The intuitions for using multiple categories and groupoids in algebraic topology.

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Higher homotopy groups are abelian

Theorem

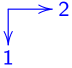
If G is a set with two monoid structures $\circ_1$, $\circ_2$ satisfying (*): each is a morphism for the other, then the structures coincide and are abelian.

Proof Condition (*) can be phrased as: for all $x, y, z, w \in G$

$$(x \circ_1 y) \circ_2 (z \circ_1 w) = (x \circ_2 z) \circ_1 (y \circ_1 w). \tag{1}$$

or that there is only one way of reading

$$\begin{bmatrix} x & z \\ y & w \end{bmatrix} \tag{2}$$

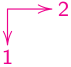


Higher homotopy groups are abelian

Now we check in turn:

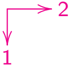
$e_1 = e_2 = e$, say.

$$\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}$$



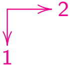
$\circ_1 = \circ_2 = \circ$, say

$$\begin{bmatrix} x & e \\ e & y \end{bmatrix}$$



$\circ$ is abelian

$$\begin{bmatrix} e & x \\ y & e \end{bmatrix}$$



Boundary of a simplex

Classic formula for the boundary of a simplex:
If x has dimension n, $\partial_i x$ is its ith face, then

$$d_n x = \sum_{i=0}^{n} (-1)^i \partial_i x,$$

This replaces older formulae involving orientation. It works in abelian situations, chain complexes, is related to integration theory, and it is easy to prove

$$d_{n-1} d_n = 0.$$

So you get homology groups

$$H_n = \text{Ker } d_n / \text{Im } d_{n+1}.$$

Homotopy addition lemma

This also gives 'the boundary of a simplex', but it takes account also of:

• a set of base points (the vertices of the simplex);

• operators of dimension 1 on dimensions $\geqslant 2$;

• nonabelian structures in dimensions 1 and 2.

In dimension 1, we have end points:

$$0 \xrightarrow{x} 1 \qquad \text{(HAL1-diagram)}$$

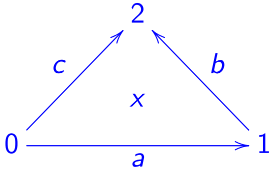
$$0 = \partial_1 x, \qquad 1 = \partial_0 x.$$

Homotopy addition lemma

In dimension 2 we have a groupoid rule:

$$\delta_2 x = -\partial_1 x + \partial_2 x + \partial_0 x, \qquad \text{(HAL2)}$$

which is represented by the diagram



(HAL2-diagram)

and the easy to understand formula (HAL2) says that

$$\delta_2 x = -c + a + b$$

Homotopy addition lemma

In dimension 3 we have the nonabelian rule:

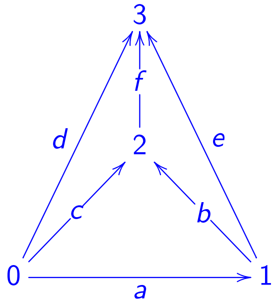
$$\delta_3 x = (\partial_3 x)^{u_3} - \partial_0 x - \partial_2 x + \partial_1 x. \qquad \text{(HAL3)}$$

Note that in HAL3 we have an exponent u_3: this is given by $f = \partial_0^2 x$.The necessity for this is that our convention is that each n-simplex x has as base point its last vertex $\partial_0^n x$. Thus the base point of the above 3-simplex x is 3, while the base point of $\partial_0 x$ is 2. The exponent f relocates $\partial_0 x$ to have base point at 3, and so obtains a well defined formula.

Homotopy addition lemma

Understanding of this is helped by considering the diagram

$x :=$



(HAL3-diagram)

We have the groupoid formula

$$-f+(-c+a+b)+f-(-e+b+f)-(-d+a+e)+(-d+c+f)=0.$$

This is a translation of the rule $\delta_2\delta_3 = 0$, provided we assume

$$\delta_2(y^f) = -f + \delta_2 y + f,$$

which is the first rule for a crossed module.

Homotopy addition lemma

In dimension $n \geqslant 4$ we have the abelian rule, but still with operators:

$$\delta_n x = (\partial_n x)^{u_n} + \sum_{i=0}^{n-1} (-1)^{n-i} \partial_i x, \qquad \text{(HAL}\geqslant 4)$$

where $u_n x = \partial_0^{n-1} x$. These, or analogous, formulae underly much nonabelian cohomology theory.

The rule $\delta_{n-1}\delta_n = 0$ is straightforward to verify for $n > 4$, through working in abelian groups,but for $n = 4$ we require the second crossed module rule, that for x, y of dimension 2

$$-y + x + y = x^{\delta_2 y}.$$

Homotopy addition lemma

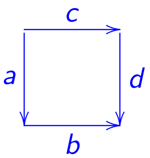
In the right context, the HAL is an inductive cone formula.

$$\delta_n x = (\partial_n x)^{u_n} - \text{formula in dimension } n-1.$$

The context for these formulae is the notion of crossed complex.
These are due to Blakers (1948), Whitehead (1949), and PJH and I set up a rich algebraic theory of these so that they can be used for calculation expression.
The HAL is then proved purely algebraically, and one also proves they model the topology.
That is the hard bit!

Commutative squares

The notion of commutative square is easy:



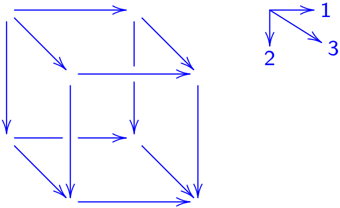
is easily translated to

$$ab = cd.$$

One can easily prove:
Any composition of commutative squares is commutative.

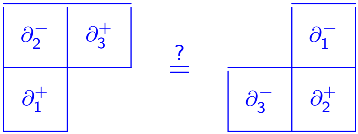
Commutative cubes?

What does it mean for the faces of a cube to commute?

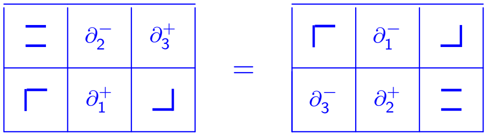


Commutative cube

A cube has six faces. We cannot equate:



We have to fill in and expand out



Connections

Need some special squares called thin:

$$\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \quad \begin{pmatrix} & 1 & \\ a & & a \\ & 1 & \end{pmatrix} \quad \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix}$$

$\square$ $=$ or $\varepsilon_2 a$ $| \; |$ or $\varepsilon_1 b$

$$\begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \quad \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}$$

$\lrcorner$ $\ulcorner$

These are *connections*

Laws for Connections

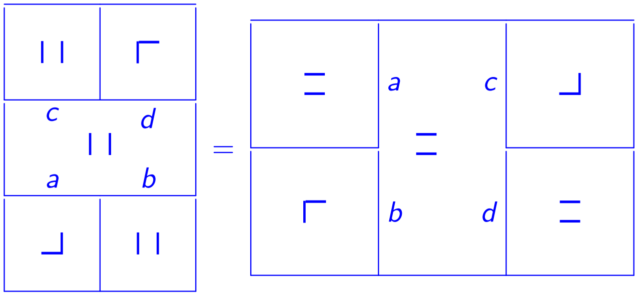
With these one can prove:

Any composition of commutative 3-cubes is commutative.

Generalisations and broader context by Philip Higgins TAC, 2005.

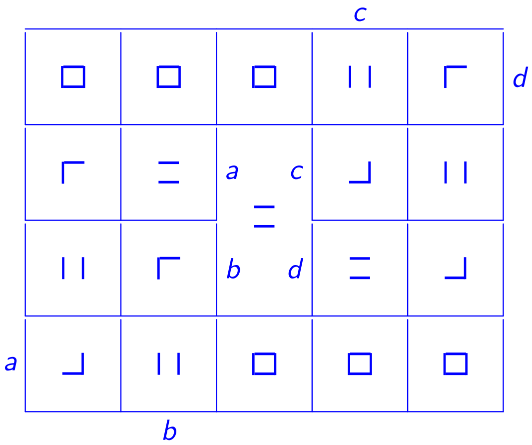
2-dimensional rewriting

Assuming $ab = cd$ we want to prove the following:



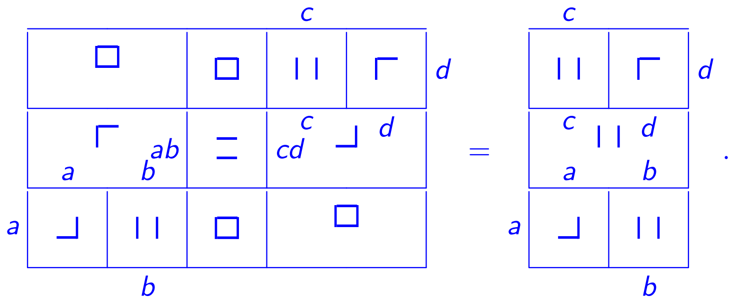
2-dimensional rewriting

To prove this we construct a common ‘subdivision’. One that is appropriate for this case is:



2-dimensional rewriting

From this diagram, we may compose parts of the second and third rows using the transport law and then rearrange things once more, getting the left hand side of the above as indicated below



Higher dimensional algebra

Algebraic systems of an hierarchical nature, with operations whose domains are defined under geometric conditions.

Applications to higher dimensional, nonabelian, local-to-global problems.

Able to express algebraic inverses to subdivision.

Applications to computer science, e.g. concurrency.

Higher order rewriting, including logged rewriting.