Galois theory and a new homotopy double groupoid of a map of spaces

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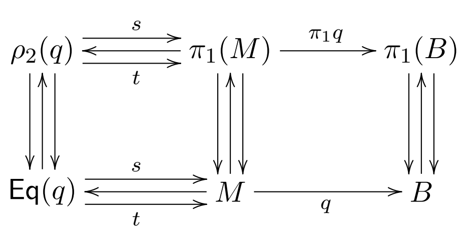
Abstract

The authors have used generalised Galois Theory to construct a homotopy double groupoid of a surjective fibration of Kan simplicial sets. Here we apply this to construct a new homotopy double groupoid of a map of spaces, which includes constructions by others of a 2-groupoid, cat^1-group or crossed module. An advantage of our construction is that the double groupoid can give an algebraic model of a foliated bundle.[1]

Introduction

Our aim is to develop for any map $q : M \to B$ of topological spaces the construction and properties of a new homotopy double groupoid which has the form of the left hand square in the following diagram, while the right hand square gives a morphism of groupoids:



(1)

where:
$\pi_1(M)$ is the fundamental groupoid of M;
$\mathsf{Eq}(q)$ is the equivalence relation determined by q; and
s, t are the source and target maps of the groupoids.
Note that $qs = qt$ and $(\pi_1 q)s = (\pi_1 q)t$, so that $\rho_2(q)$ is seen as a double groupoid analogue of $\mathsf{Eq}(q)$.

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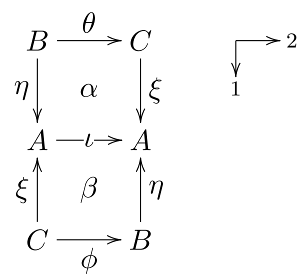
This double groupoid contains the 2-groupoid associated to a map defined by Kamps and Porter in [16], and hence also includes the cat^1-group of a fibration defined by Loday in [17], the 2-groupoid of a pair defined by Moerdijk and Svensson in [19], and the classical fundamental crossed module of a pair of pointed spaces defined by J.H.C. Whitehead. Advantages of our construction are:
(i) it contains information on the map q, and
(ii) we get different results if the topology of M is varied to a finer topology.
In particular, we can apply this construction in the case M is foliated by replacing the topology on M by a finer one so that $\pi_1 M$ is replaced by the fundamental groupoid of the foliation.

This applies in particular to the Möbius Band with its standard foliation by circles. We can extract from this double groupoid a small version $\mathcal{D}(M)$ with only three vertices, and which seems to represent well many properties of the Möbius Band. It has basic vertices, edges and squares as follows:



Note that the vertical groupoid for $\circ_1$ is an indiscrete groupoid, while the horizontal groupoid for $\circ_2$ contains a copy of the infinite cyclic group, since there are compositions

$$a_1 \circ_2 a_2 \circ_2 \cdots \circ_2 a_n$$

where the a_i are alternately $(-_1\alpha)$ and β.

The idea for this double groupoid arose from the Generalised Galois Theory of Janelidze [14, 15], which under certain conditions gives a Galois groupoid from a pair of adjoint functors. The standard fundamental group arises from the adjoint pair between topological spaces and sets given by discrete and π_0, see for example [8]. The adjoint pair between simplicial sets and crossed complexes given by nerve and π_1 was studied in [9] and shown to lead to a Galois double groupoid of a fibration of simplicial sets. We are now giving a topological version of this construction. We show that if $p : E \to B$ is a Serre fibration then the fundamental groupoid $\pi_1(E)$ has an additional compatible groupoid structure arising from the equivalence relation $\mathsf{Eq}(p)$ defined by the map p; these two groupoid structures define a double groupoid which we write $\gamma(p)$, since it is defined by methods of Galois theory. The double groupoid $\rho(q)$ arises by pullback by i where $q = pi$ is the usual factorisation of any map through a homotopy equivalence i and a fibration p. However the proof of the relation of $\rho(q)$ with classical notions and compositions is tricky, and so is given in some detail. A further reason for this detail is the possibility that a modification of this construction could be used in association with the 'thin fundamental groupoids' and their smooth structures in differential geometrical situations, as exemplified by Mackaay and Picken in [18].

Here is some background to the search for higher groupoid models of homotopical structures (for more detailed references, see [3]). Geometers in the early part of the 20th century were aware that in the connected case the first homology group was the fundamental group made abelian, and that homology groups existed in all positive dimensions. Further, the fundamental group gave more information in geometric and analytic contexts than did the first homology group. They were therefore interested in seeking higher dimensional versions of the non abelian fundamental group. E. Čech submitted to the 1932 ICM at Zurich a paper on higher homotopy groups, using maps of spheres. However these groups were quickly proved to be abelian in dimensions > 1, and on this ground Čech was asked to withdraw his paper, so that only a small paragraph appeared [12]. Thus the dream of these topologists seemed to fail, and was widely felt to be a mirage, although the abelian higher homotopy groups became and still are very important.

J.H.C. Whitehead in the 1940s introduced the notion of crossed module, using the boundary of the second relative homotopy group of a pair and the action of the fundamental group. He and Mac Lane showed that crossed modules classified (connected) homotopy 2-types. Crossed modules are indeed more complicated than groups, and they make a good candidate for '2-dimensional groups'.

In the 1960s, Brown introduced the fundamental groupoid of a space on a set of base points, and the writing of his 1968 book on topology suggested to him that all of 1-dimensional homotopy theory was better expressed in terms of groupoids rather than groups. This raised the question of the putative value of groupoids in higher homotopy theory. A relation of certain double groupoids to crossed modules was worked out with C.B. Spencer in the early 1970s, and this showed that double groupoids are indeed more complicated than groups. A definition of a homotopy double groupoid of a pair of pointed spaces was made with P.J. Higgins in 1974, and exploited to obtain a 2-dimensional Van Kampen type theorem for this double groupoid, and hence for Whitehead's crossed module of a pair (see [4]). The double groupoid constructed in [4] is edge symmetric and has a connection, and so is not the same as that constructed here.

A classification of certain double groupoids is given in [11], but this does not yield much information for the double groupoid considered here. Thus there is still a way to go in the understanding and in the use of double groupoids.

Higher homotopy groupoids were defined by Brown and Higgins for a filtered space in [6], and by Loday for an n-cube of spaces in [17]; his cat^n-groups were shown there to model connected homotopy $(n+1)$-types. These higher groupoid methods yield new calculations in homotopy theory through higher order Van Kampen theorems [6, 10], as well as suggesting new algebraic constructions.

1 Galois groupoids

Later we will be considering the category $\mathsf{C} = \mathsf{Sets}^{\Delta^{op}}$ of simplicial sets and the fundamental groupoid functor $I = \pi_1 : \mathsf{C} \to \mathsf{X}$ from the category C to the category $\mathsf{X} = \mathsf{Grpd}$ of (small) groupoids. Further, C, an internal category in C, will be the particular simplicial category (actually groupoid) $\mathsf{Eq}(p)$ which is the equivalence relation (in C) determined by p where $p : E \to B$ is a surjective fibration of Kan complexes. Here we give first the general result, using this notation.

Let $I : \mathsf{C} \to \mathsf{X}$ be an arbitrary functor between categories C and X with pullbacks, and let

$$C = \left(C_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} C_1 \substack{\longrightarrow\\\longleftarrow\\\longrightarrow} C_0 \right) \tag{2}$$

be an internal category in C. We recall

Proposition 1.1 *Suppose the canonical morphisms*

$$I(C_1 \times_{C_0} C_1) \to I(C_1) \times_{I(C_0)} I(C_1) \tag{3}$$

$$I(C_1 \times_{C_0} C_1 \times_{C_0} C_1) \to I(C_1) \times_{I(C_0)} I(C_1) \times_{I(C_0)} I(C_1) \tag{4}$$

are isomorphisms. Then:

(a)

$$I(C) = \Big(I(C_2) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} I(C_1) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} I(C_0) \Big) \tag{5}$$

is an internal category in X*;*

(b) if C *is a groupoid, then so is* $I(C)$*.*

For a morphism $p : E \to B$ in C, let $\mathsf{Eq}(p) =$

$$(E \times_B E) \times_E (E \times_B E) \approx E \times_B E \times_B E \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} E \times_B E \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} E \tag{6}$$

be the equivalence relation corresponding to p (=kernel pair of p) regarded as an internal groupoid in C. Applying Proposition 1.1 with $C = \mathsf{Eq}(p)$, we obtain:

Corollary 1.2 *Suppose the canonical morphisms*

$$I((E \times_B E) \times_E (E \times_B E)) \to I(E \times_B E) \times_{I(E)} I(E \times_B E) \tag{7}$$

$$I(E \times_B E) \times_E (E \times_B E) \times_E (E \times_B E)) \to I(E \times_B E) \times_{I(E)} I(E \times_B E) \times_{I(E)} I(E \times_B E) \tag{8}$$

are isomorphisms. Then $I(\mathsf{Eq}(p)) =$

$$I((E \times_B E) \times_E (E \times_B E)) \approx I(E \times_B E \times_B E) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} I(E \times_B E) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} I(E) \tag{9}$$

is an internal groupoid in X*.*

This fact, which goes back to A. Grothendieck's observation "the fundamental groupoids are to be defined as quotients of equivalence relations", is used in categorical Galois theory and its various special cases (see [1], [4], and other references in [1]), to define the *Galois groupoid* of (E, p) as

$$Gal_I(E, p) = I(\mathsf{Eq}(p)). \tag{10}$$

In particular this applies to the following situation studied by the authors before (see Proposition 3.5 in [2]):

Proposition 1.3 *Let* $I : \mathsf{C} \to \mathsf{X}$ *be the fundamental groupoid functor from the category* $\mathsf{C} = \mathsf{Sets}^{\Delta^{op}}$ *of simplicial sets to the category* $\mathsf{X} = \mathsf{Grpd}$ *of (small) groupoids, and* $p : E \to B$ *a surjective fibration of Kan complexes. Then the morphisms (6) and (7) are isomorphisms and so the Galois groupoid (9) is well defined. Since the internal groupoids in* Grpd *are the same as double groupoids, it is a double groupoid.*

2 From simplicial sets to topological spaces

Consider the diagram

$$\begin{array}{ccccc} \mathsf{Top} & \overset{R}{\underset{S}{\leftrightarrows}} & \mathsf{Sets}^{\Delta^{\mathrm{op}}} & \overset{I}{\underset{H}{\rightleftarrows}} & \mathsf{Grpd} \\ & {}_{r}\nwarrow & \uparrow{\scriptstyle y} & \nearrow_{i} & \\ & & \Delta & & \end{array} \tag{11}$$

where

- Top is the category of topological spaces, R is the geometric realisation functor, and S is its right adjoint, usually called the singular complex functor;
- $I \vdash H$ is the adjoint pair used in [2], i.e. I is the fundamental groupoid functor, and H the nerve functor;
- y is the Yoneda embedding, r and i are the restrictions of R and I respectively along y; explicitly, r is the singular simplex functor and i carries finite ordinals to codiscrete groupoids on the same sets of objects.

By the universal property of the Yoneda embedding, the two adjunctions of the row are uniquely (up to isomorphisms) determined by r and i; let us also recall from [9] and [14, 15]:

Proposition 2.1 *(a) The composite* $IS : \mathsf{Top} \to \mathsf{Grpd}$ *can be identified with the classical (geometric) fundamental groupoid functor* π_1*;*

(b) For every topological space X*,* $S(X)$ *is a Kan complex;*

(c) The S*-image of a morphism* p *in* Top *is a Kan fibration if and only if* p *itself is a Serre fibration.*

3 What is the Galois double groupoid of a Serre fibration?

Let $p : E \to B$ be a Serre fibration of topological spaces. By Propositions 1.3 and 2.1, the Galois (double) groupoid $Gal_I(S(E), S(p))$ is well defined. Moreover, since the functor S being a right adjoint preserves pullbacks, we can write

$$Gal_I(S(E), S(p)) \approx Gal_{SI}(E, p) \approx Gal_{\pi_1}(E, p) \tag{12}$$

and conclude that $Gal_{\pi_1}(E, p)$ also is a well-defined double groupoid. We write this double groupoid as $\gamma(p)$ to indicate the relation with Galois theory, and now describe it explicitly.

The underlying double graph has the following description:

- presented as an internal groupoid in Grpd, $\gamma(p)$ is displayed as

$$\pi_1((E \times_B E) \times_E (E \times_B E)) \approx \pi_1(E \times_B E \times_B E) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \pi_1(E \times_B E) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \pi_1(E) \tag{13}$$

from which we conclude:

- the set of objects in $\gamma(p) = Gal_{\pi_1}(E, p)$ is E;
- a horizontal arrow $e \to e'$ is a morphism $e \to e'$ in $\pi_1(E)$, i.e. a homotopy class of a path from e to e' (recall that such a homotopy $h : [0,1] \times [0,1] \to E$ is required to have $h(0,t) = e$ and $h(1,t) = e'$ for every t in $[0,1]$);
- a vertical arrow $e \to e'$ is just the pair (e, e') provided $p(e) = p(e')$;
- a square

$$\begin{array}{ccc} e_1 & \xrightarrow{\phi} & e_2 \\ \downarrow & u & \downarrow \\ e'_1 & \xrightarrow[\phi']{} & e'_2 \end{array} \tag{14}$$

  is a homotopy class of a path from (e_1, e'_1) to (e_2, e'_2) in $E \times_B E$; its vertical domain ϕ and vertical codomain ϕ' are homotopy classes of paths from e_1 to e_2 and from e'_1 to e'_2 respectively, and the horizontal are pairs (e_1, e'_1) and (e_2, e'_2) respectively.

Clearly there is no problem with the horizontal composition as well - since we know that $\pi_1(E \times_B E)$ and $\pi_1(E)$ are groupoids. The only non-trivial part of our construction is the vertical composition of squares:

Given

$$\begin{array}{ccc} e_1 & \xrightarrow{\phi} & e_2 \\ \downarrow & u & \downarrow \\ e'_1 & \xrightarrow{\phi'} & e'_2 \\ \downarrow & u' & \downarrow \\ e''_1 & \xrightarrow[\phi']{} & e''_2 \end{array} \qquad \begin{array}{cc} \cdot & \rightarrow 2 \\ \downarrow & \\ 1 & \end{array} \tag{15}$$

we have to define $u \circ_1 u'$, which must be (an equivalence class of) a path from (e_1, e''_1) to (e_2, e''_2) in $E \times_B E$. Of course we should choose a representative f of u, a representative f' of u', a homotopy h between the vertical codomain path of f and the vertical domain path of f', paste them together, and take the homotopy class of the resulting path in $E \times_B E$. However, how do we know that such an $u \circ_1 u'$ does not depend on the choices we made? The nice consequence of the results above is that we do not need to prove this. Indeed, since the morphism $\pi_1((E \times_B E) \times_E (E \times_B E)) \to \pi_1(E \times_B E) \times_{\pi_1(E)} \pi_1(E \times_B E)$ is an isomorphism, the pair (u', u) determines a path v in $(E \times_B E) \times_E (E \times_B E)$, and the desired vertical composite $u'u$ is nothing but the image of v under the functor $\pi_1((E \times_B E) \times_E (E \times_B E)) \to \pi_1(E \times_B E)$ induced by the composition map $(E \times_B E) \times_E (E \times_B E) \to E \times_B E$ of $\mathsf{Eq}(p)$. Moreover, we do not have to worry about associativity of the horizontal composition, which in fact follows from

$$\pi_1((E \times_B E) \times_E (E \times_B E) \times_E (E \times_B E)) \to \pi_1(E \times_B E) \times_{\pi_1(E)} \pi_1(E \times_B E) \times_{\pi_1(E)} \pi_1(E \times_B E)$$

being an isomorphism.

Example 3.1 Suppose for example that $p : E = (B \times F) \to B$ is the projection of a product, and so is a fibration. Then $\mathsf{Eq}(p) = E \times_B E$ is homeomorphic to the product $B \times (F \times F)$ and hence $\pi_1(\mathsf{Eq}(p))$ is easily determined, together with its double groupoid structure. The vertical edges are triples $(b, f^-, f^+), b \in B, f^\pm \in F$, the horizontal edges are pairs $(\beta, \phi) \in \pi_1 B \times \pi_1 F$ and the squares are triples

$$(\beta, \phi^-, \phi^+) \in \pi_1(B) \times \pi_1(F) \times \pi_1(F)$$

with vertical boundaries given by $\partial_1^\pm(\beta, \phi^-, \phi^+) = (\beta, \phi^\pm)$. The horizontal composition $\circ_2$ is that of the fundamental groupoids; the vertical composition $\circ_1$ reflects that of the product groupoid $B \times (F \times F)$ in which B is the discrete groupoid and $F \times F$ is the codiscrete groupoid.

In the next sections we translate the construction for $\gamma(p)$ where p is a fibration into results for an arbitrary map $q : M \to B$ by the usual factorisation process, and so relate these ideas to classical constructions.

4 The homotopy double groupoid of an arbitrary continuous map

If C is an internal category (or a groupoid) in a category X with pullbacks, and $i : M \to C_0$ a morphism into the object-of-object of C, we can always form the "induced internal category (respectively groupoid)" $i^*(C) =$

$$C_2 \times_{C_0 \times C_0 \times C_0} (M \times M \times M) \rightrightarrows C_1 \times_{C_0 \times C_0} (M \times M) \rightleftarrows M \tag{16}$$

which, in the case $\mathsf{X} = \mathsf{Sets}$, can be simply described as the category with objects all elements of M, and morphisms "as in C". In particular, for a topological space E and a subspace M one defines the relative fundamental groupoid $\pi_1(E, M)$ as $i^*(\pi_1(E))$, where $i : M \to E$ is the inclusion map; in this paper we do not consider $\pi_1(E)$ as a topological groupoid, but the same construction of $\pi_1(E, M)$ can be repeated internally to Top.

Let $p : E \to B$ be a Serre fibration and $i : M \to E$ an arbitrary map in Top. Using the morphism $\pi_1(i) : \pi_1(M) \to \pi_1(E)$ of fundamental groupoids, and having in mind that $\pi_1(E)$ is the object-of-objects of $Gal_{\pi_1}(E, p)$ (when it is regarded as an internal groupoid in Grpd), we can construct what we are going to call the relative Galois double groupoid $Gal_{\pi_1}(E, p, M, i)$ as

$$Gal_{\pi_1}(E, p, M, i) = \pi_1(i)^* Gal_{\pi_1}(E, p). \tag{17}$$

A good reason for introducing this new double groupoid is that any continuous map $q : M \to B$ can be presented as a composite $q = pi$ above with i being a homotopy equivalence, and so any q determines such a double groupoid equivalent (in an appropriate sense) to the Galois double groupoid of a Serre fibration. It is this double groupoid we have displayed in diagram (1).

In the next sections we will identify this double groupoid in terms of homotopy classes of certain maps and so relate these ideas and the compositions obtained more clearly in terms of classical homotopical ideas. One further reason for doing this is to allow the possibility of generalisations of these geometric constructions to situations where the appropriate Galois theory is not yet obtained, for example in terms of smooth structures and 'thin fundamental groupoids', as in [18]. This could lead to smooth structures on variations of the constructions given here.

5 Geometric interpretation

We now start to interpret the previous results geometrically in terms of compositions related to classical notions of relative homotopy groups. To this end, we use standard notation for the cubical singular complex KM of a space M. Here K_nM is the set of singular n-cubes in M (i.e. continuous maps $I^n \to M$ with K_0M identified with M). There are standard *face maps* $\partial_i^{\pm} : K_nM \to K_{n-1}M$ and *degeneracy maps* $\varepsilon_i : K_{n-1}M \to K_nM$ for $i = 1, \ldots, n$ (with $\varepsilon_1 : K_0M \to K_1M$ written simply ε, and giving for $x \in M$ the constant path εx at x). There are also for $i = 1, \ldots, n$ *compositions* $a \circ_i b$ defined for $a, b \in K_nM$ such that $\partial_i^+ a = \partial_i^- b$, and *inversions* $-_i : K_n \to K_n$. Finally we shall later use *connections* $\Gamma_i^{\alpha} : K_{n-1}M \to K_nM$ for $i = 1, \ldots, n, \alpha = \pm$ induced by the maps $\gamma_i^{\alpha} : I^n \to I^{n-1}$ defined by

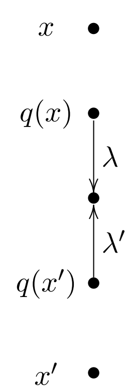
$$\gamma_i^{\alpha}(t_1, t_2, \ldots, t_n) = (t_1, t_2, \ldots, t_{i-1}, A(t_i, t_{i+1}), t_{i+2}, \ldots, t_n)$$

where $A(s,t) = \max(s,t), \min(s,t)$ as $\alpha = -, +$ respectively. The enrichment with connections Γ_i^+ for the traditional cubical sets was introduced in [5]. The full properties of these structures are set out in for example [1]. Here we will assume only the obvious geometric properties in the range $n = 0, \ldots, 3$.

Let $q : M \to B$ be a map of spaces. We recall the standard factorisation $q = pi$ where $i : M \to M_q$ is a homotopy equivalence and $p : M_q \to B$ is a fibration. Here

$$M_q = \{(x, \lambda) \mid \lambda : I \to B,\ \lambda(0) = q(x)\} \subseteq M \times B^I$$

and $p(x, \lambda) = \lambda(1)$, while $i : x \mapsto (x, \varepsilon(q(x))$ where $\varepsilon(q(x))$ is the constant path at $q(x)$ in B. A point of $M_q \times_B M_q$ is a pair $((x, \lambda), (x', \lambda'))$ with $\lambda(1) = \lambda'(1)$ as in the following picture:



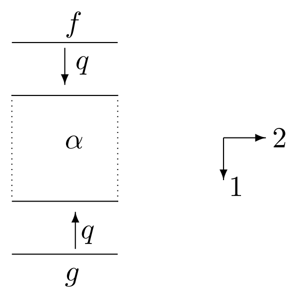
So $M_q \times_B M_q$ is homeomorphic to the space M' of triples $(x, \mu, x') \in M \times K_1B \times M$ such that $\mu(0) = q(x), \mu(1) = q(x')$. Hence we have a groupoid structure on M' with object set M_q, where the source and target maps s', t' send (x, μ, x') to (x, μ_1), $(x', -\mu_2)$ where $\mu_1, \mu_2 \in K_1B$ are respectively the rescaled forms of the first half and the second half of μ. The composition $\circ'$ in this groupoid is, when defined, given by

$$(x, \mu, x') \circ' (x', \mu', x'') = (x', \mu_1 \circ_1 \mu_2, x'')$$

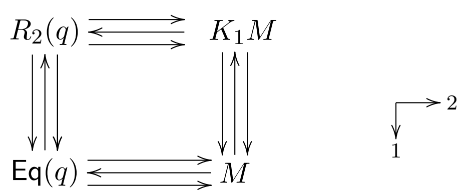
where $\circ_1$ is here the usual composition of paths. Of course this composition is defined if and only if $\mu_2 = -_1\mu_1$.

Let $j : M \times_B M \to M'$ be given by $(x, x') \mapsto (x, \varepsilon(qx), x')$. The set of paths $I \to M'$ with end points in Im j can be identified with the subset $R_2(q)$ of $K_1M \times K_2B \times K_1M$ of triples (f, α, g)

such that $qf = \partial_1^- \alpha, qg = \partial_1^+ \alpha$ and $\partial_2^- \alpha, \partial_2^+ \alpha$ are constant paths. Thus an element of $R_2(q)$ may be pictured as:



where the dotted lines show constant paths. Thus $R_2(q)$ fits in the following diagram:



(18)

The boundary maps are given by:

$$\begin{aligned}
\partial_1^-(f, \alpha, g) &= f, \\
\partial_1^+(f, \alpha, g) &= g, \\
\partial_2^-(f, \alpha, g) &= (f(0), g(0)), \\
\partial_2^+(f, \alpha, g) &= (f(1), g(1)).
\end{aligned}$$

The degeneracy maps $\varepsilon_1 : K_1 M \to R_2(q),\ \varepsilon_2 : \mathsf{Eq}(q) \to R_2(q)$ are given by:

$$\begin{aligned}
\varepsilon_1(f) &= (f, \varepsilon_1(qf), f), \\
\varepsilon_2(x, x') &= (\varepsilon(x), \varepsilon_1^2(q(x)), \varepsilon(x')),
\end{aligned}$$

for $(f, \alpha, g) \in R_2(q), (x, x') \in M \times_B M$.

Clearly (18) can be considered as a diagram of reflexive graphs. We now examine compositions on $R_2(q)$.

The set $R_2(q)$ has two partial compositions. The composition $\circ_2$ is determined by the usual composition of paths and squares in this direction:

$$(f, \alpha, g) \circ_2 (f', \alpha', g') = (f \circ_1 f', \alpha \circ_2 \alpha', g \circ_1 g').$$

The composition $\circ_1$ in the direction 1 is given by

$$(f, \alpha, g) \circ_1 (g, \beta, h) = (f, \alpha \circ_1 \beta, h). \tag{19}$$

Note that this definition generalises a construction by Kamps and Porter in [16, Section 4.1], in which they assume $f(0) = g(0), f(1) = g(1)$ whereas in our situation we have only $qf(0) = qg(0), qf(1) = qg(1)$. Hence they end up with a 2-groupoid, and we end up with a double groupoid. Their method of proving the properties of their homotopy 2-groupoid is to assume first that p is a fibration, and then apply this case to an arbitrary map q by converting it to a fibration $p = \bar{q}$. This is analogous to our methods, except that we have used Galois theory, whereas they use directly properties of fibrations.

We now form the quotient of diagram (18) by taking homotopy classes rel vertices of $R_2(q)$ and of K_1M to yield the diagram:

$$\begin{array}{ccc} \rho_2(q) & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & \pi_1(M) \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ \mathsf{Eq}(q) & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & M \end{array} \tag{20}$$

where $\pi_1(M)$ is the fundamental groupoid of M. It is clear that the horizontal composition $\circ_2$ on $R_2(q)$ is inherited by $\rho_2(q)$. Our main result of this section is a direct verification that the composition $\circ_1$ is also inherited, without going through the simplicial Galois theory of the previous sections. We also have to show that this composition is related to that derived from the equivalence relation structure on K_1M'.

In fact we prove a stronger result. The set

$$R_2(q) \;{}_{\partial_1^+}\!\times_{\partial_1^-}\; R_2(p)$$

is the domain of composition of $\circ_1$ on $R_2(q)$. A *homotopy rel vertices* on this set is a continuous family $((f_u, \alpha_u, g_u), (g_u, \beta_u, k_u)), 0 \leqslant u \leqslant 1$ of elements of this set such that $f_u(0) = f_0(0), k_u(1) = k_0(1),\ 0 \leqslant u \leqslant 1$. We use the notation π_0^v for the set of homotopy classes rel vertices.

Theorem 5.1 *The natural map*

$$\Xi : \pi_0^v(R_2(q) \;{}_{\partial_1^+}\!\times_{\partial_1^-}\; R_2(q)) \to \rho_2(q) \;{}_{\partial_1^+}\!\times_{\partial_1^-}\; \rho_2(q) \tag{21}$$

defined by the projections, is a bijection.

For the proof we use properties of the connections, and we use the following notation from [21].

We write:

$$\ulcorner \quad \underset{j}{\downarrow}\!\overset{\ \ j+1}{\rightarrow} \text{ for } \Gamma_j^+; \qquad \lrcorner \quad \underset{j}{\downarrow}\!\overset{\ \ j+1}{\rightarrow} \text{ for } \Gamma_j^-$$

$$\mid\,\mid \quad \underset{j}{\downarrow}\!\overset{\ \ j+1}{\rightarrow} \text{ for } \varepsilon_j; \qquad = \quad \underset{j}{\downarrow}\!\overset{\ \ j+1}{\rightarrow} \text{ for } \varepsilon_{j+1}.$$

Thus the thick lines denote degenerate faces. We shall use inversions applied to connections, for example

$$-_2\ulcorner, \quad \underset{1}{\downarrow}\!\overset{\ \ 2}{\rightarrow},$$

and write this also as $\urcorner$ since it coincides with $-_1 \lrcorner$.

This notation allows us to write some compositions as for example that involving 3-cubes x, y with $\partial_3^+ x = \partial_3^- y$ as

$$A = \begin{bmatrix} | \; | & \ulcorner \\ x & y \end{bmatrix} \qquad \begin{matrix} \ulcorner\!\rightarrow 3 \\ \downarrow \\ 2 \end{matrix}$$

which is an abbreviation for

$$\begin{bmatrix} \varepsilon_2 \partial_2^- x & \Gamma_2^+ \partial_2^- y \\ x & y \end{bmatrix}$$

and makes it transparent what are the faces of A. The direction arrows are omitted when convenient.

Proof of theorem 5.1 We define an inverse Φ for Ξ.

We use square brackets [ ] to denote homotopy classes. Let $([f, \alpha, g], [h, \beta, k]) \in \rho_2(q)\, {}_{\partial_1^+}\!\times_{\partial_1^-} \rho_2(q)$. Then there is a homotopy rel vertices of paths $\xi : g \simeq h : I^2 \to M$. We set

$$\Phi([f, \alpha, g], [h, \beta, k]) = [(f, \alpha, g), (g, (q\xi) \circ_1 \beta, k)] \tag{22}$$

and have to prove Φ is well defined and an inverse to Ξ.

Suppose we are given homotopies

$$\kappa : (f, \alpha, g) \equiv (f', \alpha', g') \tag{23}$$
$$\lambda : (h, \beta, k) \equiv (h', \beta', k') \tag{24}$$
$$\xi' : g' \simeq h'. \tag{25}$$

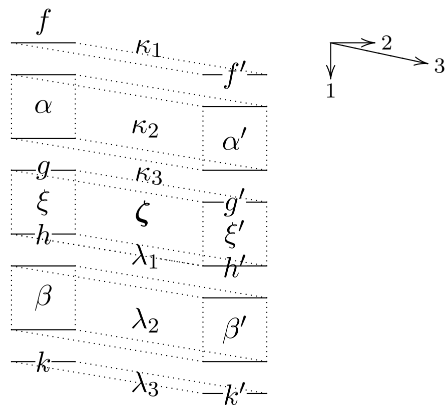
Then κ, λ are given by three component homotopies rel vertices

$$\kappa_1 : f \simeq f', \; \kappa_2 : \alpha \equiv \alpha', \kappa_3 : g \simeq g', \tag{26}$$
$$\lambda_1 : h \simeq h', \; \lambda_2 : \beta \equiv \beta', \lambda_3 : k \simeq k', \tag{27}$$

with the properties that

$$q\kappa_1 = \partial_1^- \kappa_2, \; q\kappa_3 = \partial_1^+ \kappa_2, \tag{28}$$
$$q\lambda_1 = \partial_1^- \lambda_2, \; q\lambda_3 = \partial_1^+ \lambda_1. \tag{29}$$

We now use the fact that all homotopies are rel vertices and that the maps $\alpha, \alpha', \beta, \beta' : I^2 \to B$ are constant on the edges $\partial_2^-, \partial_2^+$. So in the following picture, the dotted lines represent constant paths, and $\boldsymbol{\zeta}$ is a hollow cube not yet filled in, but has four faces well defined. Note also that the maps $I^2 \to B$ given by $\partial_2^\pm(\kappa_2)$ and $\partial_2^\pm(\lambda_2)$ are constant maps, by our definition of homotopies.



The maps $\xi, \xi', \kappa_3, \lambda_1$ define a map

$$(I^2 \times \dot{I}) \cup (\dot{I} \times I^2) \to E$$

(where $\dot{I} = \{0,1\}$) given by $(s,t,0) \mapsto \xi(s,t), (s,t,1) \mapsto \xi'(s,t)$ on $I^2 \times \dot{I}$, and by $(0,t,u) \mapsto \kappa_3(t,u), (1,t,u) \mapsto \lambda_1(t,u)$ on $\dot{I} \times I^2$ respectively. By the rel vertices condition, these maps can be extended by the constant map over $I \times \{0\} \times I$, which is $\partial_2^-(\boldsymbol{\zeta})$. So we now have maps defined on 5 faces of I^3 and agreeing on their common edges, and so these extend to a map $\zeta : I^3 \to E$. However, while this map does agree with the other homotopies, the result will not be a homotopy of the type required since $\zeta_1 = \partial_2^+(\zeta) = (\zeta|(I \times \{1\} \times I)$ is not constant as would be required. So we have to make a modification to get a homotopy between representatives of the original classes. Intuitively, we move the face ζ_1 of ζ to the right and down of our composite picture. This modification will also change λ_2, λ_3, but this does not matter for our purposes, since we need to show only that a homotopy of the required type exists.

We let $\varepsilon, \varepsilon_1, \varepsilon_2$ denote degenerate elements – the element they act on in the following formulae will be clear from the context, in order to make the compositions properly defined.

Our new homotopy

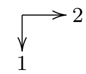
$$(f \circ \varepsilon, \alpha \circ_2 \varepsilon_2, g \circ \varepsilon), (g \circ \varepsilon, ((q\xi) \circ_1 \beta) \circ_2 \varepsilon_2, k \circ \varepsilon)) \equiv (f' \circ \varepsilon, \alpha' \circ_2 \varepsilon_2, g' \circ \varepsilon), (g' \circ \varepsilon, ((q\xi') \circ_1 \beta') \circ_2 \varepsilon_2, k' \circ \varepsilon))$$

will be given by

$$\kappa_1 \circ_2 \varepsilon : f \circ \varepsilon \simeq f' \circ \varepsilon, \tag{30}$$
$$\kappa_2 \circ_2 \varepsilon_2 : \alpha \circ_2 \varepsilon_2 \equiv \alpha' \circ_2 \varepsilon_2, \tag{31}$$
$$\kappa_3 \circ_2 \varepsilon : g \circ \varepsilon \simeq g' \circ \varepsilon, \tag{32}$$
$$\lambda_2' : ((p\xi) \circ_1 \beta) \circ_2 \varepsilon_2 \simeq ((p\xi') \circ_1 \beta') \circ_2 \varepsilon_2, \tag{33}$$
$$\lambda_3 \circ \partial_2^+\zeta : k \circ \varepsilon \simeq k' \circ \varepsilon, \tag{34}$$

where

$$\lambda_2' = \begin{bmatrix} q\zeta & \urcorner \\ \lambda_2 & |\,| \end{bmatrix}$$



where $\urcorner$ is given by $(s,t,u) \mapsto (\partial_2^+ \zeta)(\min(s, 1-t), u)$. Note that the combination of $\urcorner$ and $\varepsilon = | \; |$ in the second column of the matrix has the effect of pushing the non constant face $\partial_2^+ \zeta$ of ζ down to be able to combine with λ_3.

It is clear that compositions with degenerate elements in direction 2 do not change homotopy classes, and so this completes our geometric proof that Φ is well defined.

Next we must prove $\Phi\Xi = 1, \Xi\Phi = 1$.

Considering the formula (22) for Φ, we see that we can set

$$\Phi([f, \alpha, g], [g, \beta, k]) = [(f, \alpha, g), (g, (q\xi) \circ_1 \beta, k)]$$

where now ξ can be chosen to be a constant homotopy ε. It is then easily seen that

$$[(f, \alpha, g), (g, (q\varepsilon) \circ_1 \beta, k)] = [(f, \alpha, g), (g, \beta, k)],$$

and so that $\Phi\Xi = 1$.

To prove $\Xi\Phi = 1$ it is sufficient to show that if $\xi : g \simeq h$ is a homotopy rel vertices, then $(g, (q\xi) \circ_1 \beta, k) \equiv (h, \varepsilon_1(q\xi) \circ_1 \beta, k)$. Such a homotopy is given by $(\xi, \kappa, \varepsilon_3(k))$ where

$$\kappa = \begin{bmatrix} \lrcorner \\ \varepsilon_3(\beta) \end{bmatrix} \qquad \begin{smallmatrix} \ulcorner\!\to 3 \\ \downarrow \\ 1 \end{smallmatrix} .$$

$\square$

Corollary 5.2 *The composition $\circ_1$ on $R_2(q)$ is inherited by $\rho_2(q)$ so that $\rho(q)$ becomes a double groupoid.*

Proof The composition $\circ_1$ on $\rho_2(q)$ is the composition of the maps

$$\rho_2(q) \;{}_{\partial_1^+}\!\times_{\partial_1^-} \rho_2(q) \xrightarrow{\Phi} \pi_0^v(R_2(q) \;{}_{\partial_1^+}\!\times_{\partial_1^-} R_2(q)) \to \rho_2(q)$$

where the second map is induced by the composition on $R_2(q)$. It is easy to see that the structure $\circ_1$ gives a groupoid structure on $\rho_2(q)$.

Thus the only part remaining is the interchange law. However we easily find that a double composition can be given as

$$\begin{bmatrix} [f, \alpha, g] & [f', \alpha', g'] \\ [h, \beta, k] & [h', \beta', k'] \end{bmatrix} = \left[f \circ f', \begin{bmatrix} \alpha & \alpha' \\ (q\xi) \circ_1 \beta & (q\xi') \circ_1 \beta' \end{bmatrix}, k \circ k' \right] \qquad \begin{smallmatrix} \ulcorner\!\to 2 \\ \downarrow \\ 1 \end{smallmatrix}$$

where $\xi : g \simeq h, \xi' : g' \simeq h'$. So the interchange law follows from that for singular squares. $\square$

Finally, we have to show the relation between the composition $\circ'$ on $K_1 M'$ and the composition $\circ_1$ above. Let $(f, \alpha, g), (g, \beta, h) \in R_2(q)$. We first note that, analogously to the existence of identities in the fundamental groupoid,

$$[f, \alpha, g] = [f, \alpha \circ_1 (\varepsilon_1(qg)), g], \; [g, \beta, h] = [g, (\varepsilon_1(qg)) \circ_1 \beta, h].$$

Hence

$$\begin{aligned} [f, \alpha \circ_1 \beta, h] =& [f, \alpha, g] \circ_1 [g, \beta, h] \\ =& [f, \alpha \circ_1 \varepsilon_1(qg), g] \circ_1 [g, \varepsilon_1(qg) \circ_1 \beta, h] \\ =& [(f, \alpha \circ_1 \varepsilon_1(qg), g) \circ' (g, \varepsilon_1(qg) \circ_1 \beta, h)] \end{aligned}$$

as required.

6 Examples

In order to study the double groupoid $\rho(q)$ we need to have examples of double groupoids with which to compare it, in addition to the product fibration of Example 3.1. As we shall see, there are some sub-double groupoids of $\rho(q)$ which are familiar, but it is interesting that we have little information about the most general form of double groupoids. For example, the methods of [11] give an equivalence between double groupoids satisfying some filler conditions and what are there called *core diagrams*, but these do not seem to be helpful in this case.

Here we suggest various examples and comparisons for further investigation.

Example 6.1 Let $i : M \to B$ be the inclusion of a subspace M of B. Then the equivalence relation $\mathsf{Eq}(i)$ is discrete, and so $\rho(i)$ is a 2-groupoid. Further, if $m \in M$ then the natural map

$$\eta : \pi_2(B, M, m) \to \rho_2(i)$$

is injective.

Proof We represent $\pi_2(B, M, m)$ by maps $\alpha : I^2 \to B$ such that the face $\partial_1^- \alpha$ maps into M and the other three faces map to the base point m. The homotopy classes of α which yield an element $[\alpha]$ of $\pi_2(B, M, m)$ are through maps of the same type. Then α also yields an element $\langle\alpha\rangle$ of $\rho_2(i)$, but there the homotopies allow $\partial_1^+ \alpha$ to vary in M. We have to prove that the map $\eta : [\alpha] \mapsto \langle\alpha\rangle$ is injective. Suppose then $[\alpha_-], [\alpha_+] \in \pi_2(B, M, m)$ and $\langle\alpha_-\rangle = \langle\alpha_+\rangle \in \rho_2(i)$. Let $h : I^3 \to B$ be a homotopy determined by this equality, so that

$$\partial_3^-(h) = \alpha_-, \partial_3^+(h) = \alpha_+,$$

$\partial_2^\pm(h)$ maps to m. Let $\theta = \partial_1^+(h)$. The problem is that θ is not constant. So we change h to 'move' θ to the top face and still give a homotopy $h' : \alpha'_- \simeq \alpha'_+$ where $[\alpha_\pm] = [\alpha'_\pm]$. We can take

$$h' = \begin{bmatrix} \mid\;\mid & h \\ \llcorner & \lrcorner \end{bmatrix} \qquad \begin{array}{l} \ulcorner\!\!\longrightarrow 2 \\ \downarrow \\ 1 \end{array}$$

so that the two ends of this homotopy are

$$\partial_3^\pm(h') = \begin{bmatrix} \square & \alpha_\pm \\ \square & \square \end{bmatrix},$$

where $\square$ denotes a double identity, as required. $\square$

Note that $\rho(i)$ is the homotopy 2-groupoid of a pair discussed by Moerdijk and Svensson in [19], and is also recovered from the work of [16].

Example 6.2 The double groupoid $\rho(q)$ contains a 2-groupoid

$$\bar{\rho}(q) \rightrightarrows \pi_1(M) \rightrightarrows M$$

where $\bar{\rho}_2(q)$ is the subset of $\rho_2(q)$ of elements u such that $\partial_2^- u,\ \partial_2^+ u$ are degenerate, that is consist of pairs (x, x). This is essentially the homotopy 2-groupoid of a map discussed by Kamps and Porter in [16]. This 2-groupoid contains various cat^1-groups of the form considered by Loday in [17]. The crossed module of groupoids associated to this 2-groupoid is of the form $C \to \pi_1(M)$ where for each point $x \in M$ we have $C(x)$ is isomorphic to $\pi_1(F_x, \bar{x})$, the fundamental group of the homotopy fibre F_x of q over $q(x)$ at the base point $\bar{x}$ determined by x. If M is a subspace of B and q is the inclusion then $C(x)$ is isomorphic to the familiar relative homotopy group $\pi_2(B, M, x)$ and the crossed module $C(x) \to \pi_1(M, x)$ is essentially that first studied by J.H.C. Whitehead. However we do not have a reconstruction method for $\rho(q)$ from $\bar{\rho}(q)$, whereas the 2-groupoid can be reconstructed from the crossed module of groupoids it contains, as shown in [7]. □

Example 6.3 Foliations Let $\mathcal{F}$ be a foliation on a space M. Thus the leaves of the foliation define an equivalence relation $R = R(\mathcal{F})$. Let $q : M \to B$ be a map of spaces. The foliation defines a finer topology than that given on M to give a space $M_{\mathcal{F}}$ in which all leaves of the foliation are open components. So we also have a map $q_{\mathcal{F}} : M_{\mathcal{F}} \to B$ and hence may define the homotopy double groupoid $\rho(q_{\mathcal{F}})$. Where this differs from $\rho(q)$ is that in $\rho(q_{\mathcal{F}})$ the 'horizontal' paths, and the homotopies of paths, all lie in leaves of the foliation.

An illustrative example is the Möbius Band M with its projection $q : M \to S^1$ and foliation $\mathcal{F}$ by circles of which the centre one goes once round the Band and the other circles go twice round. Then $\rho(q_{\mathcal{F}})$ contains the double groupoid $\mathcal{D}(M)$ explained in the Introduction, and which seems to be a good discrete algebraic model of the foliated Möbius Band. □

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