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BOOK REVIEW

Ronald Brown, Philip J. Higgins, Rafael Sivera: "Nonabelian Algebraic Topology"

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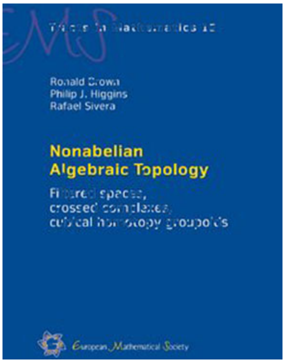

A result describing the fundamental group of a union of two connected subcomplexes of a simplicial complex goes back at least to Seifert [14], in the case when the intersection is connected. This is an early result on the fundamental group of the union of two spaces. For Seifert, this result was a technical tool for describing the fundamental groups of 3-manifolds he constructed in various ways, in particular in terms of a Heegard decomposition. In modern language, Seifert's result yields the fundamental group under discussion in terms of a pushout diagram of groups. Given an algebraic curve in the complex projective plane, using purely algebro-geometric methods, Zariski wrote down a presentation of the fundamental group of the complement and suggested to van Kampen to confirm the correctness of the presentation by purely topological methods, which he did in [17]. Thereafter van Kampen established a general theorem on the fundamental group of certain pathwise connected topological spaces [16]. This result underlies the contents of his previous paper; it is essentially the same as that established by Seifert for simplicial complexes.

The problem with the connectivity assumption of the intersection prevented the use of the theorem for deducing the result that the fundamental group of the circle is free cyclic. In [1], R. Brown could then overcome this obstacle by generalizing the statement of the theorem from the fundamental group on one base point to the

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fundamental groupoid on a system of base points chosen freely according to a given geometric situation, a key observation being that the notion of pushout diagram makes perfect sense for groupoids. The fundamental group of the circle drops out almost immediately.

In the 1920s, the definition of groupoid arose from Brandt's attempts to extend from the binary to the quaternary case Gauss's work on composition laws of quadratic forms. Groupoids appear in Reidemeister's book on topology [12] for handling the change of generators of the (combinatorially defined) fundamental group of a closed surface induced by the change of normal form of the surface, and for handling isomorphisms of a family of structures.

The book under review gives a newly organised exposition of work done over many years (since 1965), mainly of the first two authors. This subsumes some traditional algebraic topology and includes generalizations to higher dimensions of purely topological theorems of the type described above; such theorems are referred to in the book as *Seifert-Van Kampen theorems*.

The book deals with its topics in a thorough yet readable manner. It also touches on various foundational issues related to the perception and implementation of geometrical ideas, one such issue being the successful usage of the groupoid concept, as opposed to the mere notion of abstract group that is considered so central a concept of mathematics.

How do we encounter situations in mathematical nature that call for generalization of the Seifert-van Kampen theorem, what does such a generalization possibly look like and what might it signify?

Apart from the above situation where the usage of groupoids yields the solution, here are a few other such situations: Trying to unveil the structure of a second relative homotopy group, Whitehead isolated the notion of crossed module and in particular that of free crossed module [18, 19]. This structure was developed independently by Reidemeister and his student Peiffer [11]—the manuscript was submitted for publication in June 1944. Reidemeister and Peiffer explored *identities among relations* of a group presentation and thereby discovered free crossed modules. One of their aims was to develop normal forms for 3-manifolds. In a personal letter sent to me from R. Peiffer in the late 1970s I learnt that Reidemeister, as one of the founders of knot theory, was well aware of the apparent relations between the identities for crossed modules and those for knots and links. Crossed modules are typically nonabelian. For intelligibility, we recall that a crossed module $\partial\colon C \to G$ consists of two G-groups C and G where G is considered as a G-group via conjugation and a morphism ∂ of G-groups, subject to the rule

$$aba^{-1} = {}^{\partial a}b \in C, \quad a, b \in C. \tag{1}$$

Any element of C of the kind $aba^{-1}({}^{\partial a}b)^{-1}$ or the inverse of such an element is referred to in the literature as a *Peiffer element*.

Structural insight into the Cayley graph of a presentation of a group and its cousin, the geometric realization of the presentation, can be obtained in terms of the associated free crossed module. This yields, e.g., a description of the structure of the normal closure N of the relators R in the free group F on the generators as an F-group, relative to conjugation: The non-trivial *identities among relations*, that is, the identities modulo *Peiffer identities* (identities of the kind (1) above, independently of the particular presentation), yield the "correct" relations for N as an F-group generated by R. There are no non-trivial identities in this sense if and only if the second absolute homotopy group of the geometric realization is zero and hence N the free crossed F-module generated by R. A search in the MR database exhibits at present more than 50 papers that deal with "Peiffer commutators", "Peiffer elements" and the like.

The method of diagrams has become a standard tool in combinatorial group theory to explore a Cayley graph; an exposition of this theory, with a special emphasis on identities among relations, can be found in [2]. This concept goes back at least to [15]; a version thereof appears in [11] under the name *Randwegaggregat*. Such a diagram represents an element of the associated free crossed module in a geometric manner.

Schreier's approach to the extension problem for, in general, nonabelian groups, published in 1926, was reworked, clarified and completely settled by Eilenberg and Mac Lane [5]; indeed that extension problem was among the impetuses to the development of group cohomology. A key observation of Eilenberg and Mac Lane was to the effect that, given a discrete group G and a G-module A, the elements of $\mathrm{H}^3(G, A)$ correspond to classes of "abstract kernels". Quoting from the historical note by S. Mac Lane, appended to [7], "Eilenberg, Mac Lane and Whitehead all knew that the elements of $\mathrm{H}^3(G, A)$ were closely connected with Whitehead's 'crossed modules', but they missed the exact theorem, that there is a natural bijection from $\mathrm{H}^3(G, A)$ to equivalence classes of four term crossed sequences starting at A and ending at G". The history of what Mac Lane refers to as an "exact theorem", isolated only in the late 1970s, is described in that note in detail.

In [3], Brown and Higgins proved a Seifert-van Kampen theorem in dimension 2. Roughly speaking, the theorem says that, given a pair (X, A) of spaces and, furthermore, subspaces of X whose interiors cover X, under suitable connectivity assumptions, the resulting square of crossed modules is a pushout diagram. This implies Whitehead's result on free crossed modules as well as the corresponding result of Reidemeister-Peiffer. The proof uses generalized groupoid techniques (double groupoids etc.) in an essential way. Also the proof does not assume the existence of pushouts of crossed modules; instead it verifies directly the required universal property for this case, so that the requisite pushout exists. This is an instance of what was meant above by "various foundational issues".

The idea of szyzygy, applied to crossed modules, leads in an obvious manner to the more general notion of crossed complex. The classification problem of homotopy n-types was raised by J.H.C. Whitehead [19]. Crossed complexes were used implicitly by Eilenberg-Mac Lane to determine the n-type (old terminology: $(n+1)$-type) of an n-complex with non-trivial fundamental group π_1 and higher homotopy groups π_j zero for $2 \leq j < n$ (empty assumption when $n = 2$) [6]; the requisite additional invariant developed by Eilenberg-Mac Lane is the k-invariant in $\mathrm{H}^{n+1}(\pi_1, \pi_{n+1})$, identified in 1951 by Postnikov as the first of an entire family of invariants which, together with the homotopy groups, completely characterize the homotopy type of a CW complex. Crossed complexes yield an interpretation of group cohomology in arbitrary dimension; again the history thereof is described in Mac Lane's historical note in detail. In this interpretation, it is the crossed complex itself which represents the k-invariant [8].

We already pointed out that the successful cure to what might be considered an anomaly prompted the development of what the book is about: a systematic approach from the abstract topology point of view whose aim is to generalize to all dimensions the notion of fundamental group, including relaxing the connectivity assumption in dimension zero. The generalization proceeds in various ways and involves, among other items, filtered spaces, crossed complexes, ω-groupoids, cubical sets, etc. The idea of a filtered space arises somewhat naturally out of generalizing the idea of fundamental groupoid on a system of base points chosen freely according to a given geometric situation. A filtered space can be seen as an instance of a structured space in the sense of Sect. 5 of Grothendieck's *Esquisse d'un programme*, published in [13]. The idea of structured space was developed further in the work of R. Brown with Loday which involves n-cubes of spaces [4]; this links with classical work in homotopy theory on the homotopy groups of n-ads.

In Part I (Chapters 1–6), the book covers all the items mentioned so far (except the algebro-geometric origin of van Kampen's result). In particular, the 2-dimensional Seifert-van Kampen theorem is given as Theorem 2.3.1. Part II of the book (Chapters 7–12) introduces and explores a higher homotopy Seifert-van Kampen theorem. This theorem includes the 1- and 2-dimensional theorems so far explained and, in a sense, generalizes them to arbitrary dimension. The method of proof is analogous to the 2-dimensional theorem but technically more complicated and deferred to Part III (Chapters 13–16).

The higher homotopy Seifert-van Kampen theorem (Theorem 8.1.5, p. 262) is phrased as the statement that a certain diagram is the coequalizer in the category of crossed complexes. This theorem gives a mode of calculation of the fundamental crossed complex functor Π from filtered topological spaces to crossed complexes. This functor is defined homotopically, that is, in terms of suitably defined homotopy classes of certain maps. A consequence of the definition is that Π preserves coproducts; this is one of the advantages of the groupoid approach. More subtle is the application to gluing spaces, and the authors approach this concept, as for the 1- and 2-dimensional version of the theorem, through the notion of coequalizer; here again, a connectivity condition in all dimensions is needed.

The authors then show how the higher homotopy Seifert-van Kampen theorem gives some computations of homotopy groups of pairs of spaces and, as a consequence, some classical results such as the suspension theorem, the Brouwer degree theorem, and the relative Hurewicz theorem. These are basic theorems in homotopy theory but are obtained here without homology theory machinery. At the present stage, the authors' concern with foundational issues shows up clearly: A major aspect of the book is to tie in the fundamental group and higher homotopy groups without passing to the universal covering space, as is done in the more conventional approach. It is also unclear how to obtain the 1- and 2-dimensional versions of the Seifert-van Kampen theorem by covering space methods. The applications culminate in a homotopy classification theorem (Theorem 11.4.19, p. 391). To describe it, let X denote a CW complex, C a crossed complex, ΠX_* the crossed complex associated with X, and BC the cubical classifying space C. The classification theorem employs the cubical theory in an essential way to give, in terms of crossed complexes, a description of the weak homotopy type of the mapping space $(BC)^X$. This description, in turn, entails a bijection from the homotopy classes $[\Pi X_*, C]$ of crossed complex morphisms to the homotopy classes $[X, BC]$ of ordinary maps of spaces. Thereafter applications to nonabelian cohomology, using fibrations, i.e. model category kind of structures and arguments, are worked out.

In Part III the topics covered, including ω-groupoids, cubical sets, connections and compositions in cubical sets, etc., are motivated by the need to develop the requisite technology so that the authors can eventually craft a proof of the higher homotopy Seifert-van Kampen theorem and develop the key monoidal closed structures required for the homotopy classification theorem. Working through this technology may be seen as a challenge looming over the reader. But, amice lector, if you find this rather special, keep in mind that the authors need, e.g., an algebra of cubes which enables them to handle composition in all directions, and the underlying nonabelian algebra is still in the process of intense development, comparable, perhaps, to the early stage of what are now standard algebraic topology or algebraic geometry notions.

Groupoids nowadays play a significant role in various areas distinct from topology, e.g., in the Lie theory of symplectic groupoids and in the related issue of quantization, in differential Galois theory as what is known as Malgrange groupoid, in algebraic geometry, see below, etc. Crossed modules and notions related to them arise under various circumstances where, at first, one would not expect to see them. One such situation is the positive answer to a question posed by Atiyah whether there is a finite-dimensional construction of the Chern-Simons function in dimension 3; the answer in [9] involves identities among relations in an essential way. There is an intimate relation between braids and crossed modules; this relation is already present in Whitehead's original proof of the freeness of the crossed module discussed earlier. Recently I have reworked and somewhat extended that relationship [10]. Crossed modules show up in the theory of gerbes and in string theory. There are so many names attached to this activity that I prefer not to mention any of them. What is known as the Teichmüller cocycle in Galois theory admits its natural interpretation in terms of crossed modules. Lie crossed modules are nowadays studied in differential geometry.

In classical algebraic geometry, points are characterized in terms of functions, a point being an algebra map from a coordinate ring to the base field. In topology points belong to a space which is usually a continuum, and ordinary commutative algebra, so well suited to algebraic geometry, is not strong enough to recover the nonabelian phenomena that are attacked in the book. Nonabelian phenomena play as well a major role in algebraic geometry (Brauer-Severi varieties, Teichmüller groupoid, etc., to list a few instances). It may well be that, in the future, the ideas presented in the book contribute to some of the many open questions in these areas. Van Kampen is well known in algebraic geometry circles for his result on the fundamental group of the complement of a plane curve but hardly any algebraic geometer is aware of the more general topological result that underlies it and was proved two years earlier by Seifert. Likewise hardly any topologist knows that a special case of the Seifert-van Kampen theorem established, at the time, an important result in algebraic geometry. In this sense, the book will, perhaps, also contribute to the unity of mathematics.

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