Diagonal for the fundamental crossed complex of the Klein bottle.

1
26/7/92
6:00 a.

Let K be the Klein bottle, and let $C = \pi K$. Then

C_1 is the free group on a, b

C_2 is the free crossed module on e with

$$\delta e = -a + b + a + b$$

(all groups will be written additively, and operations for crossed modules will be on the right).

Then $C \otimes C$ is in dimension 1 the free group on generators a_1, b_1, a_2, b_2, and in dimension 2 is the free crossed module on generators $e_1, e_2, a_1 \otimes a_2, a_1 \otimes b_2, b_1 \otimes a_2, b_1 \otimes b_2$ with

$$\delta e_1 = -a_1 + b_1 + a_1 + b_1$$

$$\delta e_2 = -a_2 + b_2 + a_2 + b_2$$

$$\delta(a_1 \otimes a_2) = -a_1 - a_2 + a_1 + a_2$$

$$\delta(a_1 \otimes b_2) = -a_1 - b_2 + a_1 + b_2$$

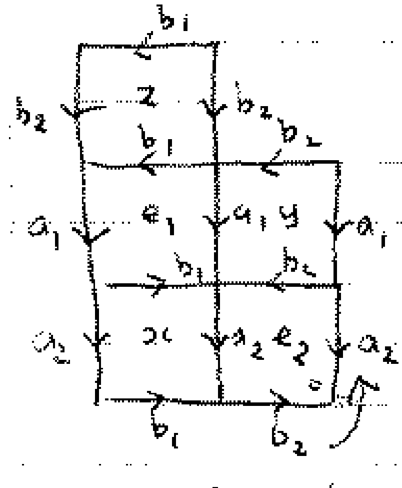
$$\delta(b_1 \otimes a_2) = -b_1 - a_2 + b_1 + a_2$$

$$\delta(b_1 \otimes b_2) = -b_1 - b_2 + b_1 + b_2$$

The diagonal $\Delta : C \to C \otimes C$ is given in dimension 1 by $\Delta a = a_1 + a_2$, $\Delta b = b_1 + b_2$.

Hence $\Delta(-a + b + a + b) = -a_2 - a_1 + b_1 + b_2 + a_1 + a_2 + b_1 + b_2$.

We need to find an element of $(C \otimes C)_2$ with boundary this element.



Let $y = a_1 \otimes (-b_2) = -(a_1 \otimes b_2)^{-b_2}$

$x = -(b_1 \otimes a_2)$

$z = -((-b_1) \otimes b_2)$

$= (b_1 \otimes b_2)^{-b_1}$

Then the diagram on the left has the correct boundary. To describe this as an element of $(C \otimes C)_2$ we apply homotopy addition lemma type arguments.

I find it easier to work as follows, not knowing a better "algorithm": write down the boundary of the figure, going round each cell in turn, setting it up for maximal calculation. So we write the expression

$$
\begin{aligned}
E = & -a_2 - a_1 + b_2 - \underline{b_2 + b_1 + b_2} - \underline{b_1} - b_2 + a_1 + a_2 \\
& - a_2 - \underline{a_1 + b_2 + a_1 - b_2} + a_2 \\
& - \underline{a_2 + b_2 + a_2 + b_2} \\
& - b_2 - \underline{a_2 - b_1 + a_2 + b_1} + b_2 \\
& - b_2 - b_1 - a_2 + b_1 \underline{- a_1 + b_1 + a_1 + b_1} - b_1 + a_2 + b_1 + b_2
\end{aligned}
$$

which does reduce to $-a_2 - a_1 + a_1 + a_2 + b_1 + b_2$ ..

The element, or an element, in $(C \otimes C)_2$ with this boundary is $w =$

$$(b_1 \otimes b_2)^{-b_1 - b_2 + a_1 + a_2} - (a_1 \otimes b_2)^{-b_2 + a_2} + e_2 - (b_1 \otimes a_2)^{-b_2} + e_1^{-b_1 + a_2 + b_1 + b_2}$$

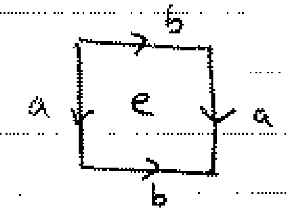
This suggests that an ω-groupoid would be more convenient and descriptive of the geometry: in that context, the above figure would be written

$$u = \begin{bmatrix} z & \gamma \\ e_1 & y \\ x & e_2 \end{bmatrix}.$$

However, we do not have the clear methods to show that this yields a morphism $\rho K \to \rho K \otimes \rho K$.

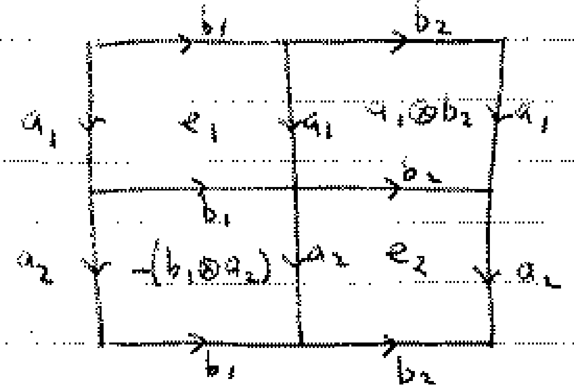
Diagonal for the Torus

The notation is similar to the above, but we now have



$$\delta e = -a - b + a + b$$

and similarly for e_1, e_2. Here we use the diagram



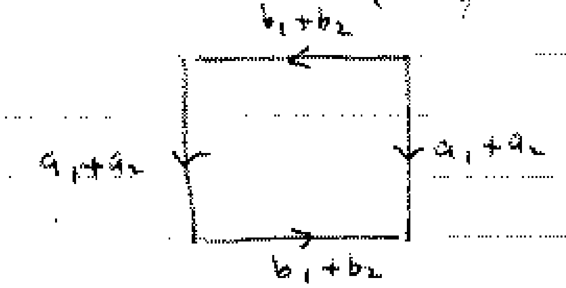
yielding

$$\Delta e = (a_1 \otimes b_2)^{a_2} + e_2 - (b_1 \otimes a_2)^{b_2} + e_1^{-b_1 + a_2 + b_1 + b_2}$$

Notice that in each of these case, all we are saying is that we have defined a morphism of crossed complexes $C \to C \otimes C$ over the obvious diagonal in dimension 1. However, by freeness and asphericity, for these complexes, such diagonals are unique up to homotopy. What we don't know (or expect?) is whether coassociativity holds.

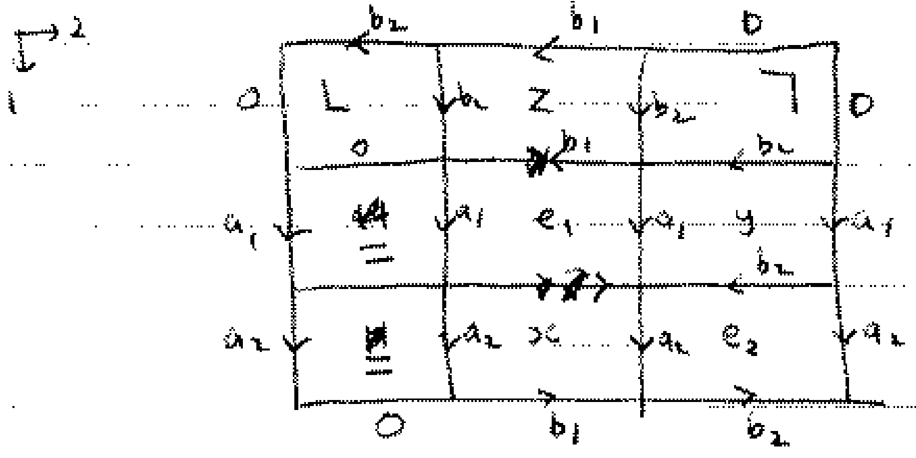
Actually, there is a cubical way of making this clearer, so that the algebra simply corresponds to the geometry.

If K is a cubical set, then ρK is the free ω-groupoid on K (this is proved in "Colimits of relative homotopy groups"). So we need to define a morphism $\rho K \to \rho K \otimes \rho K$ by its values on the elements of K. Hence we need an element of $(\rho K \otimes \rho K)_2$ with boundary

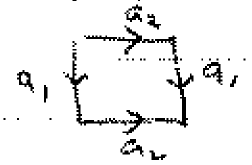


This is for the Klein bottle.

This can be obtained by a modification of our previous diagram:



Some care is needed to identify the elements x, y, z in this context. In ρK $a_1 \otimes a_2$ has boundary



so that $z = -_1 \tau(b_1 \otimes b_2)$ where τ is the anticlockwise rotation through $\pi/4$, as described by Brown and Higgins.

The above suggests that convenient descriptions of diagonals where one has more general relations, with mixtures of geometric figures, would require polyhedral T-complexes.

The fact that the crossed module formula is deduced from the figure suggests that it would be desirable to keep to the figure as long as possible. The crossed complex figure could, however, be useful for deducing the corresponding diagonal for chain complexes with operators.

An interesting example to try would be D_3, and also D_5, the latter because it involves pentagons!

Another good test would be to work out the diagonals for $C(K) \otimes C(K)$, or $C(K) \otimes C(T)$, and also to evaluate the corresponding results in the chain complexes with operators.