M(q)$ and composition

$$(x, m)(y, n) = (xy, m^y + n),$$

defined when $x \in H(p, q)$, $y \in H(q, r)$, $m \in M(q)$, $n \in M(r)$.

The projection $H \ltimes M \to H$ has as its homomorphic sections all maps $x \mapsto (x, fx)$ where $f : H \to M$ is a *derivation*, that is, it maps $H(p, q)$ to $M(q)$ and satisfies

$$f(xy) = (fx)^y + fy$$

whenever xy is defined in H. More generally, if G is any groupoid, a morphism $G \to H \ltimes M$ is of the form $x \mapsto (\psi x, fx)$ where $\psi : G \to H$ is a morphism of groupoids and $f : G \to M$ is a ψ-*derivation*, that is, f maps $G(p, q)$ to $M(\psi q)$ and satisfies

$$f(xy) = (fx)^{\psi y} + fy$$

whenever xy is defined in G.

Now the map $\kappa : G \to \overrightarrow{I}G$, given by $\kappa(x) = x - 1_q$ for $x \in G(p, q)$, is a derivation and is universal in the sense that every derivation f from G to a G-module N is uniquely of the form $f = \hat{f}\kappa$, where $\hat{f} : \overrightarrow{I}G \to N$ is a G-morphism (see Lemma 1.2 of [15] for the corresponding fact for categories). Indeed, if $\psi : G \to H$ is a morphism of groupoids and M is an H-module, then any ψ-derivation $f : G \to M$ is uniquely of the form $f = \hat{f}\kappa$ where $\hat{f} : \overrightarrow{I}G \to M$ is a morphism of modules over ψ. Thus we have a natural bijection

$$\mathsf{Mod}((\overrightarrow{I}G, G), (M, H)) \cong \mathsf{Gpd}(G, H \ltimes M),$$

that is, the functor $\overrightarrow{I} : \mathsf{Gpd} \to \mathsf{Mod}$ has a right adjoint $(M, H) \mapsto H \ltimes M$.

On the other hand, given a module (M, H), the underlying set UM of M (that is, the union of the (disjoint) sets $M(p)$, $p \in \mathrm{Ob}\,H$) has an indexing map $\beta : UM \to \mathrm{Ob}\,H$ sending $x \in M(p)$ to its basepoint $\beta x = p$. We may therefore form the *pull-back* (or *inverse image*) groupoid

$$P(M, H) = \beta^* H = \{(m, h, n);\, m, n \in UM,\, h \in H(\beta m, \beta n)\}$$

with $\mathrm{Ob}\,(\beta^* H) = UM$ and multiplication

$$(m, h, n)(n, k, p) = (m, hk, p);$$

(see Ehresmann [12], p. 245, Mackenzie [22], p. 11). This groupoid, with its canonical morphism to H, $(m, h, n) \mapsto h$, is universal for morphisms $\psi : G \to H$ of groupoids such that $\mathrm{Ob}\,\psi$ factors through

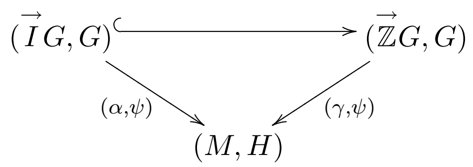
$\beta : UM \to \mathrm{Ob}\, H$. Thus groupoid morphisms $G \to P(M,H)$ are naturally bijective with pairs (α,ψ) where $\alpha : \mathrm{Ob}\, G \to UM$ is a map, $\psi : G \to H$ is a morphism and $\mathrm{Ob}\,\psi = \beta \circ \alpha$. However, since $\overrightarrow{\mathbb{Z}}G$ is freely generated as G-module by $\mathrm{Ob}\, G$ (embedded in $\overrightarrow{\mathbb{Z}}G$ as the set of identities of G), such pairs (α,ψ) are naturally bijective with morphisms of modules $(\gamma,\psi) : (\overrightarrow{\mathbb{Z}}G, G) \to (M,H)$.

Proposition 2.1 *The functors $\overrightarrow{I}$ and $\overrightarrow{\mathbb{Z}}$ from* Gpd *to* Mod *have right adjoints $(M,H) \mapsto H \ltimes M$ and $(M,H) \mapsto P(M,H)$ respectively. Hence both $\overrightarrow{I}$ and $\overrightarrow{\mathbb{Z}}$ preserve colimits. The natural transformation $\overrightarrow{I}G \hookrightarrow \overrightarrow{\mathbb{Z}}G$ is conjugate (see [24], p. 97) to the natural transformation*

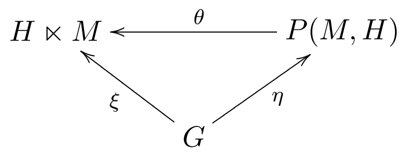
$$\theta = \theta_{(M,H)} : P(M,H) \to H \ltimes M$$

given by $\theta(m,h,n) = (h, m^h - n)$. For each module (M,H), this $\theta_{(M,H)}$ is a covering morphism.

Proof The adjointness has been established above. Any commutative triangle



in Mod corresponds to a commutative triangle



in Gpd, where θ is natural and, if $g \in G(p,q)$, then $\xi g = (\psi g, \alpha(g - 1_q))$ and $\eta g = (\gamma 1_p, \psi g, \gamma 1_q)$. Given $(m,h,n) \in P(M,H)$, we may take $G = H$, $\psi = id$, and choose γ so that $\gamma 1_p = m$, $\gamma 1_q = n$. Then

$$\begin{aligned} \theta(m,h,n) = \xi h &= (\psi h, \alpha(h - 1_q)) \\ &= (h, \gamma(h - 1_q)) \\ &= (h, \gamma(1_p h) - \gamma 1_q) \\ &= (h, m^h - n). \end{aligned}$$

Finally, let $(h,x) \in H \ltimes M$, with $h \in H(p,q)$ and $x \in M(q)$, and let $m \in M(p)$ be an object of $P(M,H)$ lying over the source p of (h,x). Then there is a unique $n \in M(q)$ such that $m^h - n = x$. Hence there is a unique arrow (m,h,n) over (h,x) with source n. □

It is perhaps worth commenting that if one restricts attention to groups, and modules over groups, the restricted functor $\overrightarrow{\mathbb{Z}}$ does not have a right adjoint since, for example, it converts the initial object

1 in the category of groups to the module $(\mathbb{Z}, 1)$ which is not initial in the category of modules over groups. However, the functor $\overrightarrow{I}$, when restricted to groups does have a right adjoint given by the split extension as above.

Definition 2.2 *The* derived module D_ϕ *of a morphism of groupoids* $\phi : H \to G$ *is a* G*-module with a* ϕ*-derivation* $h : H \to D_\phi$ *which is universal for* ϕ*-derivations, that is, every* ϕ*-derivation* $f : H \to M$ *is uniquely of the form* $f = f'h$*, where* $f' : D_\phi \to M$ *is a* G*-morphism.*

This definition is an extension to groupoids of Crowell's definition for groups [11]. The proof of existence extends easily. One constructs F, the free G-module on the family of sets $X = \{X(q),\ q \in \mathrm{Ob}\, G\}$ where $X(q)$ is the set of elements x of H such that $\phi(x)$ has target q. Then $F(q)$ has an additive basis of pairs (x, g) such that $\phi(x)g$ is defined in G, and the action of G is given by

$$(x, g)^{g_1} = (x, gg_1)$$

when gg_1 is defined in G. There is a natural map $i : H \to F$, $i(x) = (x, 1_q)$, where $\phi(x)$ has target q, and if we impose on F the relations

$$i(xy) = i(x)^{\phi(y)} + i(y)$$

whenever xy is defined in H we obtain a quotient G-module D_ϕ, a quotient morphism $s : F \to D_\phi$ and a universal ϕ-derivation $h = si : H \to D_\phi$.

Alternatively, regarding the category of G-modules as the functor category $(\mathsf{Ab})^G$, any functor $M : H \to \mathsf{Ab}$ has a left Kan extension $\phi_* M : G \to \mathsf{Ab}$ along $\phi : H \to G$. Then the derived module D_ϕ is canonically isomorphic to $\phi_*(\overrightarrow{I} H)$, the G-module *induced* from $\overrightarrow{I} H$ by $\phi : H \to G$. In the case of a group morphism ϕ, this induced module is just $IH \otimes_H \mathbb{Z}G$, where $\mathbb{Z}G$ is viewed as a left H-module via ϕ and left multiplication; however the construction is a little more subtle in the case of groupoids.

The adjointness property of the derived module is as follows. Let Gpd^2 be the category of arrows in Gpd (see [24], p. 40). Then we have a functor $D : \mathsf{Gpd}^2 \to \mathsf{Mod}$ given by $D(H \xrightarrow{\phi} G) = (D_\phi G)$.

Proposition 2.3 *The functor* D *has a right adjoint* $\mathsf{Mod} \to \mathsf{Gpd}^2$ *given by*

$$(M, K) \mapsto (K \ltimes M \xrightarrow{\pi_1} K).$$

Proof This is an immediate consequence of (2.1) and the formula $D_\phi = \phi_*(\overrightarrow{I} H)$. □

We are now able to define $\nabla : \mathsf{Crs} \to \mathsf{Chn}$. Let C be the crossed complex

$$\cdots C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta_2} C_1 \rightrightarrows C_0\ .$$

Then all the C_n $(n \geqslant 1)$ have object set C_0, which is mapped identically by δ_n if $n \geqslant 2$. Since $N = \delta_2 C_2$ is a totally intransitive normal subgroupoid of C_1, we may define $G = \pi_1(C) = C_1/N$ (with $\mathrm{Ob}\, G = C_0$) and let $\phi : C_1 \to G$ be the quotient morphism. For $n \geqslant 3$, N acts trivially on C_n, so C_n is a G-module and $\delta_{n+1} : C_{n+1} \to C_n$ is a G-morphism. Similarly, N acts on C_2 by conjugation: $a^{\delta b} = -b + a + b$ for $a, b \in C_2(q)$, so N acts trivially on C_2^{Ab}, the family of Abelianized groups $C_2(p)^{Ab}$. This makes C_2^{Ab} a G-module, and since $\delta_3 : C_3 \to C_2$ is C_1-equivariant, we have a G-morphism $\partial_3 = \alpha_2 \delta_3 : C_3 \to C_2^{Ab}$, where α_2 is the Abelianization map $C_2 \to C_2^{Ab}$.

Proposition 2.4 *There are G-morphisms*

$$C_2^{Ab} \xrightarrow{\partial_2} D_\phi \xrightarrow{\partial_1'} \overrightarrow{I}G$$

such that the diagram

Diagram 2.5

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_3 & \xrightarrow{\delta_3} & C_2 & \xrightarrow{\delta_2} & C_1 & \xrightarrow{\phi} & G \\
 & \downarrow = & & \downarrow = & & \downarrow = & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 \\
\cdots \longrightarrow & C_n & \xrightarrow[\partial_n]{} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_3 & \xrightarrow[\partial_3]{} & C_2^{Ab} & \xrightarrow[\partial_2]{} & D_\phi & \xrightarrow[\partial_1']{} & \overrightarrow{I}G
\end{array}$$

commutes and the lower line is a chain complex over G, where α_1 is the universal ϕ-derivation, α_0 is the G-derivation $x \mapsto x - 1_q$ for $x \in G(p,q)$ and $\partial_n = \delta_n$ for $n \geqslant 4$.

Proof The functor $D : \mathsf{Gpd}^2 \to \mathsf{Mod}$, applied to the sequence of morphisms

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_3 & \xrightarrow{\delta_3} & C_2 & \xrightarrow{\delta_2} & C_1 & \xrightarrow{\phi} & G \\
 & \downarrow \varepsilon_3 & & \downarrow \varepsilon_2 & & \downarrow \phi & & \downarrow = \\
\cdots \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & G & \longrightarrow & G
\end{array}$$

gives a sequence of module morphisms

$$\ldots \to (D_{\varepsilon_3}, 1) \to (D_{\varepsilon_2}, 1) \to (D_\phi, G) \to (\overrightarrow{I}G, G).$$

Since a derivation $C_n \to M$ over a null map $\varepsilon_n : C_n \to 1$ is just a morphism to an Abelian group, we may identify D_{ε_n} with C_n^{Ab} and its universal derivation with the Abelianization map. Thus we obtain a commutative diagram (2.5) in which the vertical maps are the corresponding universal derivations. This establishes all the stated properties except the G-invariance of ∂_2 and the relations $\partial_2\partial_3 = 0$, $\partial_1'\partial_2 = 0$. Clearly $\partial_2\partial_3 = \alpha_1\delta_2\delta_3 = 0$. Also $\partial_1'\partial_2\alpha_2 = \alpha_0\phi\delta_2 = 0$ and since α_2 is surjective, this implies $\partial_1'\partial_2 = 0$. Finally, if $x \in C_2^{Ab}$, $g \in G$ and x^g is defined, choose $a \in C_2$, $b \in C_1$ such that $\alpha_2 a = x$, $\phi b = g$. Then

$$\begin{aligned}
\partial_2(x^g) &= \alpha_1\delta_2(a^b) \\
&= \alpha_1(b^{-1}cb), \text{ where } c = \delta_2 a, \\
&= [(\alpha_1(b^{-1}))^{\phi c} + \alpha_1 c]^{\phi b} + \alpha_1 b, \text{ since } \alpha_1 \text{ is a } \phi\text{-derivation}, \\
&= (\alpha_1 c)^{\phi b}, \text{ since } \phi c = 1, \\
&= (\partial_2 x)^g, \text{ as required.}
\end{aligned}$$

□

Definition 2.6 *For any crossed complex C, $\nabla' C$ is the chain complex over $G = \pi_1(C)$ displayed on the lower line of diagram (2.5), and ∇C is the chain complex*

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_3 \xrightarrow{\partial_3} C_2^{Ab} \xrightarrow{\partial_2} D_\phi \xrightarrow{\partial_1} \overrightarrow{\mathbb{Z}}G$$

over G in which ∂_1 is the composite of $\partial_1' : D_\phi \to \overrightarrow{I}G$ with the inclusion of $\overrightarrow{I}G$ in $\overrightarrow{\mathbb{Z}}G$.

This definition gives functors $\nabla,\ \nabla' : \mathsf{Crs} \to \mathsf{Chn}$ and it follows easily from Propositions 2.1 and 2.3 that ∇' has right adjoint Θ' where, for a chain complex L over a groupoid H, $\Theta' L = \Theta'(L, H)$ is the crossed complex

$$\cdots \longrightarrow L_n \xrightarrow{\partial} L_{n-1} \longrightarrow \cdots \longrightarrow L_3 \xrightarrow{\partial} L_2 \xrightarrow{(0,\partial)} H \ltimes L_1 \rightrightarrows \operatorname{Ob} H.$$

Here $H \ltimes L_1$ acts on L_n $(n \geqslant 2)$ via the projection $H \ltimes L_1 \to H$, so that L_1 acts trivially. We note that $\Theta' L$ is independent of L_0; this reflects the fact that, in $\nabla' C$, the boundary map $\partial_1' : D_\phi \to \overrightarrow{I}G$ is an epimorphism.

The functor $\nabla : \mathsf{Crs} \to \mathsf{Chn}$ also has a right adjoint Θ, but now ΘL involves L_0 in an essential way. A morphism $(\beta, \psi) : (\nabla C, G) \to (L, H)$ in Chn is equivalent to a commutative diagram in Mod:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_3 & \longrightarrow & C_2^{Ab} & \longrightarrow & D_\phi & \longrightarrow & \overrightarrow{I}G & \hookrightarrow & \overrightarrow{\mathbb{Z}}G \\
 & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0'} & \swarrow{\scriptstyle \beta_0} & \\
\cdots & \longrightarrow & L_3 & \xrightarrow[\partial]{} & L_2 & \xrightarrow[\partial]{} & L_1 & \xrightarrow[\partial]{} & L_0 & &
\end{array}$$

(over some morphism $\psi : G \to H$) and hence, by Propositions 2.1, 2.3, to a commutative diagram in Gpd:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \xrightarrow{\phi} & G & \searrow & \\
 & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \bar\beta_2} & & \downarrow{\scriptstyle \gamma_1} & & {\scriptstyle \xi}\downarrow & & \\
\cdots & \longrightarrow & L_3 & \xrightarrow[\partial]{} & L_2 & \xrightarrow[(0,\partial)]{} & H \ltimes L_1 & \xrightarrow[(1,\partial)]{} & H \ltimes L_0 & \xleftarrow[\theta]{} & P(L_0, H)
\end{array}$$

where $(\ldots \beta_3, \bar\beta_2, \gamma_1)$ is a morphism of crossed complexes, and θ is the canonical covering morphism. This in turn is equivalent to a commutative diagram

Diagram 2.7

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_3 & \longrightarrow & C_2 & \xrightarrow{\delta} & C_1 & \xrightarrow{\omega} & P(L_0, H) \\
 & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \bar\beta_2} & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \theta} \\
\cdots & \longrightarrow & L_3 & \xrightarrow[\partial]{} & L_2 & \longrightarrow & H \ltimes L_1 & \longrightarrow & H \ltimes L_0
\end{array}$$

because, in any such diagram, $\theta\omega\delta = 0$ and θ is a covering morphism, so $\omega\delta = 0$, that is, ω factorizes through $\phi : C_1 \to G$.

By pulling back θ along the bottom row of (2.7), we obtain a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & P(L_0, H) \\
& \downarrow{\scriptstyle \sigma_3} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \theta} \\
\cdots \longrightarrow & L_3 & \longrightarrow & L_2 & \xrightarrow[(0,\partial)]{} & H \ltimes L_1 & \xrightarrow[(1,\partial)]{} & H \ltimes L_0
\end{array}$$

in which each E_n is a groupoid over $E_0 = UL_0$, the underlying set of L_0 (see p. 6), and each σ_n is a covering morphism. For $n \geqslant 2$, the composite map $L_n \to H \ltimes L_0$ is 0 and , since $\operatorname{Ker}\theta$ is discrete, it follows that E_n is just a family of groups each isomorphic to a group of L_n. There is also an action of E_1 on E_n ($n \geqslant 2$) induced by the action of $H \ltimes L_1$ on L_n; for if $e_1 \in E_1(x, y)$, where $x \in L_0(p)$, $y \in L_0(q)$, and if $e_n \in E_n(x)$, then $\sigma_1 e_1$ acts on $\sigma_n e_n$ to give an element of $L_n(q)$ which lifts uniquely to an element of $E_n(y)$. It is now easy to see that $E = \{E_n\}_{n\geqslant 0}$ is a crossed complex and that the σ_i form a morphism $\sigma : E \to \Theta' L$ of crossed complexes. Diagram (2.7) is therefore equivalent to a morphism of crossed complexes $C \to E$. This shows that, if we define $\Theta(L, H)$ to be the crossed complex E, then we obtain a functor $\Theta : \mathsf{Chn} \to \mathsf{Crs}$ which is right adjoint to ∇.

An explicit description of $E = \Theta(L, H)$ can be extracted from the constructions given above. The set of objects of E is $E_0 = UL_0$. An arrow of E_1 from x to y, where $x \in L_0(p)$, $y \in L_0(q)$, $p, q \in \operatorname{Ob} H$, is a triple (h, a, y), where $h \in H(p, q)$, $a \in L_1(q)$, and $x^h = y + \partial a$. Composition in E_1 is given by

$$(h, a, y)(k, b, z) = (hk, a^k + b, z)$$

whenever hk is defined in H and $y^k = z + \partial b$.

For $n \geqslant 2$, E_n is a family of groups; the group at the object $y \in L_0(q)$ has arrows (a, y) where $a \in L_n(q)$, with composition

$$(a, y) + (b, y) = (a + b, y).$$

The boundary map $\delta : E_2 \to E_1$ is given by

$$\delta(a, y) = (1_q, \partial a, y) \text{ for } a \in L_2(q),\ y \in L_0(q).$$

The boundary map $\delta : E_n \to E_{n-1}$ ($n \geqslant 3$) is given by $\delta(a, y) = (\partial a, y)$ and the action of E_1 on E_n ($n \geqslant 2$) is given by

$$(a, y)^{(k,b,z)} = (a^k, z),$$

where $k \in H(q, r)$, $a \in L_n(q)$, $y \in L_0(q)$ and $y^k = z + \partial b$.

Proposition 2.8 *The functors ∇, $\nabla' : \mathsf{Crs} \to \mathsf{Chn}$ have right adjoints Θ, Θ'. Hence both ∇ and ∇' preserve colimits.*

Remark. The construction of the adjoint pair (∇, Θ) has been put together from a variety of sources. The principal source for ∇ is [31], but Whitehead's construction requires C_1 to be a free group. (If C_1 is free on a set X, then D_ϕ is just the free G-module on X.) The general construction of $(\nabla C)_1 = D_\phi$ was suggested by [11].

The existence of an adjoint was suggested by results in [25] that the Alexander module preserves colimits. Special cases of the groupoid $E_1 = (\Theta L)_1$ appear in [10], [17] and [16].

The fact that $\nabla : \mathsf{Crs} \to \mathsf{Chn}$ preserves all colimits implies that the Van Kampen theorem proved in [6] for the fundamental crossed complex ΠX_* of a filtered space X_* can be converted into a similar theorem for the chain complex $CX_* = \nabla \Pi X_*$. The interpretation of this result will be discussed in Section 5. The following simple example illustrates some of the interesting features that arise in computing colimits in Crs and Chn. Note that if all the crossed complexes in a diagram $\{C^\lambda\}$ are reduced then the colimit of $\{C^\lambda\}$ is reduced provided that the diagram is connected, in which case the colimit of $\{\nabla C^\lambda\}$ can be computed in the category of chain complexes over groups instead of groupoids.

Example 2.9 *Let $M \to P$, $N \to P$ be crossed modules over a group P. Their coproduct in the category of crossed modules over P is given by the pushout in Crs:*

$$\begin{array}{ccccc} & & (\cdots 0 \to M \to P \to *) & & \\ & \nearrow & & \searrow & \\ (\cdots 0 \to 0 \to P \rightrightarrows *) & & & & (\cdots 0 \to M \bowtie N \rightrightarrows P \to *) \\ & \searrow & & \nearrow & \\ & & (\cdots 0 \to N \to P \rightrightarrows *) & & \end{array}$$

where the group $M \bowtie N$ is the Peiffer product described in [2], [14]. To find the corresponding chain complexes let $G = P/\delta M$, $H = P/\delta N$ and write ϕ, ψ for the quotient maps $P \to G$, $P \to H$. Then the corresponding derived modules are $D_\phi = IP \otimes_P \mathbb{Z}G$ and $D_\psi = IP \otimes_P \mathbb{Z}H$ and we wish to compute the pushout in Chn (or in chain complexes over groups) of

$$\begin{array}{ccc} & & (\cdots 0 \to M^{Ab} \to IP \otimes_P \mathbb{Z}G \to \mathbb{Z}G, G) \\ & \nearrow & \\ (\cdots 0 \to 0 \to IP \to \mathbb{Z}P, P) & & \\ & \searrow & \\ & & (\cdots 0 \to N^{Ab} \to IP \otimes_P \mathbb{Z}H \to \mathbb{Z}H, H) \end{array}$$

To do this, we first form the pushout K of

$$\begin{array}{ccc} & & G \\ & \nearrow & \\ P & & \\ & \searrow & \\ & & H \end{array}$$

namely $K = P/(\delta M.\delta N)$; this is the group acting on the pushout chain complex. Next we form the induced modules over K of each module in the diagram and then form pushouts of K-modules in each dimension. This gives the chain complex

$$(\cdots 0 \to (M^{Ab} \otimes_P \mathbb{Z}K) \oplus (N^{Ab} \otimes_P \mathbb{Z}K) \to IP \otimes_P \mathbb{Z}K \to \mathbb{Z}K, K).$$

Since $K = P/\delta M \delta N$, and δM acts trivially on M^{Ab}, we have $M^{Ab} \otimes_P \mathbb{Z}K = M^{Ab}/[M^{Ab}, N]$; similarly $N^{Ab} \otimes_P \mathbb{Z}K = N^{Ab}/[N^{Ab}, M]$. Thus the pushout in dimension 2 is

$$M^{Ab}/[M^{Ab}, N] \oplus N^{Ab}/[N^{Ab}, M],$$

which is easily identifiable as $(M \bowtie N)^{Ab}$, confirming that ∇ preserves this pushout.

In [8] an internal hom functor $\mathsf{CRS}(-,-)$ was defined for crossed complexes similar to that defined in Section 1 for chain complexes over groupoids. The crossed complex $\mathsf{CRS}(B, C)$ has as its objects all morphisms of crossed complexes $B \to C$, and its elements in dimension $n \geqslant 1$ are suitably defined n-fold homotopies $B \to C$. This functor, together with the appropriate tensor product, defines a symmetric monoidal closed structure on the category of crossed complexes. The relationship between the two monoidal closed structures is best described in terms of the adjoint functors ∇ and Θ.[2]

Theorem 2.10 *For crossed complexes B, C and chain complexes L there are natural isomorphisms*

(i) $\mathsf{CRS}(C, \Theta L) \cong \Theta \mathsf{CHN}(\nabla C, L)$,

(ii) $\nabla(B \otimes C) \cong \nabla B \otimes \nabla C$.

Proof The two natural isomorphisms are equivalent because

$$\begin{aligned} \mathsf{Chn}(\nabla(B \otimes C), L) &\cong \mathsf{Crs}(B \otimes C, \Theta L) \\ &\cong \mathsf{Crs}(B, \mathsf{CRS}(C, \Theta L)), \end{aligned}$$

while

$$\begin{aligned} \mathsf{Chn}(\nabla B \otimes \nabla C, L) &\cong \mathsf{Chn}(\nabla B, \mathsf{CHN}(\nabla C, L)) \\ &\cong \mathsf{Crs}(B, \Theta \mathsf{CHN}(\nabla C, L)). \end{aligned}$$

The isomorphism (i) is easier to verify than (ii) because we have explicit descriptions of the elements of both sides, whereas in (ii) we have only presentations.

In dimension 0 we have on the left of (i) the set $\mathsf{Crs}(C, \Theta L)$ of morphisms $\hat{f} : C \to \Theta L$; on the right we have the set $\mathsf{Chn}(\nabla C, L)$ of morphisms $(\tilde{f}, \psi) : \nabla C \to L$, where ψ is a morphism of groupoids from $G = \pi_1 C$ to H, the operator groupoid for L. These sets are in one-one correspondence, by adjointness, and their elements are also equivalent to pairs (f, ψ) where $\psi : G \to H$ and f is a family

Diagram 2.11

$$\begin{array}{ccccccc} \cdots \xrightarrow{\delta} & C_2 & \xrightarrow{\delta} & C_1 & \overset{\delta^0}{\underset{\delta^1}{\rightrightarrows}} & C_0 \\ & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ \cdots \xrightarrow{\partial} & L_2 & \xrightarrow{\partial} & L_1 & \xrightarrow{\partial} & L_0 \end{array}$$

[2]The result (ii) of the following theorem gives a useful description of $B \otimes C$ in dimensions > 2, complementing that in dimensions $\leqslant 2$ in [8].

such that

(i) $f_0(p) \in L_0(\psi(p))\,(p \in C_0)$,

(ii) f_1 is a $\psi\phi$-derivation, where ϕ is the quotient map $C_1 \to G$,

(iii) f_n is a ψ-morphism for $n \geqslant 2$,

(iv) $\partial f_{n+1} = f_n \delta$ $(n \geqslant 1)$,

(v) $\partial f_1(x) = (f_0 \delta^0 x)^{\psi\phi x} - (f_0 \delta^1_x)\,(x \in C_1)$.

Such a family will be called a ψ-derivation $f : C \to L$.

We recall from [8] that an element of dimension i in $\mathsf{CRS}(C, E)$ is an i-fold homotopy $(\hat{h}, \hat{f}) : C \to E$, where $\hat{f}$ is a morphism $C \to E$ and $\hat{h}$ is a family of maps

Diagram 2.12

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_2 & \longrightarrow & C_1 & \rightrightarrows & C_0 \\
 & \downarrow{\scriptstyle \hat{h}_2} & & \downarrow{\scriptstyle \hat{h}_1} & & \downarrow{\scriptstyle \hat{h}_0} \\
\cdots & E_{i+2} & & E_{i+1} & & E_i
\end{array}$$

satisfying

(i) $\hat{h}_0(p) \in E_i(\hat{f}_0(p))\,(p \in C_0)$;

(ii) $\hat{h}_1$ is a $\hat{f}_1$-derivation;

(iii) $\hat{h}_n$ is a $\hat{f}_1$-morphism for $n \geqslant 2$.

In the case $E = \Theta L$, where L is a chain complex over H, it is easy to see that, if $i \geqslant 2$, such a homotopy is equivalent to the following data: a morphism of groupoids $\psi : G \to H$; a ψ-derivation $f : C \to L$ as in (2.11); and a family h of maps

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_2 & \longrightarrow & C_1 & \overset{}{\underset{\delta_1}{\rightrightarrows}} & C_0 \\
 & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} \\
\cdots & L_{i+2} & & L_{i+1} & & L_i
\end{array}$$

satisfying

(i) $h_0(p) \in L_i(\psi p)\,(p \in C_0)$;

(ii) h_1 is a $\psi\phi$-derivation;

(iii) h_j is a ψ-morphism for $j \geqslant 2$.

The maps $\hat{h}_j$ of (2.12) are then given by

$$\begin{array}{rcl}\hat{h}_j(x) & = & (h_j(x), f_0(q)) \text{ if } x \in C_j(q),\ j \geqslant 2,\\ \hat{h}_1(x) & = & (h_1(x), f_0(q)) \text{ if } x \in C_1(p,q),\\ \hat{h}_0(q) & = & (h_0(x), f_0(q)) \text{ if } q \in C_0.\end{array}$$

In the case $i = 1$, because of the special form of E_1, we also need a map $\tau : C_0 \to H$ satisfying

(iv) $\tau(q) \in H(\psi'(q), \psi(q))$ for some $\psi'(q) \in \operatorname{Ob} H$,

and in this case $\hat{h}_0(q) = (\tau(q), h_0(q), f_0(q))$.

It is now an easy matter to see that these data are equivalent to an element of dimension i in $\Theta\mathsf{CHN}(\nabla C, L)$. In the case $i = 1$, the map τ defines a natural transformation $\tilde{\tau} : \psi' \to \psi$, where $\psi'(g) = \tau(p)\psi(g)\tau(q)^{-1}$ for $g \in G(p,q)$. This $\tilde{\tau}$ is an element of the groupoid $\mathsf{GPD}(G,H)$ (the operator groupoid for $\mathsf{CHN}(\nabla C, L)$) and provides the first component of the triple $(\tilde{\tau}, \tilde{h}, \tilde{f})$ which is the required element of $\Theta_1\mathsf{CHN}(\nabla C, L)$; the other components are $\tilde{f} : \nabla C \to L$, the morphism of chain complexes induced by f, and $\tilde{h}$, the 1-fold homotopy $\nabla C \to L$ induced by h. Here $\tilde{h}_0(1_p) = h_0(p)$ and $\tilde{h}_n\alpha_n = h_n$ for $n \geqslant 1$, where the α_i are as in (2.5). The rest of the proof is straightforward. □

3 Exactness and lifting properties of ∇

Our first proposition gives an extension of the exact module sequence of Crowell [10, 11]; see also [23], p. 120.

Proposition 3.1 *Let $C = \{C_r\}$ be a crossed complex and suppose that the sequence of groupoids*

$$C_3 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_1 \xrightarrow{\phi} G \to 1$$

is exact. Then, in $\nabla' C$, the sequence of G-modules

$$C_3 \xrightarrow{\partial} C_2^{Ab} \xrightarrow{\partial} D_\phi \xrightarrow{\partial'} \vec{I}G \to 0$$

is exact.

Proof The exactness of $C_2 \to C_1 \xrightarrow{\phi} G \to 1$ implies that

$$\begin{array}{ccc}(C_2 \to 1) & \longrightarrow & (C_1 \xrightarrow{\phi} G)\\ \downarrow & & \downarrow\\ (1 \to 1) & \longrightarrow & (G \xrightarrow{=} G)\end{array}$$

is a pushout square in the arrow category Gpd^2. Applying $D : \mathsf{Gpd}^2 \to \mathsf{Mod}$, as in the proof of (2.4), and noting that D preserves colimits by (2.3), we obtain a pushout square

$$\begin{array}{ccc} C_2^{Ab} & \xrightarrow{\partial} & D_\phi \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \vec{I}G \end{array} \qquad \text{over} \qquad \begin{array}{ccc} 1 & \longrightarrow & G \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & G \end{array}$$

in Mod. Since $\partial : C_2^{Ab} \to D_\phi$ is in fact a G-morphism, it follows that

$$C_2^{Ab} \to D_\phi \to \vec{I}G \to 0$$

is an exact sequence of G-modules.

To prove exactness of $C_3 \to C_2^{Ab} \to D_\phi$, write $N = \operatorname{Ker}\phi = \delta C_2$ and note that the exactness of

$$C_3 \to C_2 \to N \to 1$$

implies the exactness of

$$C_3 \to C_2^{Ab} \to N^{Ab} \to 1.$$

It remains, therefore, to show that the map $\gamma : N^{Ab} \to D_\phi$ induced by $\partial : C_2^{Ab} \to D_\phi$ is injective.

Now $\phi : C_1 \to G$ is a quotient morphism of groupoids with totally intransitive kernel N. In these circumstances the additive groupoid structure of D_ϕ is given by generators $[c] \in D_\phi(q)$ for $c \in C_1(p, q)$, with defining relations

$$[cy] = [c] + [y] \text{ for } c \in C_1(p,q),\ y \in N(q);$$

the groupoid C_1 acts on this additive groupoid by

$$[c]^x = [cx] - [x]$$

and N acts trivially, making D_ϕ a G-module; the canonical ϕ-derivation $\alpha_1 : C_1 \to D_\phi$ is given by $\alpha_1(c) = [c]$.

Choose coset representatives $t(c) \in cN$ of N in C_1 with $t(1_q) = 1_q$. Then for all $c \in C_1$, $c = t(c)s(c)$ where $s(c) \in N$. The map $s : C_1 \to N$ satisfies $s(y) = y$ for $y \in N$ and

$$s(cy) = s(c)y \text{ for all } c \in C_1(p,q),\ y \in N(q).$$

Consequently, there is an additive map $s^* : D_\phi \to N^{Ab}$ defined by $s^*[c] = \alpha s(c)$, where α is the canonical map $N \to N^{Ab}$. Since, for any $u = \alpha y$ in N^{Ab},

$$s^*\gamma u = s^*\gamma\alpha y = s^*\alpha_1 y = \alpha s(y) = \alpha y = u,$$

γ is injective, as required. □

Definition 3.2 *The crossed complex C is* regular *if $K \cap [C_2, C_2] = 0$, where K is the kernel of $\delta : C_2 \to C_1$.*

Corollary 3.3 *If C is regular, then the map $C_2 \to C_2^{Ab}$ maps the kernel of δ_2 isomorphically to the kernel of $\partial : C_2^{Ab} \to D_\phi$*[3]*.*

Proof The exactness result Proposition 3.1 shows the map of kernels is surjective. Regularity is precisely the condition needed for injectivity. □

We note in passing a sufficient condition for regularity due in the group case to Whitehead [31]:

Proposition 3.4 *If in the crossed complex C, the groupoid C_1 is free, then C is regular. In particular, the fundamental crossed complex $\pi(\mathbf{X})$ of a CW-complex $\mathbf{X}$ is regular.*

Proof Since $N = \delta C_2$ is a subgroupoid of C_1, it is a free groupoid (in fact a family of free groups). Hence the map $\delta : C_2 \to N$ has a homomorphic section s. But the kernel K of δ is in the centre of C_2, since C_2 is a crossed module over C_1. Hence $C_2 = K \times_{C_0} s(N)$ is a groupoid, that is, for each $p \in C_0$, $C_2(p) = K(p) \times sN(p)$. This implies that $[C_2, C_2] = [sN, sN]$ and hence that $K \cap [C_2, C_2] = 0$. □

We now consider the realisability problem for maps $(\theta, \psi) : \nabla B \to \nabla C$, that is, to find conditions which ensure that such a map can be realised as ∇f for some $f : B \to C$, and also that chain homotopies can be realised as homotopies in Crs. The unit adjunction morphism $\eta : C \to \Theta\nabla C$ plays an important role in such liftings, so we first examine it in detail.

We recall from [30] that a normal crossed subcomplex K of C consists of normal subgroupoids K_n of C_n for $n \geqslant 1$ such that (i) K_n admits the action of C_1 for $n \geqslant 2$, (ii) δ maps K_n into K_{n-1} for $n \geqslant 2$, and (iii) the object groups of K_1 act trivially on the quotient groupoid C_n/K_n for $n \geqslant 2$. Kernels of morphisms are of this type. In the case when K_1 is totally intransitive, (i.e. a family of groups), the sequence of quotient groupoids

$$\cdots \to C_n/K_n \to \ldots \to C_2/K_2 \to C_1/K_1 \rightrightarrows C_0$$

is a crossed complex, denoted C/K. In the general case a quotient complex C/K is formed by killing the action of K_1 on each C_n/K_n $(n \geqslant 2)$, which involves identifying objects $p, q \in C_0$ which are joined in K_1 and identifying the corresponding groups over p, q in each C_n/K_n $(n \geqslant 2)$.

Definition 3.5 *For any crossed complex C we introduce two auxiliary crossed complexes C', $\bar{C}$ as follows, where $N = \delta C_2$:*

$$C' : \cdots \to 0 \to \cdots \to 0 \to [C_2, C_2] \to [N, N] \rightrightarrows C_0,$$

$$\bar{C} : \cdots \to C_n \to \cdots \to C_3 \to C_2/[C_2, C_2] \to C_1/[N, N] \rightrightarrows C_0.$$

Note that $N = \delta C_2$ acts trivially on C_n for $n \geqslant 3$ and acts by conjugation on C_2, so $[N, N]$, which is totally intransitive, acts trivially on C_n $(n \geqslant 3)$ and on $C_2/[C_2, C_2]$. Thus C' is a normal subcomplex of C and $\bar{C}$ is the quotient complex C/C'. We write $\varepsilon : C \to \bar{C}$ for the quotient morphism.

Proposition 3.6 *Let C be a crossed complex and let $\eta : C \to E = \Theta\nabla C$ be the unit adjunction morphism. Then*

[3] This is a correction to the original.

(i) *the image of η is the full subcomplex $E^\dagger$ of E on the objects $\eta_0(C_0) = \{1_p; p \in C_0\}$;*

(ii) *the kernel of η is the crossed complex C' above.*

Proof We use the notations of Proposition 3.1 and its proof; in particular, $N = \delta C_2$ and $G = C_1/N$.

According to the description of Θ in Section 2, we have $E_0 = U(\overrightarrow{\mathbb{Z}G}) = \bigcup_{q \in C_0} \overrightarrow{\mathbb{Z}G}(q)$. Also, $E_0^\dagger = \eta_0(C_0)$ is the set of identities $1_q \in \overrightarrow{\mathbb{Z}G}$ for $q \in C_0$.

The elements of E_1 are triples (g, a, y), where $g \in G(p,q)$, $a \in D_\phi(q)$, $y \in \overrightarrow{\mathbb{Z}G}(q)$. The image in E of $c \in C_1(p,q)$ is $\eta_1(c) = (\phi(c), \alpha_1(c), 1_q)$ and this is in $E_1^\dagger(p,q)$. Conversely, any element of $E_1^\dagger(p,q)$ has source 1_p and target 1_q and is of the form $(g, a, 1_q)$ where $g \in G(p,q)$, $a \in D_\phi(q)$ and $1_p^g = 1_q + \partial a$, that is, $\partial a = g - 1_q \in \overrightarrow{IG}$. Choose $c \in C_1(p,q)$ with $\phi(c) = g$. Then $\partial \alpha_1 c = \alpha_0 \phi c = g - 1_q = \partial a$, so $a - \alpha_1 c$ is in the kernel of $\partial' : D_\phi \to \overrightarrow{IG}$. By (3.1), this kernel is $\gamma N^{Ab} = \alpha_1 N$, so there is $y \in N$ with $a - \alpha_1 c = \alpha_1 y$. Putting $c' = cy$, we have

$$\eta_1(c') = (\phi(cy), \alpha_1(cy), 1_q) = (g, a, 1_q)$$

since $\alpha_1(cy) = (\alpha_1 c)^{\phi y} + \alpha_1 y = a$. Thus $\eta_1(C_1) = E_1^\dagger$. If c is in the kernel of η_1, then $\phi(c) = 1$ and $\alpha_1 c = 0$, so $c \in N$ and $\gamma \alpha c = 0$. Since $\gamma : N^{Ab} \to D_\phi$ is an injection we deduce that $c \in [N,N] = \delta[C_2, C_2]$, and the converse is clear. This proves (i) and (ii) in dimension 1.

In dimension 2, $E_2^\dagger$ is the set of all $(b, 1_q)$, $b \in C_2^{Ab}(q)$, and for $c \in C_2(q)$ we have $\eta_2(c) = (\alpha_2(c), 1_q)$ where α_2 is the canonical map $C_2 \to C_2^{Ab}$. For $n \geqslant 3$, $E_n^\dagger$ is the set of all $(c, 1_q)$, $c \in C_n(q)$, and $\eta_n(c) = (c, 1_q)$. The proposition follows immediately. □

Corollary 3.7 $\eta(C) \cong \bar{C}$.

Proof The morphism $\eta : C \to E$ satisfies the first isomorphism theorem for crossed complexes because $\eta_0 : C_0 \to E_0$ is injective. □

If C is a crossed complex and $\nabla C = (K, G)$, then $K_0 = \overrightarrow{\mathbb{Z}G}$ and there is an injection $G_0 \to K_0$, $p \mapsto 1_p$, whose image is a *preferred basis* for K_0 as G-module. A morphism $f : B \to C$ of crossed complexes induces a morphism $\nabla f = (\theta, \psi) : \nabla B \to \nabla C$ where $\nabla B = (L, H)$ and $\psi : H \to G$. The map $\theta : L \to K$ respects the preferred basis in the sense that $\theta_0(1_p) = 1_q$ where $q = \psi_0(p)$. Any morphism $(\theta, \psi) : \nabla B \to \nabla C$ satisfying this condition will be called a *preferred morphism.* Because only preferred morphisms $\nabla B \to \nabla C$ are candidates for realisation as ∇f, we wish to restrict attention to preferred morphisms. However, we cannot do this at the level of chain complexes without losing some of the structure: the collection of all morphisms $\nabla B \to \nabla C$ is a module over the groupoid $F = \mathsf{GPD}(H, G)$, but the preferred morphisms do not form a submodule (they admit the action of F but not the addition).

By Theorem 2.10 there is a natural isomorphism of crossed complexes

Diagram 3.8

$$\mathsf{CRS}(B, \Theta\nabla C) \cong \Theta\mathsf{CHN}(\nabla B, \nabla C).$$

The objects of the right hand side are arbitrary morphisms $\nabla B \to \nabla C$ and we write $\Theta^{Pr}\mathsf{CHN}(\nabla B, \nabla C)$ for the full subcomplex of $\Theta\mathsf{CHN}(\nabla B, \nabla C)$ whose objects are all the preferred morphisms.

Theorem 3.9 *The isomorphism (3.8) induces an isomorphism of crossed complexes*

$$\mathsf{CRS}(B, \bar{C}) \cong \Theta^{Pr}\mathsf{CHN}(\nabla B, \nabla C).$$

Proof One checks easily that, if $E^\dagger$ is a full subcomplex of a crossed complex E, then $\mathrm{CRS}(B, E^\dagger)$ can be identified with the full subcomplex of $\mathrm{CRS}(B, E)$ whose objects are those morphisms $f : B \to E$ taking values in $E^\dagger$. Putting $E = \Theta\nabla C$ and $E^\dagger = \eta(C)$ as in 3.6, we find that $CRS(B, \eta(C))$ is a full subcomplex of $\mathrm{CRS}(B, \Theta\nabla C)$ whose objects are morphisms $f : B \to \Theta\nabla C$ such that, for $p \in B_0$, $f(p)$ is of the form $1_q \in U\overrightarrow{\mathbb{Z}}G$, where $G = \pi_1(C)$. Under the isomorphism (3.8) these morphisms correspond precisely to the preferred morphisms $\nabla B \to \nabla C$. Hence (3.8) induces

$$\mathsf{CRS}(B, \eta(C)) \cong \Theta^{Pr}\mathsf{CHN}(\nabla B, \nabla C)$$

and the result follows from Corollary 3.7. □

The information contained in Theorem 3.9 in dimensions 1 and 0 will be used, by applying the functor π_0, to relate homotopy classes of maps of crossed complexes to homotopy classes of maps of chain complexes over groupoids.

For crossed complexes B, C, we define $[B, C]$ to be $\pi_0\mathrm{CRS}(B, C)$, the set of components of the groupoid $\mathsf{CRS}_1(B, C) \rightrightarrows \mathsf{CRS}_0(B, C)$. Thus $[B, C]$ is the set of equivalence classes of morphisms $B \to C$ under the relation of 1-fold homotopy, which corresponds in the topological context to free homotopy (see [7]). For convenience we recall from [8] the definition of this equivalence relation (originally due to J. H. C. Whitehead [31]). A 1-fold (left) homotopy $h : f' \simeq f$, where f', $f : B \to C$, is a family of maps $h_n : B_n \to C_{n+1}$ $(n \geqslant 0)$ such that: $\beta h_n(b) = \beta f_n(b) \in B_0$ for all $b \in B_n$; h_1 is a derivation over f_1; h_n $(n \geqslant 2)$ is an operator morphism over f_1; and, for $b \in B_n$,

Diagram 3.10

$$f'_n(b) = \begin{cases} [f_n(b) + h_{n-1}\delta b + \delta h_n b]^{-h_0\beta b} & \text{if } n \geqslant 2, \\ (h_0\delta^0 b)(f_1 b)(\delta h_1 b)(h_0\delta^1 b)^{-1} & \text{if } n = 1, \\ \delta h_0 b & \text{if } n = 0. \end{cases}$$

For chain complexes (M, G), (N, H), where G, H are groupoids, $\mathsf{CHN}(M, N)$ is a chain complex over $F = \mathsf{GPD}(G, H)$, so

$$H_0\mathsf{CHN}(M, N) = \mathrm{coker}(\partial : \mathsf{CHN}_1(M, N) \to \mathsf{CHN}_0(M, N))$$

is an F-module. We define

$$[M, N] = (H_0\mathsf{CHN}(M, N))/F,$$

the set of orbits of the F-action. The additive structure of $H_0\mathsf{CHN}(M, N)$ is not inherited by $[M, N]$; all that survives is a fibring over $\pi_0 F$, the set of homotopy classes of morphisms $G \to H$. It is easy to see that two morphisms (f, ψ), $(f', \psi') : (M, G) \to (N, H)$ represent the same element of $[M, N]$ if and only if there is a chain homotopy $h \in \mathsf{CHN}_1(M, N)$ and a homotopy $\alpha : \psi \to \psi'$ in F such that $f' = f^\alpha + \partial h$, where f^α is defined by $f^\alpha(m) = f(m)^{\alpha(p)}$ for $m \in M_i(p)$. This is the appropriate notion of free homotopy for chain complexes over groupoids and we write $(f, \psi) \simeq (f', \psi')$.

Lemma 3.11 *For any chain complex (L, F) there is a natural bijection*

$$\pi_0\Theta(L, F) \cong H_0(L, F)/F.$$

In particular, for chain complexes (M, G), (N, H), there is a natural bijection

$$\pi_0\Theta\mathsf{CHN}(M, N) \cong [M, N].$$

Proof The objects of $\Theta(L, F)$ are the elements of L_0 and there is an arrow in $\Theta_1(L, F)$ from $x \in L_0(p)$ to $y \in L_0(q)$ if and only if $x^\alpha = y + \partial a$ for some $\alpha \in F(p, q)$ and $a \in L_1(q)$. □

Let $[\nabla B, \nabla C]^{Pr}$ denote the set of those free homotopy classes which can be represented by preferred morphisms. Combining Theorem 3.9 with Lemma 3.11, we now obtain

Corollary 3.12 *Let B, C be arbitrary crossed complexes. Then there is a natural bijection*

$$[B, \bar{C}] \cong [\nabla B, \nabla C]^{Pr}.$$

Proposition 3.9 and its corollary, 3.12, show what information on morphisms and homotopies is lost in passing from crossed complexes to the corresponding chain complexes. We now show that, in special circumstances, $[B, C]$ is actually determined by ∇B and ∇C.

We say that a morphism $f : C \to D$ of crossed complexes is a *quotient morphism* if it induces an isomorphism $C/\operatorname{Ker} f \to D$. Necessary and sufficient conditions for this are that (i) $f_0 : C_0 \to D_0$ is surjective, (ii) f_1 maps $C_1(p, q)$ surjectively to $D_1(f_0p, f_0q)$ for all $p, q \in C_0$, and (iii) f_n maps $C_n(p)$ surjectively to $D_n(f_0p)$ for all $p \in C_0$ and $n \geqslant 2$.

A crossed complex B is *of free type* if B_1 is a free groupoid, B_2 is a free crossed module over B_1 and, for $n \geqslant 3$, B_n is a free module over $\pi_1(B)$.

Proposition 3.13 *Let B, C be crossed complexes and suppose that C is regular and B is of free type. Let C', $\bar{C}$ be as in (3.5). Then the morphism of crossed complexes*

$$\varepsilon_\# : \mathsf{CRS}(B, C) \to \mathsf{CRS}(B, \bar{C})$$

induced by the quotient morphism $\varepsilon : C \to \bar{C}$ is itself a quotient morphism with kernel $\mathsf{CRS}(B, C')$.

Proof The quotient morphism $\varepsilon : C \to \bar{C}$ is a fibration of crossed complexes [18]. By Proposition 2.2 of [4], it is a trivial fibration (that is, a fibration and a weak equivalence), because criterion (ii) of that proposition is satisfied when C is regular. We now use the lifting properties for trivial fibrations established in [4].

We first have to prove that $\varepsilon_\#$ is surjective on objects. This is just the condition that a morphism $B \to \bar{C}$ lifts to a morphism $B \to C$, which holds since B is of free type and ε is a trivial fibration.

In dimension 1 we have to show that any diagram

$$\begin{array}{ccc} B \otimes \{0,1\} & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ B \otimes \mathsf{J} & \longrightarrow & C \end{array}$$

has a completion $B \otimes \mathsf{J} \to C$. This follows from the fact that ε is a trivial fibration and i is a cofibration. Finally, in dimension $n \geqslant 2$ we have to show that any diagram

$$\begin{array}{ccc} B \otimes \{1\} & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ B \otimes \mathbb{C}(n) & \longrightarrow & C \end{array}$$

has a completion $B \otimes \mathbb{C}(n) \to C$, and this follows for similar reasons. □

Theorem 3.14 *Let B, C be crossed complexes with C regular and B of free type. Then there is a natural isomorphism of crossed complexes*

$$\mathsf{CRS}(B, C)/\mathsf{CRS}(B, C') \cong \Theta^{Pr}\mathsf{CHN}(\nabla B, \nabla C).$$

In particular, application of the functor π_0 gives a natural bijection

$$[B, C] \cong [\nabla B, \nabla C]^{Pr},$$

induced by the map $f \mapsto \nabla f$.

Proof The isomorphism of crossed complexes follows from Proposition 3.13 and Proposition 3.9. When π_0 is applied to this isomorphism we get $[\nabla B, \nabla C]^{Pr}$ on the right, by Lemma 3.11. On the left we get π_0 of a quotient, which only depends on the groupoids in dimension 1. However, for a quotient of groupoids G/H, we have $\pi_0(G/H) \cong \pi_0(G)$. (Indeed $\pi_0(G)$ can be defined to be the quotient G/G.) Thus, on the left, we get $\pi_0\mathsf{CRS}(B, C) = [B, C]$. □

Remarks 3.15 (i) A closer analysis of the conditions which hold on ε when C is regular shows that Proposition 3.13, and so Theorem 3.14, require of B only that B_1 is a free groupoid.

(ii) Theorem 3.14 is essentially the algebraic content of J. H. C. Whitehead's paper [31], but, particularly taking account of the last Remark, with weaker assumptions about the freeness of the complexes.

(iii) Baues in [1] also considers a generalization of Whitehead's results, to (reduced) crossed complexes of free type, and which are under a fixed group G. He obtains obstruction type conditions involving the vanishing of certain cohomology groups with coefficients in $R = Ker\,\delta_2 \cap [C_2, C_2]$. It would be interesting to obtain these results from the fact $C \to \bar{C}$ is a fibration whose kernel is up to weak homotopy equivalence a $K(R, 2)$ (in the category of crossed complexes). Note also that Baues always works with pointed spaces and pointed homotopies, whereas Whitehead uses free homotopies. Baues uses his theorem to obtain homotopy classification theorems for cell complexes under a fixed space D.

A slightly different version of Theorem 3.14, whose proof uses the Remark 3.15(i), can be stated as follows. Let $\mathsf{Crs}^{(1)}$ denote the category of crossed complexes which are free in dimension 1, and all morphisms between them. Let Chn^{Pr} denote the category of preferred chain complexes (that is, complexes (M,G) with $M_0 = \overrightarrow{\mathbb{Z}}G)$ and preferred morphisms. Let $\mathsf{Crs}^{(1)}/\simeq$ and $\mathsf{Chn}^{Pr}/\simeq$ be the corresponding homotopy categories, where the homotopies are as previous defined for Crs and Chn. Then, combining Theorem 3.14 and Proposition 3.4, we obtain

Corollary 3.16 *The functor* $\nabla : \mathsf{Crs} \to \mathsf{Chn}$ *induces a full and faithful functor* $(\mathsf{Crs}^{(1)}/\simeq) \to (\mathsf{Chn}^{Pr}/\simeq)$.

4 The pointed case

As in [8], a crossed complex is pointed if it has a distinguished object $*$, called the base-point, and a morphism $f : B \to C$ is pointed if $f_0(*) = *$. All pointed crossed complexes and pointed morphisms form a category Crs_*. An n-fold homotopy $(h,f) : B \to C$ is pointed if f is pointed and $h_0(*) = 1_*$ if $n = 1$, or $h_0(*) = 0_*$ if $n \geqslant 2$. The pointed morphisms and pointed homotopies $B \to C$ form a pointed crossed complex $\mathsf{CRS}_*(B,C)$ and the functor $\mathsf{CRS}_*(-,-)$ is the internal Hom functor for a monoidal closed structure on Crs_*.

Similarly, a pointed chain complex is a chain complex over a pointed groupoid and a morphism $(f,\psi) : (L,G) \to (M,H)$ or a homotopy $(h,\psi) : (L,G) \to (M,H)$ is pointed if $\psi(*) = *$ (from which follows $f_n(0_*) = 0_*$, $h_n(0_*) = 0_*$ for all $n \geqslant 0$). Together, these pointed morphisms and pointed homotopies $L \to M$ make up a pointed complex $\mathrm{CHN}_*(L,M)$ which we view as a chain complex over the groupoid $F_* = \mathrm{GPD}_*(G,H)$ of pointed morphisms $G \to H$ and pointed natural transformations between them. This construction gives an internal Hom functor in Chn_*, the category of pointed chain complexes.

To obtain a pointed version of Theorem 3.14 in the case when B and C are pointed crossed complexes, we re-examine the morphisms

$$\mathsf{CRS}(B,C) \xrightarrow{\eta_\#} \mathsf{CRS}(B,\eta(C)) \xrightarrow{\cong} \Theta^{Pr}\mathsf{CHN}(\nabla B,\nabla C)$$

of Section 3 to determine their effect on the sub-crossed complex $\mathsf{CRS}_*(B,C)$ of $\mathrm{CRS}(B,C)$. The adjunction morphism $\eta : C \to \Theta\nabla C$ sends $*$ to $1_* \in \overrightarrow{\mathbb{Z}}H$, where $H = \pi_1 C$. We therefore assign 1_* to be the base-point of $\Theta\nabla C$ so that η is a pointed morphism and $\eta(C)$ is a pointed crossed complex.

In dimension 0, and object $f : B \to C$ of $\mathsf{CRS}(B,C)$ goes to $\eta \circ f$ in $\mathsf{CRS}(B,\eta(C))$ and to ∇f in $\Theta^{Pr}\mathsf{CHN}(\nabla B,\nabla C)$. Clearly $\eta \circ f$ and ∇f are pointed morphisms if and only if f is pointed.

In dimension 1, a 1-homotopy $(h,f) : B \to C$ goes to $(\eta \circ h, \eta \circ f) : B \to \eta(C)$ in $\mathsf{CRS}(B,\eta(C))$ and again this is pointed if and only if (h,f) is pointed. As for the image of (h,f) in $E = \Theta^{Pr}\mathsf{CHN}(\nabla B,\nabla C)$, we recall that $mathsfCHN(\nabla B,\nabla C)$ is a chain complex over $F = \mathsf{GPD}(G,H)$, where $G = \pi_1 B$, $H = \pi_1 C$, and therefore an element of E_1 from λ' to λ is a triple (τ,u,λ) where:

(i) λ', λ are morphisms $\nabla B \to \nabla C$ over $\psi', \psi : G \to H$;

(ii) $\tau \in F(\psi',\psi)$;

(iii) u is a 1-homotopy $\nabla B \to \nabla C$ over ψ with $(\lambda')^\tau = \lambda + \partial u$.

In this notation, the image of (h,f) in E_1 is given by:

(i) $\lambda = \nabla f$;

(ii) $\tau = \phi \circ h_0$, where ϕ is the quotient morphism $C_1 \to H$;

(iii) u is the unique homotopy $\nabla h : \nabla B \to \nabla C$ over ψ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{h} & C \\ \downarrow & & \downarrow \\ \nabla B & \xrightarrow[\nabla h]{} & \nabla C \end{array}$$

commutes, where the vertical arrows are the canonical derivations. If (h, f) is a pointed homotopy then τ, u and λ are all pointed and this means that (τ, u, λ) is an element of $(E_*)_1$, where $E_* = \Theta^{Pr}\mathsf{CHN}_*(\nabla B, \nabla C)$, provided that we view $\mathsf{CHN}_*(\nabla B, \nabla C)$ as a chain complex over F_* rather than over F. Conversely, if $(\tau, u, \lambda) \in (E_*)_1$, then (h, f) is pointed.

In dimension $n \geqslant 2$ the argument is similar, except that the natural transformation τ does not now appear. The images of (h, f) are $(\eta \circ h, \eta \circ f)$ in $\mathsf{CRS}(B\eta(C))$ and $(\nabla h, \nabla f)$ in E and the following are equivalent:

(i) (h, f) is pointed;

(ii) $(\eta \circ h, \eta \circ f)$ is pointed;

(iii) $(\nabla h, \nabla f) \in (E_*)_n$.

An immediate consequence of the above is that $\mathsf{CRS}_*(B, \eta(C)) \cong E_* = \Theta^{Pr}\mathsf{CHN}_*(\nabla B, \nabla C)$. By (3.7) this implies

Theorem 4.1 *Let B, C be pointed crossed complexes. Then the isomorphism (3.8) induces an isomorphism*

$$\mathsf{CRS}_*(B, \bar{C}) \cong \Theta^{Pr}\mathsf{CHN}_*(\nabla B, \nabla C).$$

Note. The base-point of $\mathsf{CRS}_*(B, \bar{C})$ is the morphism $* : B \to \bar{C}$ sending all elements to $*$, 1_* or 0_*. The image of this base-point in $\Theta^{Pr}\mathsf{CHN}_*(\nabla B, \nabla C)$ is the augmentation map $\nabla B \to \nabla C$ which sends everything to 0_* in dimensions $n \geqslant 1$ but in dimension 0 maps $\overrightarrow{\mathbb{Z}}G$ to $\overrightarrow{\mathbb{Z}}H$ by sending all groupoid elements g to 1_*. In general $\Theta(L, F)$ for a pointed chain complex L over F does not have a natural base-point except possibly $0_* \in L(*)$, which is unsuitable in the above situation.

For pointed crossed complexes B, C we define $[B, C]_*$ to be the set of pointed homotopy classes of pointed morphisms $B \to C$, that is

$$[B, C]_* = \pi_0 \mathsf{CRS}_*(B, C).$$

Similarly, for pointed chain complexes L, M over G, H, we define $[L, M]_*$ to be the set of classes of pointed morphisms $L \to M$ under the equivalence relation: $f' \simeq f$ if there exist $\tau \in F_* = \mathrm{GPD}_*(G, H)$ and a pointed 1-homotopy $u : L \to M$ such that $(f')^\tau = f + \partial u$. Thus

$$[L, M]_* = (H_0 \mathsf{CHN}_*(L, M))/F_*.$$

In the case $L = \nabla B$, $M = \nabla C$ we denote by $[\nabla B, \nabla C]_*^{Pr}$ the set of such classes which can be represented by preferred morphisms $\nabla B \to \nabla C$. Applying the functor π_0 to the isomorphism of Theorem 4.1, and using Lemma 3.11, we obtain

Corollary 4.2 *For arbitrary pointed crossed complexes B, C there is a natural bijection*

$$[B, \bar{C}]_* \cong [\nabla B, \nabla C]_*^{Pr}.$$

Finally, if B, C are pointed crossed complexes with B of free type and C regular, we know from Proposition 3.13 that $\varepsilon_\# : \mathsf{CRS}(B, C) \to \mathsf{CRS}(B, \bar{C})$ is a quotient morphism with kernel $\mathsf{CRS}(B, C')$. We have also seen that the inverse image of $\mathsf{CRS}_*(B, \bar{C}) \cong \mathsf{CRS}_*(B, \eta(C))$ under $\varepsilon_\#$ is precisely $\mathsf{CRS}_*(B, C)$. Under these circumstances we deduce that the induced morphism $\mathsf{CRS}_*(B, C) \to \mathsf{CRS}_*(B, \bar{C})$ is a quotient morphism, and its kernel is clearly $\mathsf{CRS}_*(B, C) \cap \mathsf{CRS}(B, C') = \mathsf{CRS}_*(B, C')$. This proves

Theorem 4.3 *Let B, C be pointed crossed complexes with B of free type and C regular. Then there is a natural isomorphism of crossed complexes*

$$\mathsf{CRS}_*(B, C)/\mathsf{CRS}_*(B, C') \cong \Theta^{Pr}\mathsf{CHN}_*(\nabla B, \nabla C)$$

and hence a natural bijection

$$[B, C]_* \cong [\nabla B, \nabla C]_*^{Pr}$$

induced the map $f \mapsto \nabla f$.

Corollary 4.4 *If B and C are reduced crossed complexes with B of free type and C regular, then*

$$[B, C]_* \cong H_0\mathsf{CHN}(\nabla B, \nabla C)^{Pr},$$

where the right-hand side is the set of homology classes represented by preferred morphisms $\nabla B \to \nabla C$.

Proof In this special case of Theorem 4.3, all morphisms and homotopies $\nabla B \to \nabla C$ are pointed and the groupoid $F_* = \mathsf{GPD}_*(G, H)$ is a discrete groupoid. So two preferred morphisms f', $f : \nabla B \to \nabla C$ are in the same class if and only if there exists a 1-homotopy $u : \nabla B \to \nabla C$ with $f' = f + \partial u$. □

Remark. *As in the unpointed case, Theorem 4.3 and Corollary 4.4 are in fact true whenever B_1 is free and C is regular.*

5 The chain complex of a filtered space

Our object in this section is to identify the chain complex $\nabla\Pi X_*$ in terms of chains of universal covers for certain filtered spaces X_*. We first need an analysis of connectivity conditions for filtered spaces. In stating these it is convenient to write X_∞ for the space X of a filtered space X_*, so that in the following conditions, the case $r \geqslant i$ will include the case $r = \infty$.

Proposition 5.1 *For a filtered space X_* the following conditions (i),(ii) and (iii) are equivalent:*

(i) $(\phi)_0$*: The function $\pi_0 X_0 \to \pi_0 X_r$ induced by inclusion is surjective for all $r \geqslant 0$; and, for all $i \geqslant 1$,*

$$(\phi_i) : \pi_i(X_r, X_i, v) = 0 \textit{ for all } r > i \textit{ and } v \in X_0.$$

(ii) (ϕ'_0): *The function* $\pi_0 X_s \to \pi_0 X_r$ *induced by inclusion is surjective for all* $0 = s < r$ *and bijective for all* $1 \leqslant s \leqslant r$*; and, for all* $i \geqslant 1$,

$$(\phi'_i): \pi_j(X_r, X_i, v) = 0 \text{ for all } v \in X_0 \text{ and all } j, r \text{ such that } 1 \leqslant j \leqslant i < r.$$

(iii) (ϕ'_0) *and, for all* $i \geqslant 1$,

$$(\phi''_i): \pi_j(X_{i+1}, X_i, v) = 0 \text{ for all } j \leqslant i, \text{ and } v \in X_0.$$

The proof is a straightforward argument on the exact homotopy sequences of various pairs and triples and is omitted.

The conditions (i), (ii) and (iii) are also equivalent to the condition called homotopy full in [6], which was expressed in a form suitable for use in the proof of the Generalized Van Kampen Theorem (Theorems B and C of [6]). It is convenient here, by analogy with the use of the term in [9], to call a filtered space satisfying these conditions *connected*. We note that the skeletal fibration of any CW-complex is connected in this sense.

All spaces which arise will now be assumed to be Hausdorff and to have universal covers.

Let X_* be a filtered space. For $v \in X_0$, let $p : \tilde{X}(v) \to X$ denote the universal cover of X and let $\hat{\mathbf{X}}(v)$ denote the filtered space consisting of $\tilde{X}(v)$ and the family of subspaces

$$\hat{X}_i(v) = p^{-1}(X_i)$$

for all $i \geqslant 0$.

Suppose X_* is a connected filtered space. The connectivity assumption implies that $\hat{X}_i(v)$ is the universal cover of X_i based at v for $i \geqslant 2$.

Proposition 5.2 *If* X_* *is a connected filtered space, then* $\nabla\Pi X_*$ *has operating groupoid* $\pi_1(X, X_0)$ *and has chain complex* $C(v)$ *at* $v \in X_0$ *given by*

$$C_i(v) = H_i(\hat{X}_i(v), \hat{X}_{i-1}(v))$$

for all $i \geqslant 1$.

Proof Let $v \in X_0$ and let $i \geqslant 3$. The pair $(\hat{X}_i(v), \hat{X}_{i-1}(v))$ is $(i-1)$-connected, and so

$$\begin{aligned} \pi_i(X_i(v), X_{i-1}(v), v) &\cong \pi_i(\hat{X}_i(v), \hat{X}_{i-1}(v), v) \text{ since } p \text{ is a covering,} \\ &\cong H_i(\hat{X}_i(v), \hat{X}_{i-1}(v)) \text{ by the relative Hurewicz theorem,} \end{aligned}$$

since $\hat{X}_i(v)$ and $\hat{X}_{i-1}(v)$ are in fact the universal covers at v of X_i and X_{i-1} respectively. If $i = 2$, a similar argument applies but in this case $\pi_1(\hat{X}_1, v) = \delta\pi_2(X_2, X_1, v)$. So the relative Hurewicz theorem now gives

$$H_2(\hat{X}_2(v), \hat{X}_1(v)) \cong \pi_2(\hat{X}_2(v), \hat{X}_1(v), v)^{Ab} = C_2(v).$$

The case $i = 1$ is essentially the result of [11], section 4. □

In view of the above we define for a filtered space X_* the chain complex with operators CX_* to have groupoid of operators $\pi_1(X, X_0)$ and to have $C_iX_*(v) = H_i(\hat{X}_i(v), \hat{X}_{i-1}(v))$. This defines the functor

$$C : \mathsf{FTop} \to \mathsf{Chn}.$$

The result given above is that if X_* is connected then $CX_* = \nabla\pi X_*$.

Corollary 5.3 *Let X_* be a filtered space and suppose that X is the union of a family $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of open sets such that $\mathcal{U}$ is closed under finite intersection. Let U^λ_* be the filtered space obtained from X_* by intersection with U^λ. Suppose that each U^λ_* is a connected filtered space. Then X_* is connected and the natural morphism in* Chn

$$\mathrm{colim}^\lambda CU^\lambda_* \to CX_*$$

is an isomorphism.

Proof This is a consequence of the Union Theorem (Theorem C) of [6] which gives a similar result for Π rather than C, and the fact that ∇ has a right adjoint and so preserves colimits. □

We note that results such as this have been used by various workers ([20, 27]) in the case X_* is the skeletal filtration of a CW-complex and the family $\mathcal{U}$ is a family of subcomplexes, although usually in simple cases. The general form of this 'Van Kampen Theorem' for CX_* does not seem to have been noticed, and this is probably due to the unfamiliar form of colimits in the category Chn of chain complexes over varying groupoids. Even in the group case these colimits are not quite what might be expected (see Example 2.9).

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References

[1] H. J. Baues. *Algebraic Homotopy* (Cambridge University Press, 1988).

[2] R. Brown. Coproducts of crossed P-modules: applications to the homology of groups and to second homotopy groups. *Topology* **23** (1984), 337-345.

[3] R. Brown. Some non-Abelian methods in homotopy theory and homological algebra. In *Proceedings International Conference Categorical Topology, Toledo, 1983* (ed. H.L.Bentley *et al.*), (Heldermann-Verlag, Berlin, 1984), pp. 108-146.

[4] R. Brown and M. Golasinski. A model structure for the homotopy category of crossed complexes. *Cahiers Topologie Géom. Différentielle Catégoriques* **30** (1989) (to appear).

[5] R. Brown and P. J. Higgins. The algebra of cubes. *J. Pure Appl. Algebra* **21** (1981), 233-260.

[6] R. Brown and P. J. Higgins. Colimit theorems for relative homotopy groups. *J. Pure Appl. Algebra* **22** (1981), 11-41.

[7] R. Brown and P. J. Higgins. Crossed complexes and non-Abelian extensions. In *International Conference on Category Theory, Gummersbach* (1981), Lecture Notes in Math. vol. 962 (Springer-Verlag), pp. 39-50.

[8] R. Brown and P. J. Higgins. Homotopies and tensor products for ω-groupoids and crossed complexes. *J. Pure Appl. Algebra* **47** (1987), 1-33.

[9] R. Brown and J. -L. Loday. Van Kampen Theorems for diagrams of spaces. *Topology* **26** (1987), 311-335.

[10] R. H. Crowell. Corresponding group and module sequences. *Nagoya Math. J.* **19** (1961), 27-40.

[11] R. H. Crowell. The derived module of a homomorphism. *Adv. in Math.* **6** (1971), 210-238.

[12] Ch. Ehresmann. Catégories topologiques et catégories differentiables. *Cahiers Topologie Géom. Différentielle Catégoriques* **25** (1983), 237-250.

[13] G. J. Ellis. Homotopy classification the J.H.C. Whitehead way. *Exposition. Math.* **6** (1988), 97-110.

[14] N. D. Gilbert and P. J. Higgins. The non-Abelian tensor product of groups and related constructions. *Glasgow Math. J.* **31** (1989), 17-29.

[15] M. Golasinski. Homotopy groups of small categories and derived functors. *Cahiers Topologie Géom. Différentielle Catégoriques* **28** (1987), 89-98.

[16] A. Grothendieck. Catégories cofibrés additif et complexes cotangent relatif. Lecture Notes in Math. vol. 79 (Springer-Verlag, 1968).

[17] K. W. Gruenberg and K. W. Roggenkamp. Extension categories of groups and modules II. Stem extensions. *J. Algebra* **67** (1980), 342-368.

[18] J. Howie. Pullback functors and crossed complexes. *Cahiers Topologie Géom. Différentielle Catégoriques* **20** (1979), 284-296.

[19] F. J. Korkes and T. Porter. Continuous derivations, profinite crossed complexes and pseudocompact chain complexes. (U.C.N.W. Pure Math. Preprint No. 87.18.)

[20] S. J. Lomanaco, Jr. The homotopy groups of knots I. How to compute the algebraic 2-type. *Pacific J. Math.* **95** (1981), 343-390.

[21] A. S. -T. Lue. Cohomology of groups relative to a variety. *J. Algebra* **69** (1981), 155-174.

[22] K. Mackenzie. *Lie groupoids and Lie Algebroids in Differential Geometry.* London. Math. Soc. Lecture Notes Ser. no. 124 (Cambridge University Press, 1987).

[23] S. MacLane. *Homology* (Springer-Verlag, 1963).

[24] S. MacLane. *Categories for the Working Mathematician.* Graduate Texts in Math. no. 5 (Springer-Verlag, 1971).

[25] J. M. McPherson. On the nullity and enclosure genus of wild knots. *Trans. Amer. Math. Soc.* **144** (1969), 545-555.

[26] P. Olum. Mappings of manifolds and the notion of degree. *Ann. of Math.* **58** (1953), 458-480.

[27] S. P. Plotnick and A. L. Suciu. k-invariants of knotted 2-spheres. *Comment. Math. Helv.* **60** (1985), 54-84.

[28] T. Porter. Some categorical results in the theory of crossed modules in commutative algebras. *J. Algebra* **109** (1987), 415-420.

[29] B. Schellenberg. The group of self-equivalences of some compact CW-complexes. *Math. Ann.* **200** (1973), 253-266.

[30] J. Taylor. Group actions on ω-groupoids and crossed complexes, and the homotopy groups of orbit spaces. Ph.D. thesis, University of Durham (1982).

[31] J. H. C. Whitehead. Combinatorial homotopy II. *Bull. Amer. Math. Soc.* **55** (1949), 453-496.