

‘Double modules’, double categories and groupoids, and a new homotopical double groupoid

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Abstract

We give a rather general construction of double categories and so, under further conditions, double groupoids, from a structure we call a ‘double module’.

We also give a homotopical construction of a double groupoid from a triad consisting of a space, two subspaces, and a set of base points, under a condition which also implies that this double groupoid contains two second relative homotopy groups.¹

Introduction

Double categories were introduced by Ehresmann [Ehr65] as an example of structured category. Examples of double groupoids were shown to arise from crossed modules in [BS76b, BS76a] and this result was applied in [BH78] to give a 2-dimensional version of the Van Kampen Theorem for the fundamental group, namely a colimit theorem for the fundamental crossed module of a based pair. These results were generalised to crossed modules of groupoids, and to higher dimensions, in [BH81b]. A more general version of the algebraic construction of double groupoids was given in [BM92] in terms of a *core diagram*. Another algebraic construction of double groupoids due to the author was taken up by Lu and Weinstein in [LW89] for purposes of Poisson groupoids.

The first aim (Section 1) of this note is to give a generalisation of this last construction. We do not obtain an equivalence of categories, and so in effect the construction shows that double groupoids can be quite complex objects. Perhaps they should be considered among the basic structures in mathematics. A classification of double groupoids is given in [AN].

The second aim (Section 2) is to give a new construction of a double groupoid for a topological space X with three subspaces A, B, C such that $C \subseteq A \cap B$. We write $X_* = (X; A, B; C)$. Here C is thought of as a set of base points, and the double groupoid is well defined if the two induced morphisms $\pi_2(A, c) \rightarrow \pi_2(X, c)$, $\pi_2(B, c) \rightarrow \pi_2(X, c)$ have the same image. Under this condition, the double groupoid ρX_* also contains the two second relative homotopy groups $\pi_2(X, A, c)$, $\pi_2(X, B, c)$ for all $c \in C$, and gives a diagram of four groupoids

$$\begin{array}{ccc}
 (\rho X_*, \circ_1, \circ_2) & \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \rightleftarrows \\ \rightrightarrows \end{array} & (\pi_1(B, C), \circ_2) \\
 \begin{array}{c} \updownarrow \\ \updownarrow \\ \updownarrow \end{array} & & \begin{array}{c} \updownarrow \\ \updownarrow \\ \updownarrow \end{array} \\
 (\pi_1(A, C), \circ_1) & \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \rightleftarrows \\ \rightrightarrows \end{array} & C
 \end{array} \tag{1}$$

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This construction is an extension of the construction of the homotopy double groupoid of a pointed pair in [BH78] which gave the case $A = B$; such a homotopy double groupoid has the advantage of generality, symmetry, and multiple compositions in either directions, advantages not available for the traditional second relative homotopy groups, and which are essential for the 2-dimensional van Kampen theorem of that paper, which allows some previously unobtainable computations of non abelian second relative homotopy groups.

The relation between the two constructions of the first and second sections is unclear, and it is hoped that this paper will encourage further study of the area.

The bibliography gives other uses of double categories and groupoids, [Spe77, BM99, DP93, Ehr63a, BJ04, Mac99], but is not intended to be exhaustive.

1 Double modules and double categories

We are trying to find a mathematical expression for the following diagram and associated equations:

$$\begin{array}{ccccc}
 & a & & b & \\
 u & \begin{array}{|c|c|} \hline m & n \\ \hline \end{array} & & w & \\
 & c & & d & \\
 x & \begin{array}{|c|c|} \hline p & q \\ \hline \end{array} & & z & \\
 & e & & f &
 \end{array} \quad \begin{array}{c} \rightarrow 2 \\ \downarrow 1 \end{array} \quad (2)$$

We interpret each square as a 2-cell: thus m is thought of as

$$av \Leftarrow uc : m$$

This might be formalised as the equation

$$av = ucm.$$

So our four squares give us *boundary equations*

$$\begin{aligned}
 av &= ucm, & bw &= vdn, \\
 cy &= xep, & dz &= yfq.
 \end{aligned}$$

We would like to ‘compose’ such squares. The base point of each square is thought of as the bottom right hand corner, so that is where the centre values m, n, p, q are ‘located’. So in order to ‘compose’ m and n we need to translate m to the same corner as n . Thus we assume an action $(m, d) \mapsto m^d$ satisfying $md = dm^d$, and similarly $my = ym^y$. (The data and axioms for this will be explained below: here we are concerned with formulae!)

Thus we deduce from the above rules that

$$\begin{aligned}
 abw &= avdn & avy &= ucmy \\
 &= ucmdn & &= ucym^y \\
 &= ucdm^d n & &= uxepm^y.
 \end{aligned}$$

We now construct from the above ‘data’ a double category D . The squares of D will be quintuples $(m : u \overset{a}{c} v)$ such that $av = ucm$ (compare [Ehr63b, Spe77, BM99]).

The horizontal and vertical compositions are given by:

$$\begin{aligned}
 (m : u \overset{a}{c} v) \circ_2 (n : v \overset{b}{d} w) &= (m^d n : u \overset{ab}{cd} w), \\
 (m : u \overset{a}{c} v) \circ_1 (p : x \overset{c}{e} y) &= (pm^y : ux \overset{a}{e} vy).
 \end{aligned}$$

In virtue of the above calculations, given the data of diagram (2), these give compositions with the correct boundary equations.

We also would like the interchange law, namely that the two possible ways of composing the four squares in diagram (2) give the same answer. A direct calculation shows that this is equivalent to the rule

$$m^{yf}q = qm^{dz}, \quad (3)$$

again given the boundary equation $dz = yfq$, and this makes geometric sense in terms of the choices of ‘transporting’ m to the bottom right hand corner of the square involving q .

Now we need to give a structure in which the above calculations make sense.

Our notation for categories will be that we write $s, t : C \rightarrow \text{Ob } C$ for the source and target maps of the category C so that an arrow c of C is an arrow $c : sc \rightarrow tc$, and composition cd is defined if and only if $tc = sd$.

Definition 1.1 A *double P -module* consists of three morphisms of categories all over the identity on objects:

$$\begin{array}{ccc} M & & H \\ & \searrow \mu & \downarrow \phi \\ V & \xrightarrow{\psi} & P \end{array}$$

such that M is totally intransitive, i.e. is a union of monoids, so that $s = t$ on M . We write $M(x)$ for $M(x, x)$. Further, there are given right actions of both H and V on M . This means that if $m \in M(x)$ and $d \in H(x, y)$ then there is defined $m^d \in M(y)$ and the usual axioms for an action are satisfied, namely $(mm_1)^h = m^h m_1^h$, $m^{hk} = (m^h)^k$, whenever these make sense, and $1^h = 1$, $m^1 = m$, and similarly for the action of V on M .

We do not suppose these actions commute, but nonetheless we agree to write m^{dz} when $tm = sd$ and $td = sz$. However dz is here interpreted formally, or, if you like, as an arrow of the free product category $H * V$ over the same set of objects as H, V, M . Thus $tm^{dz} = tz$.

In order to write our axioms in a way which agrees with the above calculations, we agree ‘evaluate in P ’ means apply the morphisms μ, ϕ, ψ to the given equation to give an equation in P . Thus the equation ‘ $ucm = av$ evaluated in P ’ means not only that

$$(\psi u)(\phi c)(\mu m) = (\psi a)(\phi v)$$

but also that the equation makes sense in that

$$sa = su, ta = sv, tu = sc, tc = tvtm.$$

Similarly,

$$md = dm^d \text{ evaluated in } P$$

means that $tm = sd$ and

$$(\mu m)(\phi d) = (\phi d)(\mu m^d).$$

With this agreed, the axioms are: if $m, q \in M, d, f \in H, y, z \in V$

(i) then

$$md = dm^d, my = ym^y, \text{ evaluated in } P ;$$

(ii) if also $yfq = dz$ in P , then in M ,

$$m^{yf}q = qm^{dz}.$$

□

Now we have our main result, which follows from what was written above:

Theorem 1.2 Given a double P -module as above, then the compositions \circ_1, \circ_2 give the structure of double category, which is a double groupoid if all of M, H, V, P are groupoids.

Some special cases are of interest.

Example 1.3 If M consists only of identities, then D is the double groupoid of squares from H and V which “commute in P ”. This is used in [LW89]. \square

Example 1.4 Suppose that $V = P$ and ψ is the identity. Then we obtain a diagram

$$M \xrightarrow{\mu} P \xleftarrow{\phi} H$$

together with actions of P and H on M . The axioms now imply that μ is a crossed module. \square

Example 1.5 Let H and V be subgroups of the group P and let M be a subgroup of P which is normal in both H and V . Then the inclusions, and the conjugation actions of H and V on M give the structure of a double P -module, from which we can obtain a double groupoid. \square

Example 1.6 A semicore diagram is defined in [BM92] to consist of a commutative diagram of morphisms of groupoids

$$\begin{array}{ccc} M & \xrightarrow{\eta} & H \\ & \searrow \mu & \downarrow \phi \\ & & P \end{array}$$

and an action of P on M such that μ is a crossed module, η is an inclusion of a totally intransitive subgroupoid, and if $m \in M, h \in H$ and $h^{-1}mh$ is defined, then $h^{-1}mh = m^{\phi h}$. It follows that M is normal in H . Conversely, given such a morphism of crossed modules, we obtain a double P -module as in 1.1 with $V = P$. Notice also that if $N = \text{Ker } \phi$, then N operates trivially on M . \square

Remark 1.7 This double category also has a *thin structure* in the sense that the set of quintuples of the form $(1 : u \stackrel{a}{\circlearrowleft} v)$ form a subdouble category of the main double category. \square

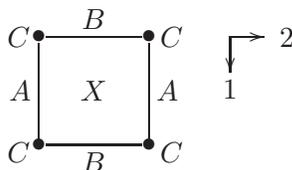
Remark 1.8 It is shown in [BS76a] that from a double groupoid one can recover two crossed modules, a kind of horizontal one and a vertical one. However the groupoid conditions are needed to recover the actions, and we know no way to do this in the category case. From a double category one can recover two 2-categories, by restricting to the subdouble categories where either the horizontal, or vertical, edge categories are discrete. \square

2 A new construction of a homotopical double groupoid

We write I for the unit interval $[0, 1]$, and use the following notation for subspaces of $I^2 = I \times I$:

$$\begin{aligned} \partial_1^- I^2 &= \{0\} \times I, & \partial_1^+ I^2 &= \{1\} \times I, \\ \partial_2^- I^2 &= I \times \{0\}, & \partial_2^+ I^2 &= I \times \{1\}, \\ J_{-,1}^2 &= (\partial_1^+ \cup \partial_2^\pm) I^2, & \partial \partial I^2 &= \{0, 1\} \times \{0, 1\}. \end{aligned}$$

Let $X_* = (X; A, B; C)$ be a space X with three subspaces A, B, C such that $C \subseteq A \cap B$. Let RX_* be the space of maps $\alpha : I^2 \rightarrow X$ which map the edges $\partial_1^\pm I^2$ in direction 1 into B , the edges $\partial_2^\pm I^2$ in direction 2 into A , and map all the vertices $\partial \partial I^2$ into C . This is shown in the following diagram:



The boundary maps and degeneracies give the following geometric structure to RX_* , in which $R(A, C)$ is the set of maps $(I, \{0, 1\}) \rightarrow (A, C)$ and $R(B, C)$ is the set of maps $(I, \{0, 1\}) \rightarrow (B, C)$:

$$\begin{array}{ccc} (RX_*, \circ_1, \circ_2) & \begin{array}{c} \rightrightarrows \\ \leftleftarrows \end{array} & (R(B, C), \circ_2) \\ \begin{array}{c} \updownarrow \\ \updownarrow \end{array} & & \begin{array}{c} \updownarrow \\ \updownarrow \end{array} \\ (R(A, C), \circ_1) & \begin{array}{c} \rightrightarrows \\ \leftleftarrows \end{array} & C \end{array} \quad (4)$$

Clearly the set RX_* obtains two compositions \circ_1, \circ_2 from the usual composition of squares in the two directions, while $R(A, C), R(B, C)$ have each just one composition. These compositions are of course not associative, nor do they have identities, but they do have reverses, $-_1, -_2$. They do however satisfy the interchange law. Further the face and degeneracies respect the compositions.

The boundary and degeneracy maps of diagram (4) are inherited by $\rho X_* = (RX_*)/\equiv$ where \equiv is the relation of homotopy rel vertices of I^2 and through the elements of RX_* , to give the diagram:

$$\begin{array}{ccc} (\rho X_*, \circ_1, \circ_2) & \begin{array}{c} \rightrightarrows \\ \leftleftarrows \end{array} & (\pi_1(B, C), \circ_2) \\ \begin{array}{c} \updownarrow \\ \updownarrow \end{array} & & \begin{array}{c} \updownarrow \\ \updownarrow \end{array} \\ (\pi_1(A, C), \circ_1) & \begin{array}{c} \rightrightarrows \\ \leftleftarrows \end{array} & C \end{array} \quad (5)$$

The following theorem generalises Propositions 3 and 7 from [BH78].²

Theorem 2.1 *Suppose given the following condition:*

(Con) *For all $c \in C$, the induced morphisms*

$$\pi_2(A, c) \rightarrow \pi_2(X, c), \pi_2(B, c) \rightarrow \pi_2(X, c)$$

have the same image.

Then the compositions \circ_1, \circ_2 are inherited by ρX_ to give it the structure of double groupoid. Further, under this condition, the natural morphisms of second relative homotopy groups*

$$\pi_2(X, A; C) \rightarrow (\rho X_*, \circ_1), \quad \pi_2(X, B; C) \rightarrow (\rho X_*, \circ_2)$$

are injective.

Proof The proof is a small elaboration of a similar proof for the case $A = B$ in [BH78]. Details of this special case are also given in [Bro99].

The homotopy class rel vertices of an element α of RX_* or of $R(A, C), R(B, C)$ is written $\langle\langle \alpha \rangle\rangle$.

We develop only the horizontal case; the other follows by symmetry. So, let us consider two elements $\langle\langle \alpha \rangle\rangle, \langle\langle \beta \rangle\rangle \in \rho X_*$ such that $\langle\langle \partial_2^+ \alpha \rangle\rangle = \langle\langle \partial_2^- \beta \rangle\rangle$, i.e. we have continuous maps

$$\alpha, \beta : (I^2; \partial_2^\pm I^2, \partial_1^\pm I^2; \partial^2 I^2) \rightarrow (X; A, B; C)$$

and a homotopy

$$h : (I, \partial(I)) \times I \rightarrow (A, C)$$

from $\alpha|_{\{1\} \times I}$ to $\beta|_{\{0\} \times I}$ rel vertices, i.e. $h(0 \times I) = y$ and $h(1 \times I) = x$. We define now the composition by

$$\langle\langle \alpha \rangle\rangle +_2 \langle\langle \beta \rangle\rangle = \langle\langle \alpha +_2 h +_2 \beta \rangle\rangle = \langle\langle [\alpha, h, \beta] \rangle\rangle.$$

This is given in a diagram by

$$\begin{array}{ccccccc} & & B & & c & & B & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ A & & \alpha & A & h & A & \beta & A & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & B & & d & & B & & \end{array}$$

²In that paper we assumed a further condition on loops in C , because we were not taking homotopies relative to the vertices of I^2 .

To prove this is independent of the choices made we chose two other representatives $\alpha' \in \langle\langle\alpha\rangle\rangle$ and $\beta' \in \langle\langle\beta\rangle\rangle$ and a homotopy h' from $\alpha'|_{\{1\} \times I}$ to $\beta'|_{\{0\} \times I}$. Using them, we get

$$\begin{array}{ccccc}
 & B & c & B & \\
 A & \alpha' & A & h' & A & \beta' & A \\
 & B & d & B & & &
 \end{array}$$

which should give the same composition in ρX_* . Let $\phi : \alpha \simeq \alpha', \psi : \beta \simeq \beta'$ be homotopies of the required type. They with h, h' give rise to a diagram of the following kind.

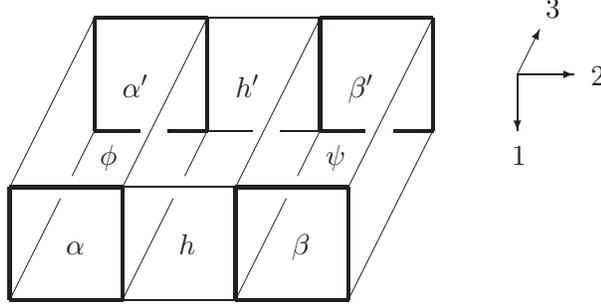


Figure 1: Filling the hole in the middle

We seem to have a hole in the middle. The key point is that all homotopies are rel vertices. So the bottom face of this hole may be filled by a constant homotopy. Then we can use a retraction to fill the hole, and this will give a cube in A , whose top face is a map $(I^2, \partial\partial I^2) \rightarrow (A, c)$. By our assumption (Con), this is deformable rel boundary and in X to a map $(I^2, \partial\partial I^2) \rightarrow (B, c)$. This homotopy is now added in direction 1 to the homotopy of the middle hole, and squashed down to give another filler of the hole; the composition in direction 2 of these three cubes is now a homotopy rel vertices through maps $(I^2; \partial_2^\pm I^2, \partial_1^\pm I^2; \partial^2 I^2) \rightarrow (X; A, B; C)$ as required.

The verification of the groupoid axioms is entirely analogous to the case of the fundamental groupoid. We now verify the interchange law.

Suppose given an array of composable elements of ρX_* :

$$\begin{pmatrix} \langle\langle\alpha\rangle\rangle & \langle\langle\beta\rangle\rangle \\ \langle\langle\gamma\rangle\rangle & \langle\langle\delta\rangle\rangle \end{pmatrix}$$

This gives rise to a partially filled array

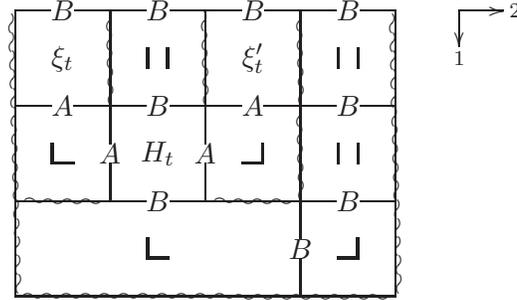
$$\begin{pmatrix} \alpha & h & \beta \\ k & & k' \\ \gamma & h' & \delta \end{pmatrix}$$

However because of the rel vertices hypothesis on the homotopies h, h', k, k' the hole in the middle can be filled with a constant map. Reading the resulting matrix in two ways gives the required interchange law.

For the last part, we identify the second relative homotopy group $\pi_2(X, B, c)$ with the homotopy classes of maps $(I^2, \partial_1^- I^2, J_{-1}^2) \rightarrow (X, B, c)$, where J_{-1}^2 is the union of all faces of I^2 except ∂_1^- , with composition given by addition in direction 2. So we have a morphism $\eta : \pi_2(X, B, c) \rightarrow \rho_2(X; A, B; C)$ which we have to prove injective. Suppose then $\eta(\alpha)$ is nulhomotopic in $R_2(X; A, B; C)$ and let H be such a nullhomotopy. We construct another nulhomotopy H' of α which is a nullhomotopy of maps $(I^2, \partial_1^- I^2, J_{-1}^2) \rightarrow (X, B, c)$.

To this end we use the *connections* Γ_i^\pm which are available in the cubical singular set $S^\square(X)$ of a space, as in [BH81a, AABS02, GM03], for example. The main point is that a connection $\Gamma : S^\square(X)_n \rightarrow$

$S^\square(X)_{n+1}$, defined using the functions \max, \min , gives a kind of degeneracy in which two adjacent faces of $\Gamma(f)$ coincide. It is convenient to represent these symbolically as $\llcorner, \lrcorner, \ulcorner, \urcorner$. The traditional cubical degeneracies are analogously represented by \lll, \llr, \llb, \llt . In our current situation we surround the homotopy H by connections and constant homotopies, and also using the hypothesis (Conn) to obtain another homotopy from this time α surrounded by constant maps or \lll . In particular, the hypothesis (Conn) is used twice to obtain homotopies ξ, ξ' as part of the following picture of the new homotopy. To show this the following picture gives the picture at $t \in [0, 1]$, but the connections are actually applied on 2-dimensional faces in directions 1 and 2 of H . The wiggly lines denote constant homotopies. This also illustrates that one of the aims of bringing in connections and 2-dimensional rewriting was to give a more algebraic method of constructing homotopies than previously available. For another application of such rewriting, to rotations, see [Bro82].



□

Remark 2.2 Even in the case C is a singleton, the condition (Conn) is a non trivial condition needed to make $\rho X_* = \pi_0(RX_*)$ a double groupoid. Of course it is satisfied if $A = B$, giving the double groupoid used in [BH78] to prove a 2-dimensional van Kampen theorem. However it is proved in [Lod82], see also [Gil87], that even without this condition the compositions \circ_1, \circ_2 are inherited by the group $\pi_1(RX_*, \hat{c})$, where \hat{c} denotes the constant map at c , to give this group the structure of cat^2 -group, i.e. a double groupoid internal to the category of groups. This fact is the foundation for the work of [Lod82, BL87]. □

References

- [AABS02] Al-Agl, F. A., Brown, R. and Steiner, R. ‘Multiple categories: the equivalence of a globular and a cubical approach’. *Adv. Math.* **170** (1) (2002) 71–118. [6](#)
- [AN] Andruskiewitsch, N. and Natale, S. ‘The structure of double groupoids’. *J. Pure Appl. Algebra* (arXiv:math/0602497) **28**. [1](#)
- [Bro82] Brown, R. ‘Higher dimensional group theory’. In ‘Low-dimensional topology (Bangor, 1979)’, *London Math. Soc. Lecture Note Ser.*, Volume 48. Cambridge Univ. Press, Cambridge (1982), 215–238. [7](#)
- [Bro99] Brown, R. ‘Groupoids and crossed objects in algebraic topology’. *Homology Homotopy Appl.* **1** (1999) 1–78 (electronic). [5](#)
- [BH78] Brown, R. and Higgins, P. J. ‘On the connection between the second relative homotopy groups of some related spaces’. *Proc. London Math. Soc. (3)* **36** (2) (1978) 193–212. [1](#), [2](#), [5](#), [7](#)
- [BH81a] Brown, R. and Higgins, P. J. ‘On the algebra of cubes’. *J. Pure Appl. Algebra* **21** (3) (1981) 233–260. [6](#)
- [BH81b] Brown, R. and Higgins, P. J. ‘Colimit theorems for relative homotopy groups’. *J. Pure Appl. Algebra* **22** (1) (1981) 11–41. [1](#)

- [BJ04] Brown, R. and Janelidze, G. ‘Galois theory and a new homotopy double groupoid of a map of spaces’. *App. Cat. Struct.* **12** (2004) 63–80. [2](#)
- [BL87] Brown, R. and Loday, J.-L. ‘Van Kampen theorems for diagrams of spaces’. *Topology* **26** (3) (1987) 311–335. With an appendix by M. Zisman. [7](#)
- [BM92] Brown, R. and Mackenzie, K. C. H. ‘Determination of a double Lie groupoid by its core diagram’. *J. Pure Appl. Algebra* **80** (3) (1992) 237–272. [1](#), [4](#)
- [BM99] Brown, R. and Mosa, G. H. ‘Double categories, 2-categories, thin structures and connections’. *Theory Appl. Categ.* **5** (1999) No. 7, 163–175 (electronic). [2](#)
- [BS76a] Brown, R. and Spencer, C. B. ‘Double groupoids and crossed modules’. *Cahiers Topologie Géom. Différentielle* **17** (4) (1976) 343–362. [1](#), [4](#)
- [BS76b] Brown, R. and Spencer, C. B. ‘ G -groupoids, crossed modules and the fundamental groupoid of a topological group’. *Nederl. Akad. Wetensch. Proc. Ser. A* **79**=*Indag. Math.* **38** (4) (1976) 296–302. [1](#)
- [DP93] Dawson, R. and Paré, R. ‘General associativity and general composition for double categories’. *Cahiers Topologie Géom. Différentielle Catég.* **34** (1) (1993) 57–79. [2](#)
- [Ehr63a] Ehresmann, C. ‘Catégories doubles et catégories structurées’. *C. R. Acad. Sci. Paris* **256** (1963) 1198–1201. [2](#)
- [Ehr63b] Ehresmann, C. ‘Catégorie double des quintettes; applications covariantes’. *C. R. Acad. Sci. Paris* **256** (1963) 1891–1894. [2](#)
- [Ehr65] Ehresmann, C. *Catégories et structures*. Dunod, Paris (1965). [1](#)
- [Gil87] Gilbert, N. D. ‘On the fundamental cat^n -group of an n -cube of spaces’. In ‘Algebraic topology, Barcelona, 1986’, *Lecture Notes in Math.*, Volume 1298. Springer, Berlin (1987), 124–139. [7](#)
- [GM03] Grandis, M. and Mauri, L. ‘Cubical sets and their site’. *Theory Applic. Categories* **11** (2003) 185–201. [6](#)
- [Lod82] Loday, J.-L. ‘Spaces with finitely many nontrivial homotopy groups’. *J. Pure Appl. Algebra* **24** (2) (1982) 179–202. [7](#)
- [LW89] Lu, J.-H. and Weinstein, A. ‘Groupoïdes symplectiques doubles des groupes de Lie-Poisson’. *C. R. Acad. Sci. Paris Sér. I Math.* **309** (18) (1989) 951–954. [1](#), [4](#)
- [Mac99] Mackenzie, K. C. H. ‘On symplectic double groupoids and the duality of Poisson groupoids’. *Internat. J. Math.* **10** (4) (1999) 435–456. [2](#)
- [Spe77] Spencer, C. B. ‘An abstract setting for homotopy pushouts and pullbacks’. *Cahiers Topologie Géom. Différentielle* **18** (4) (1977) 409–429. [2](#)