ALGEBRAIC COLIMIT CALCULATIONS IN HOMOTOPY THEORY USING FIBRED AND COFIBRED CATEGORIES

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Abstract. Higher Homotopy van Kampen Theorems allow some colimit calculations of certain homotopical invariants of glued spaces. One corollary is to describe homotopical excision in critical dimensions in terms of induced modules and crossed modules over groupoids. This paper shows how fibred and cofibred categories give an overall context for discussing and computing such constructions, allowing one result to cover many cases. A useful general result is that the inclusion of a fibre of a fibred category preserves connected colimits. The main homotopical applications are to pairs of spaces with several base points; we also describe briefly applications to triads.

Contents

1. Introduction

Our first aim is to give the framework of fibred and cofibred categories to certain colimit calculations of algebraic homotopical invariants for spaces with parts glued together: the data here is information on the invariants of the parts, and of the gluing process.

The second aim is to advertise the possibility of such calculations, which are based on various Higher Homotopy van Kampen Theorems (HHvKTs)[1], published in 1978-87.

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[1]This name for certain generalisations of van Kampen's Theorem was recently suggested by Jim Stasheff.

These are of the form that a *homotopically defined* functor

$$\Pi : (\text{topological data}) \to (\text{algebraic data})$$

preserves certain colimits [BH78, BH81b, BL87a], and where the algebraic data contains non-Abelian information.

This method gives for example some computations of the homotopy group $\pi_n(X, a), n \geqslant 2$ as a module over the fundamental group $\pi_1(X, a)$. However we would also like to take advantage of the extensions from groups to groupoids [Bro67, Bro06, BRS84] of the van Kampen in the form given by Crowell in [Cro59]. Thus we need to consider many base points and the extra variety of morphisms that the category of groupoids has over the category of groups. Among these are what are called *universal morphisms* in [Hig71, Bro06]; these correspond topologically to identifications of some elements of a discrete subspace of a topological space X, as shown in [Bro06, 9.1.2 (Corollary 3)].

Consider the following maps of spaces involving the n-sphere S^n, the first an inclusion and the second an identification:

$$S^n \xrightarrow{i} S^n \vee [0, 2] \xrightarrow{p} S^n \vee S^1 \vee S^1. \tag{1}$$

Here $n \geqslant 2$, the base point $\mathbf{0}$ of S^n is in the second space identified with 0, and the map p identifies $0, 1, 2$ in the interval $[0, 2]$ to give the wedge of two circles $S^1 \vee S^1$. The result previously cited gives that $\pi_1(S^n \vee S^1 \vee S^1, 0) \cong F_2$, the free group on 2 elements, regarded as formed from the indiscrete groupoid on $\{0, 1, 2\}$ by identifying $\{0, 1, 2\}$ to a point. The HHvKT, [BH81b, Theorem D], implies that $\pi_n(S^n \vee S^1 \vee S^1, 0)$ is the free F_2 module on one generator. More generally, if Y is connected and satisfies $\pi_r(Y, a) = 0$ for $1 < r < n$ then we can identify $\pi_n(Y \vee K(G, 1), a)$ where G is a group as the module *induced* from the $\pi_1(Y, a)$-module $\pi_n(Y, a)$ by the morphism of groups $\pi_1(Y, a) \to \pi_1(Y, a) * G$. This is a special case of a Homotopical Excision Theorem 9.3 which we give here in a many base point version of that stated in [BH81b, Theorem E]. This example could be obtained using universal coverings and homology, but these traditional methods are awkward for analogous results using many base points, and cannot give nonabelian results on induced crossed modules and crossed squares, i.e. for second relative homotopy groups and for third triad homotopy groups respectively.

The proofs of the theorems of [BH81b] which we apply require the construction and properties given there of the cubical higher homotopy ω-groupoid ρX_*, and its relation to the fundamental crossed complex ΠX_*, of the filtered space of X_*.

Fibrations and cofibrations of categories are useful for understanding this range of hierarchical constructions and their applications.

Our categorical results are for a fibration of categories $\Phi : \mathsf{X} \to \mathsf{B}$. The main new result (Theorem 2.7) is that the inclusion $\mathsf{X}_I \to \mathsf{X}$ of a fibre X_I of Φ preserves colimits of connected diagrams. Our second set of results relates pushouts in X to the construction of the 'inducing' functors $u_* : \mathsf{X}_I \to \mathsf{X}_J$ for u a morphism in B (Proposition 4.2). We show how the combination of these results uses the computation of colimits in B and in each X_I to give the computation of colimits in X (Theorem 4.4).

These general results are developed in the spirit of 'categories for the working mathematician' in sections 2 to 4. We illustrate the use of these results for homotopical calculations not only in groupoids (section 5), but also for crossed modules (section 7) and for modules, in both cases over groupoids. The natural invariant to consider when X is obtained from A by adding n-cells at various base points is $\pi_n(X, A, A_0)$ where A_0 is an appropriate set of base points. We discuss the relation between a module over a groupoid and the restriction to the vertex groups in section 8 in the general context of crossed complexes.

Section 9 gives the applications to excision and the relative Hurewicz theorem.

Finally in section 10 we give a brief account of some applications to crossed squares, as an indication of a more extensive theory in which these ideas can be applied.

We are grateful to Thomas Streicher for his Lecture Notes [Str99], on which the following account of fibred categories is based, and for helpful comments leading to improvements in the proofs. For further accounts of fibred and cofibred categories, see [Gra66, Bor94, Vis04] and the references there. The first paper gives analogies between fibrations of categories and Hurewicz fibrations of spaces.

2. Fibrations of categories

The new result here is Theorem 2.7. We recall the definition of fibration of categories.

2.1. Definition. Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a functor. A morphism $\varphi : Y \to X$ in X over $u := \Phi(\varphi)$ is called *cartesian* if and only if for all $v : K \to J$ in B and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\psi : Z \to Y$ with $\Phi(\psi) = v$ and $\theta = \varphi\psi$.

This is illustrated by the following diagram:

$$\begin{tikzcd} Z \arrow[r, dotted, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, "\varphi"'] & X & \\ & & & {} \arrow[d, "\Phi"] \\ & & & {} \\ K \arrow[r, "v"'] \arrow[rr, bend left, "uv"] & J \arrow[r, "u"'] & I & \end{tikzcd}$$

■

It is straightforward to check that cartesian morphisms are closed under composition, and that φ is an isomorphism if and only if φ is a cartesian morphism over an isomorphism.

A morphism $\alpha : Z \to Y$ is called *vertical* (with respect to Φ) if and only if $\Phi(\alpha)$ is an identity morphism in B. In particular, for $I \in \mathsf{B}$ we write X_I, called the *fibre over* I, for the subcategory of X consisting of those morphisms α with $\Phi(\alpha) = \mathrm{id}_I$.

2.2. Definition. The functor $\Phi : \mathsf{X} \to \mathsf{B}$ is a *fibration* or *category fibred over* B if and only if for all $u : J \to I$ in B and $X \in \mathsf{X}_I$ there is a cartesian morphism $\varphi : Y \to X$ over u: such a φ is called a *cartesian lifting* of X along u. ■

Notice that cartesian liftings of $X \in \mathsf{X}_I$ along $u : J \to I$ are unique up to vertical isomorphism: if $\varphi : Y \to X$ and $\psi : Z \to X$ are cartesian over u, then there exist vertical arrows $\alpha : Z \to Y$ and $\beta : Y \to Z$ with $\varphi\alpha = \psi$ and $\psi\beta = \varphi$ respectively, from which it follows by cartesianness of φ and ψ that $\alpha\beta = id_Y$ and $\beta\alpha = \mathrm{id}_Z$ as $\psi\beta\alpha = \varphi\alpha = \psi = \psi\,\mathrm{id}_Y$ and similarly $\varphi\beta\alpha = \varphi\,\mathrm{id}_Y$.

2.3. Example. The forgetful functor, $\mathrm{Ob} : \mathsf{Gpd} \to \mathsf{Set}$, from the category of groupoids to the category of sets is a fibration. We can for a groupoid G over I and function $u : J \to I$ define the cartesian lifting $\varphi : H \to G$ as follows: for $j, j' \in J$ set

$$H(j, j') = \{(j, g, j') \mid g \in G(uj, uj')\}$$

with composition

$$(j_1, g_1, j_1')(j, g, j') = (j_1, g_1 g, j'),$$

with φ given by $\varphi(j, g, j') = g$. The universal property is easily verified. The groupoid H is usually called the *pullback* of G by u. This is a well known construction (see for example [Mac05, §2.3], where pullback by u is written $u^{\downarrow\downarrow}$). ■

2.4. Definition. If $\Phi : \mathsf{X} \to \mathsf{B}$ is a fibration, then using the axiom of choice for classes we may select for every $u : J \to I$ in B and $X \in \mathsf{X}_I$ a cartesian lifting of X along u

$$u^X : u^* X \to X.$$

Such a choice of cartesian liftings is called a *cleavage* or *splitting* of Φ.

If we fix the morphism $u : J \to I$ in B, the splitting gives a so-called *reindexing functor*

$$u^* : \mathsf{X}_I \to \mathsf{X}_J$$

defined on objects by $X \mapsto u^* X$ and the image of a morphism $\alpha : X \to Y$ is $u^*\alpha$ the unique vertical arrow commuting the diagram:

$$\begin{tikzcd}
u^*X \arrow[r, "u^X"] \arrow[d, dashed, "u^*\alpha"'] & X \arrow[d, "\alpha"] \\
u^*Y \arrow[r, "u^Y"'] & Y
\end{tikzcd}$$

■

We can use this reindexing functor to get an adjoint situation for each $u : J \to I$ in B.

2.5. Proposition. *Suppose $\Phi : \mathsf{X} \to \mathsf{B}$ is a fibration of categories, $u : J \to I$ in B, and a reindexing functor $u^* : \mathsf{X}_I \to \mathsf{X}_J$ is chosen. Then there is a bijection*

$$\mathsf{X}_J(Y, u^* X) \cong \mathsf{X}_u(Y, X)$$

natural in $Y \in \mathsf{X}_J$, $X \in \mathsf{X}_I$ where $\mathsf{X}_u(Y, X)$ consists of those morphisms $\alpha \in \mathsf{X}(Y, X)$ with $\Phi(\alpha) = u$.

Proof. This is just a restatement of the universal properties concerned. ■

In general for composable maps $u : J \to I$ and $v : K \to J$ in B it does not hold that

$$v^* u^* = (uv)^*$$

as may be seen with the fibration of Example 2.3. Nevertheless there is a natural equivalence $c_{u,v} : v^*u^* \simeq (uv)^*$ as shown in the following diagram in which the full arrows are cartesian and where $(c_{u,v})_X$ is the unique vertical arrow making the diagram commute:

$$\begin{tikzcd}
v^*u^*X \arrow[r, "v^{u^*X}"] \arrow[d, dashed, "(c_{u,v})_X"', "\cong"] & u^*X \arrow[d, "u^X"] \\
(uv)^*X \arrow[r, "(uv)^X"'] & X
\end{tikzcd}$$

Let us consider this phenomenon for our main examples:

2.6. Example. 1.- Typically, for $\Phi_{\mathsf{B}} = \partial_1 : \mathsf{B}^2 \to \mathsf{B}$, the fundamental fibration for a category with pullbacks, we do not know how to choose pullbacks in a functorial way.

2.- In considering groupoids as a fibration over sets, if $u : J \to I$ is a function, we have a reindexing functor, also called *pullback*, $u^* : \mathsf{Gpd}_I \to \mathsf{Gpd}_J$. We notice that v^*u^*Q is naturally isomorphic to, but not identical to $(uv)^*Q$. ■

A result which aids understanding of our calculation of pushouts and some other colimits of groupoids, modules, crossed complexes and higher categories is the following. Recall that a category C is *connected* if for any $c, c' \in C$ there is a sequence of objects $c_0 = c, c_1, \ldots, c_{n-1}, c_n = c'$ such that for each $i = 0, \ldots, n-1$ there is a morphism $c_i \to c_{i+1}$ or $c_{i+1} \to c_i$ in C. The sequence of morphisms arising in this way is called a *zig-zag* from c to c' of length n.

2.7. Theorem. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a fibration, and let $J \in \mathsf{B}$. Then the inclusion $i_J : \mathsf{X}_J \to \mathsf{X}$ preserves colimits of connected diagrams.*

Proof. We will need the following diagrams:

$$
\begin{tikzcd}
T(c) \arrow[d, "\psi(c)"] \arrow[dr, "\theta(c)"] \arrow[dd, bend right=60, "T(f)"'] & \\
Y \arrow[r, "\varphi"] & X \\
T(c') \arrow[u, "\psi(c')"] \arrow[ur, "\theta(c')"'] & \\
& {} \arrow[d, "\Phi"] \\
J \arrow[rr, dotted, "u"'] & {} & I
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T(c) \arrow[d, "\psi(c)"] \arrow[dr, "\theta(c)"] \arrow[dd, bend right=60, "\gamma(c)"'] & \\
Y \arrow[r, "\varphi"] & X \\
L \arrow[u, dotted, "\psi'"] \arrow[ur, dotted, "\theta'"'] & \\
& {} \arrow[d, "\Phi"] \\
J \arrow[rr, dotted, "u"'] & {} & I
\end{tikzcd}
$$

(a) (b)

Let $T : \mathsf{C} \to \mathsf{X}_J$ be a functor from a small connected category C and suppose T has a colimit $L \in \mathsf{X}_J$. So we regard L as a constant functor $L : \mathsf{C} \to \mathsf{X}_J$ which comes with a universal cocone $\gamma : T \Rightarrow L$ in X_J. Let $i_J : \mathsf{X}_J \to \mathsf{X}$ be the inclusion. We prove that the natural transformation $i_J\gamma : i_J T \Rightarrow i_J L$ is a colimit cocone in X.

We use the following lemma.

2.8. Lemma. *Let $X \in \mathsf{X}$, with $\Phi(X) = I$, be regarded as a constant functor $X : \mathsf{C} \to \mathsf{X}$ and let $\theta : i_J T \Rightarrow X$ be a natural transformation, i.e. a cocone. Then*
(i) for all $c \in \mathsf{C}$, $u = \Phi(\theta(c)) : J \to I$ in B is independent of c, and
(ii) the cartesian lifting $Y \to X$ of u determines a cocone $\psi : T \Rightarrow Y$.

Proof. The natural transformation θ gives for each object the morphism $\theta(c) : T(c) \to X$ in X. Since C is connected, induction on the length of a zig-zag shows it is sufficient to prove (i) when there is a morphism $f : c \to c'$ in C. By naturality of θ, $\theta(c')T(f) = \theta(c)$. But $\Phi T(f)$ is the identity, since T has values in X_J, and so $\Phi(\theta(c)) = \Phi(\theta(c'))$. Write $u = \Phi(\theta(c))$.

Since Φ is a fibration, there is a $Y \in \mathsf{X}_J$ and a cartesian lifting $\varphi : Y \to X$ of u. Hence for each $c \in \mathsf{C}$ there is a unique vertical morphism $\psi(c) : T(c) \to Y$ in X_J such that $\varphi\psi(c) = \theta(c)$. We now prove that ψ is a natural transformation $T \Rightarrow Y$ in X_J, where Y is regarded as a constant functor.

To this end, let $f : c \to c'$ be a morphism in X_J. We need to prove $\psi(c) = \psi(c')T(f)$.

The outer part of diagram (a) commutes, since θ is a natural transformation. The upper and lower triangles commute, by construction of φ. Hence

$$\varphi\psi(c) = \theta(c) = \theta(c')T(f) = \varphi\psi(c')T(f).$$

Now $T(f)$, $\psi(c)$ and $\psi(c')$ are all vertical. By the universal property of φ, $\psi(c) = \psi(c')T(f)$, i.e. the left hand cell commutes. That is, ψ is a natural transformation $T \Rightarrow Y$ in X_J. ■

To return to the theorem, since L is a colimit in X_J, there is a unique vertical morphism $\psi' : L \to Y$ in the right hand diagram (b) such that for all $c \in C$, $\psi'\gamma(c) = \psi(c)$. Let $\theta' = \varphi\psi' : L \to X$. This gives a morphism $\theta' : L \to X$ such that $\theta'\gamma(c) = \theta(c)$ for all c, and, again using universality of φ, this morphism is unique. ■

2.9. Remark. The connectedness assumption is essential in the Theorem. Any small category C is the disjoint union of its connected components. If $T : \mathsf{C} \to \mathsf{X}$ is a functor, and X has colimits, then $\operatorname{colim} T$ is the coproduct (in X) of the $\operatorname{colim} T_i$ where T_i is the restriction of T to a component C_i. But given two objects in the same fibre of $\Phi : \mathsf{X} \to \mathsf{B}$, their coproduct in that fibre is in general not the same as their coproduct in X. For example, the coproduct of two groups in the category of groups is the free product of groups, while their coproduct as groupoids is their disjoint union. ■

2.10. Remark. A common application of the theorem is that the inclusion $\mathsf{X}_J \to \mathsf{X}$ preserves- pushouts. This is relevant to our application of pushouts in section 4. Pushouts are used to construct free crossed modules as a special case of induced crossed modules, [BH78], and to construct free crossed complexes as explained in [BH91, BG89]. ■

2.11. Remark. George Janelidze has pointed out a short proof of Theorem 2.7 in the case Φ has a right adjoint, and so preserves colimits, which applies to our main examples here. If the image of T is inside $\Phi(b)$, then ΦT is the constant diagram whose value is $\{b, 1_b\}$, and if C is connected this implies that $\operatorname{colim} \Phi T = b$. But if $\Phi(\operatorname{colim} T) = \operatorname{colim} \Phi T = b$, then $\operatorname{colim} T$ is inside $\Phi(b)$. ■

3. Cofibrations of categories

We now give the duals of the above results.

3.1. Definition. Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a functor. A morphism $\psi : Z \to Y$ in X over $v := \Phi(\psi)$ is called *cocartesian* if and only if for all $u : J \to I$ in B and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\varphi : Y \to X$ with $\Phi(\varphi) = u$ and $\theta = \varphi\psi$.

This is illustrated by the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\varphi"'] & X & & {} \arrow[d, "\Phi"] \\
K \arrow[r, "v"'] \arrow[rr, bend left, "uv"] & J \arrow[r, "u"'] & I & & {}
\end{tikzcd}$$

■

It is straightforward to check that cocartesian morphisms are closed under composition, and that ψ is an isomorphism if and only if ψ is a cocartesian morphism over an isomorphism.

3.2. Definition. The functor $\Phi : \mathsf{X} \to \mathsf{B}$ is a *cofibration* or *category cofibred over* B if and only if for all $v : K \to J$ in B and $Z \in \mathsf{X}_K$ there is a cartesian morphism $\psi : Z \to Z'$ over v: such a ψ is called a *cocartesian lifting* of Z along v. ■

The cocartesian liftings of $Z \in \mathsf{X}_K$ along $v : K \to J$ are also unique up to vertical isomorphism.

3.3. Remark. As in Definition 2.4, if $\Phi : \mathsf{X} \to \mathsf{B}$ is a cofibration, then using the axiom of choice for classes we may select for every $v : K \to J$ in B and $Z \in \mathsf{X}_K$ a cocartesian lifting of Z along v

$$v_Z : Z \to v_* Z.$$

Under these conditions, the functor v_* is commonly said to give the objects *induced* by v. Examples of induced crossed modules of groups are developed in [BW03], following on from the first definition of these in [BH78]. ■

We now have the dual of Proposition 2.5.

3.4. Proposition. *For a cofibration $\Phi : \mathsf{X} \to \mathsf{B}$, a choice of cocartesian liftings of $v : K \to J$ in B yields a functor $v_* : \mathsf{X}_K \to \mathsf{X}_J$, and an adjointness*

$$\mathsf{X}_J(v_* Z, Y) \cong \mathsf{X}_v(Z, Y)$$

for all $Y \in \mathsf{X}_J$, $Z \in \mathsf{X}_K$.

We now state the dual of Theorem 2.7.

3.5. Theorem. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a category cofibred over B. Then the inclusion of each fibre of Φ into X preserves limits of connected diagrams.*

Many of the examples we are interested in are both fibred and cofibred. For them we have an adjoint situation.

3.6. Proposition. *For a functor $\Phi : \mathsf{X} \to \mathsf{B}$ which is both a fibration and cofibration, and a morphism $u : J \to I$ in B, a choice of cartesian and cocartesian liftings of u gives an adjointness*

$$\mathsf{X}_J(Y, u^* X) \cong \mathsf{X}_I(u_* Y, X)$$

for $Y \in \mathsf{X}_J$, $X \in \mathsf{X}_I$.

The following result can be interpreted as saying that for a fibration of categories, 'locally cocartesian morphism implies cocartesian morphism'.

3.7. Proposition. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a fibration of categories. Then $\psi : Z \to Y$ in X over $v : K \to J$ in B is cocartesian if only if for all $\theta' : Z \to X'$ over v there is a unique morphism $\psi' : Y \to X'$ in X_J with $\theta' = \psi' \psi$.*

Proof. The 'only if' part is trivial. So to prove 'if' we have to prove that for any $u : J \to I$ and $\theta : Z \to X$ such that $\Phi(\theta) = uv$, there exists a unique $\varphi : Y \to X$ over u completing the diagram

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\varphi"'] & X & {} \arrow[d, "\Phi"] \\
K \arrow[r, "v"'] & J \arrow[r, "u"'] & I. & {}
\end{tikzcd}$$

Since Φ is a fibration there is a cartesian morphism $\kappa : X' \to X$ over u. By the cartesian property, there is a unique morphism $\theta' : Z \to X'$ over v such that $\kappa\theta' = \theta$, as in the diagram

$$\begin{tikzcd}
Z \arrow[r, dashed, "\theta'"'] \arrow[rr, bend left, "\theta"] & X' \arrow[r, "\kappa"'] & X.
\end{tikzcd}$$

Now, suppose $\varphi : Y \to X$ over $u : J \to I$ satisfies $\varphi\psi = \theta$, as in the diagram:

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] \arrow[dr, dashed, "\theta'"'] & Y \arrow[r, dotted, "\varphi"'] & X \\
& X' \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

By the given property of ψ there is a unique morphism $\psi' : Y \to X'$ in X_J such that $\psi'\psi = \theta'$. By the cartesian property of κ, there is a unique morphism φ' in X_J such that $\kappa\varphi' = \varphi$. Then

$$\kappa\psi'\psi = \kappa\theta' = \theta = \varphi\psi = \kappa\varphi'\psi.$$

By the cartesian property of κ, and since $\psi'\psi$, $\varphi'\psi$ are over uv, we have $\psi'\psi = \varphi'\psi$. By the given property of ψ, and since φ', ψ' are in X_J, we have $\varphi' = \psi'$. So $\varphi = \kappa\psi'$, and this proves uniqueness.

But we have already checked that $\kappa\psi'\psi = \theta$, so we are done. ∎

The following Proposition allows us to prove that a fibration is also a cofibration by constructing the adjoints u_* of u^* for every u.

3.8. Proposition. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a fibration of categories. Let $u : J \to I$ have reindexing functor $u^* : \mathsf{X}_I \to \mathsf{X}_J$. Then the following are equivalent:*

(i) *For all $Y \in \mathsf{X}_J$, there is a morphism $u_Y : Y \to u_*Y$ which is cocartesian over u;*

(ii) *there is a functor $u_* : \mathsf{X}_J \to \mathsf{X}_I$ which is left adjoint to u^*.*

Proof. That (ii) implies (i) is clear, using Proposition 3.7, since the adjointness gives the bijection required for the cocartesian property.

To prove that (i) implies (ii) we have to check that the allocation $Y \mapsto u_*(Y)$ gives a functor that is adjoint to u^*. As before the adjointness comes from the cocartesian property.

We leave to the reader the check the details of the functoriality of u_*. ∎

To end this section, we give a useful result on compositions.

3.9. Proposition. *The composition of fibrations, (cofibrations), is also a fibration (cofibration).*

Proof. We leave this as an exercise. ∎

4. Pushouts and cocartesian morphisms

Here is a small result which we use in this section and section 9, as it applies to many examples, such as the fibration $\mathrm{Ob} : \mathsf{Gpd} \to \mathsf{Set}$.

4.1. Proposition. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a functor that has a left adjoint D. Then for each $K \in \mathrm{Ob}\,\mathsf{B}$, $\mathsf{D}(K)$ is initial in X_K. In fact if $v : K \to J$ in B, then for any $X \in \mathsf{X}_J$ there is a unique morphism $\varepsilon_K : DK \to X$ over v.*

Proof. This follows immediately from the adjoint relation $\mathsf{X}_v(\mathsf{D}K, X) \cong \mathsf{B}(K, \Phi X)$ for all $X \in \mathrm{Ob}\,\mathsf{X}_J$. ∎

Special cases of cocartesian morphisms are used in [Bro06, BH78, BH81b], and we review these in section 9. A construction which arose naturally from the various Higher Homotopy van Kampen theorems is given a general setting as follows:

4.2. Theorem. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a fibration of categories which has a left adjoint D. Suppose that X admits pushouts. Let $v : K \to J$ be a morphism in B, and let $Z \in \mathsf{X}_K$. Then a cocartesian lifting $\psi : Z \to Y$ of v is given precisely by the pushout in X:*

$$\begin{tikzcd} \mathsf{D}(K) \arrow[r, "\mathsf{D}(v)"] \arrow[d, "\varepsilon_K"'] & \mathsf{D}(J) \arrow[d, "\varepsilon_J"] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd} \qquad (*)$$

Proof. Suppose first that diagram (*) is a pushout in X. Let $u : J \to I$ in B and let $\theta : Z \to X$ satisfy $\Phi(\theta) = uv$, so that $\Phi(X) = I$. Let $f : D(J) \to X$ be the adjoint of

$u : J \to \Phi(X)$.

$$\begin{tikzcd}
\mathsf{D}(K) \arrow[r, "\mathsf{D}(v)"] \arrow[d, "\varepsilon_K"'] & \mathsf{D}(J) \arrow[d, "\varepsilon_J"'] \arrow[ddr, "f"] & \\
Z \arrow[r, "\psi"] \arrow[drr, "\theta"'] & Y \arrow[dr, dotted, "\varphi"] & \\
& & X
\end{tikzcd} \tag{**}$$

$$K \xrightarrow[v]{} J \xrightarrow[u]{} I$$

Then $\Phi(fD(v)) = uv = \Phi(\theta\varepsilon_K)$ and so by Proposition 4.1, $fD(v) = \theta\varepsilon_K$. The pushout property implies there is a unique $\varphi : Y \to X$ such that $\varphi\psi = \theta$ and $\varphi\varepsilon_J = f$. This last condition gives $\Phi(\varphi) = u$ since $u = \Phi(f) = \Phi(\varphi\varepsilon_J) = \Phi(\varphi)\,\mathrm{id}_J = \Phi(\varphi)$.

For the converse, we suppose given $f : \mathsf{D}(J) \to X$ and $\theta : Z \to X$ such that $\theta\varepsilon_K = f\mathsf{D}(v)$. Then $\Phi(\theta) = uv$ and so there is a cocartesian lifting $\varphi : Y \to X$ of u. The additional condition $\varphi\varepsilon_J = f$ is immediate by Proposition 4.1. ∎

4.3. Corollary. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a fibration which has a left adjoint and suppose that X admits pushouts. Then Φ is also a cofibration.*

In view of the construction of hierarchical homotopical invariants as colimits from the HHvKTs in [BH81b] and [BL87a], the following is worth recording, as a consequence of Theorem 2.7.

4.4. Theorem. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be fibred and cofibred. Assume B and all fibres X_I are cocomplete. Let $T : \mathsf{C} \to \mathsf{X}$ be a functor from a small connected category. Then* $\operatorname{colim} T$ *exists and may be calculated as follows:*

(i) *First calculate $I = \operatorname{colim}(\Phi T)$, with cocone $\gamma : \Phi T \Rightarrow I$;*

(ii) *for each $c \in \mathsf{C}$ choose cocartesian morphisms $\gamma'(c) : T(c) \to X(c)$, over $\gamma(c)$ where $X(c) \in \mathsf{X}_I$;*

(iii) *make $c \mapsto X(c)$ into a functor $X : \mathsf{C} \to \mathsf{X}_I$, so that γ' becomes a natural transformation $\gamma' : T \Rightarrow X$;*

(iv) *form $Y = \operatorname{colim} X \in \mathsf{X}_I$ with cocone $\mu : X \Rightarrow Y$.*

Then Y with $\mu\gamma' : T \Rightarrow Y$ is $\operatorname{colim} T$.

Proof. We first explain how to make X into a functor.

We will in stages build up the following diagram:

$$
\begin{tikzcd}
& T(c) \arrow[rr, "\gamma'(c)"'] \arrow[dr, "T(f)"'] \arrow[rrrr, bend left, "\eta"] & & X(c) \arrow[r, "\mu(c)"] \arrow[d, dashed, "X(f)"] & Y \arrow[r, dotted, "\tau"'] \arrow[d, "1"] & Z \\
{} \arrow[d, "\Phi"] & & T(c') \arrow[r, "\gamma'(c')"'] & X(c') \arrow[r, "\mu(c')"'] & Y \arrow[r, dotted, "\tau'"'] & Z \\
{} & K \arrow[r, "\Phi T(f)"'] & J \arrow[r, "\gamma(c')"'] & I \arrow[r, "1"'] & I \arrow[r, "w"'] & H
\end{tikzcd}
\tag{2}
$$

Let $f : c \to c'$ be a morphism in C, $K = \Phi T(c), J = \Phi T(c')$. By cocartesianness of $\gamma'(c)$, there is a unique vertical morphism $X(f) : X(c) \to X(c')$ such that $X(f)\gamma'(c) = \gamma'(c')T(f)$. It is easy to check, again using cocartesianness, that if further $g : c' \to c''$, then $X(gf) = X(g)X(f)$, and $X(1) = 1$. So X is a functor and the above diagram shows that γ' becomes a natural transformation $T \Rightarrow X$.

Let $\eta : T \Rightarrow Z$ be a natural transformation to a constant functor Z, and let $\Phi(Z) = H$. Since $I = \mathrm{colim}(\Phi T)$, there is a unique morphism $w : I \to H$ such that $w\gamma = \Phi(\eta)$.

By the cocartesian property of γ', there is a natural transformation $\eta' : X \Rightarrow Z$ such that $\eta'\gamma' = \eta$.

Since Y is also a colimit in X of X, we obtain a morphism $\tau : Y \to Z$ in X such that $\tau\mu = \eta'$. Then $\tau\mu\gamma' = \eta'\gamma' = \eta$.

Let $\tau' : Y \to Z$ be another morphism such that $\tau'\mu\gamma' = \eta$. Then $\Phi(\tau) = \Phi(\tau') = w$, since I is a colimit. Again by cocartesianness, $\tau'\mu = \tau\mu$. By the colimit property of Y, $\tau = \tau'$. ■

This with Theorem 4.4 shows how to compute colimits of connected diagrams in the examples we discuss in sections 5 to 10, and in all of which a van Kampen type theorem is available giving colimits of algebraic data for some glued topological data.

4.5. Corollary. *Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a functor satisfying the assumptions of theorem 4.4. Then X is connected complete, i.e. admits colimits of all connected diagrams.*

5. Groupoids bifibred over sets

We have already seen in Example 2.3 that the functor Ob : $\mathsf{Gpd} \to \mathsf{Set}$ is a fibration. The proof that this functor is also a cofibration is given in essence in [Hig71, Bro06], where the cocartesian lifting of $u : I \to J$ to a groupoid G over I is written $u_* : G \to U_u(G)$ and called a *universal morphism*. The constructions given there in terms of words are useful in showing that u_* is injective on non-identity elements.

The functor Ob has both left and right adjoints, assigning respectively to a set I the discrete groupoid and indiscrete, or codiscrete, groupoids on I.

It also follows from general theorems on algebraic theories that the category Gpd is complete and cocomplete.

Clear examples of these cocartesian liftings in applications of the groupoid van Kampen theorem are given in [Bro06, Chapter 9]. We extend these to the module case in the next section.

6. Groupoid modules bifibred over groupoids

Modules over groupoids are a useful generalisation of modules over groups, and also form part of the basic structure of crossed complexes. Homotopy groups $\pi_n(X; X_0), n \geqslant 2$, of a space X with a set X_0 of base points form a module over the fundamental groupoid $\pi_1(X, X_0)$, as do the relative homotopy groups $\pi_n(Y, X : X_0), n \geqslant 3$, of a pair (Y, X).

6.1. Definition. A *module over a groupoid* is a pair (M, G), where G is a groupoid with set of objects G_0, M is a totally disconnected abelian groupoid with the same set of objects as G, and with a given action of G on M. Thus M comes with a *target function* $t : M \to G_0$, and each $M(x) = t^{-1}(x), x \in G_0$, has the structure of Abelian group. The G-action is given by a family of maps

$$M(x) \times G(x, y) \to M(y)$$

for all $x, y \in G_0$. These maps are denoted by $(m, p) \mapsto m^p$ and satisfy the usual properties, i.e. $m^1 = m$, $(m^p)^{p'} = m^{(pp')}$ and $(m + m')^p = m^p + m'^p$, whenever these are defined. In particular, any $p \in G(x, y)$ induces an isomorphism $m \mapsto m^p$ from $M(x)$ to $M(y)$.

A *morphism of modules* is a pair $(\theta, f) : (M, G) \to (N, H)$, where $f : G \to H$ and $\theta : M \to N$ are morphisms of groupoids and preserve the action. That is, θ is given by a family of group morphisms

$$\theta(x) : M(x) \to N(f(x))$$

for all $x \in G_0$ satisfying $\theta(y)(m^p) = (\theta(x)(m))^{f(p)}$, for all $p \in G(x, y), m \in M(x)$.

This defines the category Mod having modules over groupoids as objects and the morphisms of modules as morphisms. If (M, G) is a module, then $(M, G)_0$ is defined to be G_0.

We have a forgetful functor $\Phi_{\mathsf{M}} : \mathsf{Mod} \to \mathsf{Gpd}$ in which $(M, G) \mapsto G$.

6.2. Proposition. *The forgetful functor* $\Phi_{\mathsf{M}} : \mathsf{Mod} \to \mathsf{Gpd}$ *has a left adjoint and is fibred and cofibred.*

Proof. The left adjoint of Φ_{M} assigns to a groupoid G the module written $0 \to G$ which has only the trivial group over each $x \in G_0$.

Next, we give the pullback construction to prove that Φ_{M} is fibred. This is entirely analogous to the group case, but taking account of the geometry of the groupoid.

So let $v : G \to H$ be a morphism of groupoids, and let (N, H) be a module. We define $(M, G) = v^*(N, H)$ as follows. For $x \in G_0$ we set $M(x) = \{x\} \times N(vx)$ with addition given by that in $N(vx)$. The operation is given by $(x, n)^p = (y, n^{vp})$ for $p \in G(x, y)$.

The construction of $N = v_*(M,G)$ for (M,G) a G-module is as follows.

For $y \in H_0$ we let $N(y)$ be the abelian group generated by pairs (m,q) with $m \in M, q \in H$, and $t(q) = y, s(q) = v(t(m))$, so that $N(y) = 0$ if no such pairs exist. The operation of H on N is given by $(m,q)^{q'} = (m, qq')$, addition is $(m,q) + (m',q) = (m+m',q)$ and the relations imposed are $(m^p, q) = (m, v(p)q)$ when these make sense. The cocartesian morphism over v is given by $\psi : m \mapsto (m, 1_{vt(m)})$. ∎

We now describe free modules over groupoids in terms of the inducing construction. The interest of this is two fold. Firstly, induced modules arise in homotopy theory from a HHvKT, and we get new proofs of results on free modules in homotopy theory. Secondly, this indicates the power of the HHvKT since it gives new results.

6.3. Definition. Let Q be a groupoid. A *free basis* for a module (N,Q) consists of a pair of functions $t_B : B \to Q_0$, $i : B \to N$ such that $t_N i = t_B$ and with the universal property that if (L,Q) is a module and $f : B \to L$ is a function such that $t_L f = t_N$ then there is a unique Q-module morphism $\varphi : N \to L$ such that $\varphi i = f$.

6.4. Proposition. *Let B be an indexing set, and Q a groupoid. The free Q-module $(FM(t),Q)$ on $t : B \to Q_0$ may be described as the Q-module induced by $t : B \to Q$ from the discrete free module $\mathbb{Z}B = (\mathbb{Z} \times B, B)$ on B, where B denotes also the discrete groupoid on B.*

Proof. This is a direct verification of the universal property. ∎

6.5. Remark. Proposition 3.7 shows that the universal property for a free module can also be expressed in terms morphisms of modules $(FM(t),Q) \to (L,R)$. ∎

7. Crossed modules bifibred over groupoids

Our homotopical example here is the family of second relative homotopy groups of a pair of spaces with many base points.

A *crossed module*, [BH81a], consists first of a morphism of groupoids $\mu : M \to P$ of groupoids with the same set P_0 of objects such that μ is the identity on objects, and M is a family of groups $M(x), x \in P_0$; second, there is an action of P on the family of groups M so that if $m \in M(x)$ and $p \in P(x,y)$ then $m^p \in M(y)$. This action must satisfy the usual axioms for an action with the additional properties:
CM1) $\mu(m^p) = p^{-1}\mu(m)p$, and
CM2)[2] $m^{-1}nm = n^{\mu m}$
for all $p \in P$, $m, n \in M$ such that the equations make sense. Crossed modules form the objects of the category XMod in which a morphism is a commutative square of morphisms

[2]An early appearance of this rule is in footnote 25 on p. 422 of [Whi41].

of groups

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ P & \xrightarrow[f]{} & Q \end{array}$$

which preserve the action in the sense that $g(m^p) = (gm)^{fp}$ whenever this makes sense.

The category XMod is equivalent to the well known category $2\text{-}\mathsf{Gpd}$ of 2-groupoids, [BH81c]. However the advantages of XMod over 2-groupoids are:

- crossed modules are closer to the classical invariants of relative homotopy groups;
- the notion of freeness is clearer in XMod and models a standard topological situation, that of attaching 1- and 2-cells;
- the category XMod has a monoidal closed structure which helps to define a notion of homotopy; these constructions are simpler to describe in detail than those for 2-groupoids, and they extend to all dimensions.

We have a forgetful functor $\Phi_1 : \mathsf{XMod} \to \mathsf{Gpd}$ which sends $(M \to P) \mapsto P$. Our first main result is:

7.1. Proposition. *The forgetful functor $\Phi_1 : \mathsf{XMod} \to \mathsf{Gpd}$ is fibred and has a left adjoint.*

Proof. The left adjoint of Φ_1 assigns to a groupoid P the crossed module $0 \to P$ which has only the trivial group over each $x \in P_0$.

Next, we give the pullback construction to prove that Φ_1 is fibred. So let $f : P \to Q$ be a morphism of groupoids, and let $\nu : N \to Q$ be a crossed module. We define $M = \nu^*(N)$ as follows.

For $x \in P_0$ we set $M(x)$ to be the subgroup of $P(x) \times N(fx)$ of elements (p, n) such that $fp = \nu n$. If $p_1 \in P(x, x'), n \in N(fx)$ we set $(p, n)^{p_1} = (p_1^{-1}pp_1, n^{f(p_1)})$, and let $\mu : (p, n) \mapsto p$. We leave the reader to verify that this gives a crossed module, and that the morphism $(p, n) \mapsto n$ is cartesian. ∎

The following result in the case of crossed modules of groups appeared in [BH78], described in terms of the crossed module $\partial : u_*(M) \to Q$ *induced* from the crossed module $\mu : M \to P$ by a morphism $u : P \to Q$.

7.2. Proposition. *The forgetful functor $\Phi_1 : \mathsf{XMod} \to \mathsf{Gpd}$ is cofibred.*

Proof. We prove this by a direct construction.

Let $\mu : M \to P$ be a crossed module, and let $f : P \to Q$ be a morphism of groupoids. The construction of $N = f_*(M)$ and of $\partial : N \to Q$ requires just care to the geometry of the partial action in addition to the construction for the group case initiated in [BH78] and pursued in [BW03] and the papers referred to there.

Let $y \in Q_0$. If there is no $q \in Q$ from a point of $f(P_0)$ to y, then we set $N(y)$ to be the trivial group.

Otherwise, we define $F(y)$ to be the free group on the set of pairs (m, q) such that $m \in M(x)$ for some $x \in P_0$ and $q \in Q(fx, y)$. If $q' \in Q(y, y')$ we set $(m, q)^{q'} = (m, qq')$. We define $\partial' : F(y) \to Q(y)$ to be $(m, q) \mapsto q^{-1}(fm)q$. This gives a precrossed module over $\partial : F \to Q$, with function $i : M \to F$ given by $m \mapsto (m, 1)$ where if $m \in M(x)$ then 1 here is the identity in $Q(fx)$.

We now wish to change the function $i : M \to F$ to make it an operator morphism. For this, factor F out by the relations

$$\begin{aligned} (m, q)(m', q) &= (mm', q), \\ (m^p, q) &= (m, (fp)q), \end{aligned}$$

whenever these are defined, to give a projection $F \to F'$ and $i' : M \to F'$. As in the group case, we have to check that $\partial' : F \to Q$ induces $\partial'' : F' \to H$ making this a precrossed module. To make this a crossed module involves factoring out Peiffer elements, whose theory is as for the group case in [BHu82]. This gives a crossed module morphism $(\varphi, f) : (M, P) \to (N, Q)$ which is cocartesian. ∎

We recall the algebraic origin of free crossed modules, but in the groupoid context.

Let P be a groupoid, with source and target functions written $s, t : P \to P_0$. A subgroupoid N of P is said to be *normal* in P, written $N \lhd P$, if N is wide in P, i.e. $N_0 = P_0$, and for all $x, y \in P_0$ and $a \in P(x, y)$, $a^{-1}N(x)a = N(y)$. If N is also totally intransitive, i.e. $N(x, y) = \emptyset$ when $x \neq y$, as we now assume, then the quotient groupoid P/N is easy to define. (It may also be defined in general but we will need only this case.)

Now suppose given a family $R(x), x \in P_0$ of subsets of $P(x)$. Then the normaliser $N_P(R)$ of R is well defined as the smallest normal subgroupoid of P containing all the sets $R(x)$. Note that the elements of $N_P(R)$ are all *consequences* of R in P, i.e. all well defined products of the form

$$c = (r_1^{\varepsilon_1})^{a_1} \dots (r_n^{\varepsilon_n})^{a_n}, \quad \varepsilon_i = \pm 1, a_i \in P, n \geqslant 0 \tag{3}$$

and where b^a denotes $a^{-1}ba$. The quotient $P/N_P(R)$ is also written P/R, and called the *quotient* of P by the relations $r = 1, r \in R$.

As in group theory, we need also to allow for repeated relations. So we suppose given a set R and a function $\omega : R \to P$ such that $s\omega = t\omega = \beta$, say. This 'base point function', saying where the relations are placed, is a useful part of the information.

We now wish to obtain 'syzygies' by replacing the normal subgroupoid by a 'free object' on the relations $\omega : R \to P$. As in the group case, this is done using *free crossed modules*.

7.3. Remark. There is a subtle reason for this use of crossed modules. A normal subgroupoid N of P (as defined above) gives a quotient object P/N in the category Gpd_X of groupoids with object set $X = P_0$. Alternatively, N defines a congruence on P, which is a particular kind of equivalence relation. Now an equivalence relation is in general a particular kind of subobject of a product, but in this case, we must take the product in the category Gpd_X. As a generalisation of this, one should take a groupoid object in the category Gpd_X. Since these totally disconnected normal subgroupoids determine equivalence relations on each $P(x, y)$ which are congruences, it seems clear that a groupoid object internal to Gpd_X is equivalent to a 2-groupoid with object set X. ∎

7.4. Definition. A *free basis* for a crossed module $\partial : C \to P$ over a groupoid P is a set R, function $\beta : R \to P_0$ and function $i : R \to C$ such that $i(r) \in C(\beta r), r \in R$, with the universal property that if $\mu : M \to P$ is a crossed module and $\theta : R \to M$ a function over the identity on P_0 such that $\mu\theta = \partial i$, then there is a unique morphism of crossed P-modules $\varphi : C \to M$ such that $\varphi i = \theta$. ∎

7.5. Example. Let R be a set and $\beta : R \to P_0$ a function. Let $\mathrm{id} : F_1(R) \to F_2(R)$ be the identity crossed module on two copies of $F(R)$, the disjoint union of copies $\mathsf{C}(r)$ of the infinite cyclic group C with generator $c_r \in \mathsf{C}(r)$. Thus $F_2(R)$ is a totally intransitive groupoid with object set R. Let $i : R \to F_1(R)$ be the function $r \mapsto c_r$. Let $\beta : R \to R$ be the identity function. Then $\mathrm{id} : F_1(R) \to F_2(R)$ is the free crossed module on i. The verification of this is simple from the diagram

$$\begin{tikzcd} R \arrow[r, "i"'] \arrow[rr, bend left, "\theta"] & F_1(R) \arrow[r, dashed, "f"'] \arrow[d, "\mathrm{id}"'] & M \arrow[d, "\mu"] \\ & F_2(R) \arrow[r, "\mathrm{id}"'] & F_2(R) \end{tikzcd}$$

The morphism f simply maps the generator c_r to θr. ∎

7.6. Proposition. *Let R be a set, and $\mu : M \to P$ a crossed module over the groupoid P. Let $\beta : R \to P_0$ be a function. Then the functions $i : R \to M$ such that $s\mu = t\mu = \beta$ are bijective with the crossed module morphisms (f, g)*

$$\begin{tikzcd} F_1(R) \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & M \arrow[d, "\mu"] \\ F_2(R) \arrow[r, "g"'] & P \end{tikzcd}$$

such that $sg = \beta$.

Further, the free crossed module $\partial : C(\omega) \to P$ on a function $\omega : R \to P$ such that $s\omega = t\omega = \beta$ is determined as the crossed module induced from $\mathrm{id} : F_1(R) \to F_2(R)$ *by the extension of ω to the groupoid morphism $F_2(R) \to P$.*

Proof. The first part is clear since $g = \mu f$ and f and i are related by $f(c_r) = i(r), r \in R$.

The second part follows from the first part and the universal property of induced crossed modules as shown in the following diagram:

$$\begin{tikzcd}
F_1(R) \arrow[r, "f"'] \arrow[d, "\mathrm{id}"'] \arrow[rr, bend left, "\theta"] & C(\omega) \arrow[r, dashed, "\varphi"'] \arrow[d, "\partial"] & M \arrow[d, "\mu"] \\
F_2(R) \arrow[r, "g"'] & P \arrow[r, "="'] & P
\end{tikzcd}$$

■

8. Crossed complexes and an HHvKT

Crossed complexes are analogous to chain complexes but also generalise groupoids to all dimensions and with their base points and operations relate dimensions 0,1 and n. The structure and axioms for a crossed complex are those universally satisfied by the main topological example, the *fundamental crossed complex* ΠX_* of a filtered space X_*, where $(\Pi X_*)_1$ is the fundamental groupoid $\pi_1(X_1, X_0)$ and for $n \geqslant 2$ $(\Pi X_*)_n$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, x_0)$ for all $x_0 \in X_0$, with associated operations of the fundamental groupoid and boundaries.

Crossed complexes fit into our scheme of algebraic structures over a range of dimensions satisfying a HHvKT in that the fundamental crossed complex functor

$$\Pi : (\text{filtered spaces}) \to (\text{crossed complexes})$$

preserves certain colimits. We state below the special case for pushouts.

A crossed complex C is in part a sequence of the form

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

where all the $C_n, n \geqslant 1$, are groupoids over C_0. Here $\delta_2 : C_2 \to C_1$ is a crossed module and for $n \geqslant 3$ (C_n, C_1) is a module. The further axioms are that δ_n is an operator morphism for $n \geqslant 2$ and that $\delta_2 c$ operates trivially on C_n for $c \in C_2$ and $n \geqslant 3$. We assume the basic facts on crossed complexes as surveyed in for example [Bro99, Bro04]. The category of crossed complexes is written Crs. A full exposition of the theory of crossed complexes will be given in [BHS09].

To state the *Higher Homotopy van Kampen Theorem for relative homotopy groups*, namely Theorem C of [BH81b, Section 5], we need the following definition:

8.1. Definition. A filtered space X_* is said to be *connected* if the following conditions hold for each $n \geqslant 0$:
- $\varphi(X_*, 0)$: If $r > 0$, the map $\pi_0 X_0 \to \pi_0 X_r$, induced by inclusion, is surjective; i.e. X_0 meets all path connected components of all stages of the filtration X_*.
-$\varphi(X_*, n)$(for $n \geqslant 1$): If $r > n$ and $x \in X_0$, then $\pi_n(X_r, X_n, x) = 0$. ■

8.2. THEOREM. [Higher Homotopy van Kampen Theorem] [BH81b, Theorem D] *Let X_* be a filtered space and suppose:*

(i) *X is the union of the interiors of subspaces U, V;*

(ii) *the filtrations U_*, V_* and W_*, formed by intersection with X_*, and where $W = U \cap V$, are connected filtrations. Then*

(Conn) *the filtration X_* is connected, and*
(Pushout) *the following diagram of morphisms of crossed complexes induced by inclusions*

$$\begin{tikzcd} \Pi W_* \arrow[r] \arrow[d] & \Pi U_* \arrow[d] \\ \Pi V_* \arrow[r] & \Pi X_* \end{tikzcd}$$

is a pushout of crossed complexes.

8.3. REMARK. The algebraic conclusion is the most important but the connectivity conclusion is also significant. The implications in dimension 2 are in general nonabelian and so unreachable by the traditional abelian methods. This theorem is proved without recourse to homology or simplicial approximation. Instead the proof uses the construction of ρX_*, the *cubical homotopy ω–groupoid of the filtered space X_**, defined in dimension n to be the set of filter homotopy classes rel vertices[3] of I^n of maps $I^n_* \to X_*$. The properties of this construction enable the proof of the 1-dimensional theorem van Kampen theorem to be generalised to higher dimensions, and the theorem on crossed complexes is deduced using [BH81b, Theorem 5.1], a non trivial equivalence between the two constructions. The connectivity conditions are used in the following way: a filtered map $I^n_* \to X_*$ may be subdivided into small cubes each of which maps into one of U, V; however such a small cube will not usually be a map $I^n_* \to X_*$; the connectivity condition is used to deform the subdivision by induction on skeleta into one which is such a map. The cubical homotopy ω-groupoid with connections ρX_* is found convenient for this inductive process in [BH81b]; cubical methods allow *algebraic inverses to subdivision*, and are convenient for constructing homotopies. ∎

8.4. REMARK. Colimits of crossed complexes may be computed from the colimits of the groupoids, crossed modules and modules from which they are constituted. A warning has to be given that some of the algebra is not as straightforward as that in traditional homological and homotopical algebra. For example in an abelian category, a pushout of

[3] The paper [BH81b] also assumes a J_0 condition on the filtered spaces; but this can be relaxed by the refined definition of making filter homotopies of maps $I^n_* \to X_*$ to be rel vertices, as has been advertised in [BH91]. The 1981 paper also proves a more general theorem, in which arbitrary unions lead to a coequaliser rather than a pushout.

the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & 0 \arrow[d] \\ B \arrow[r, "p"'] & C \end{tikzcd}$$

is equivalent to an exact sequence

$$A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

However in the category Mod_* of modules over groups a pushout of the form

$$\begin{tikzcd} (M,G) \arrow[r] \arrow[d, "i"'] & (0,1) \arrow[d] \\ (N,H) \arrow[r, "p"'] & (P,K) \end{tikzcd}$$

is equivalent to a pair of an exact sequence of groups

$$G \xrightarrow{i} H \xrightarrow{p} K \to 1,$$

and an exact sequence of induced modules over K

$$M \otimes_{\mathbb{Z}G} \mathbb{Z}K \xrightarrow{i} N \otimes_{\mathbb{Z}H} \mathbb{Z}K \xrightarrow{p} P \to 0.$$

This shows that pushouts give much more information in our case, but also shows that handling the information may be more difficult, and that misinterpretations could lead to false conjectures or proofs. ■

8.5. Proposition. *The truncation functor $tr^1 : \mathsf{Crs} \to \mathsf{Gpd}$, $C \mapsto C_1$, is a bifibration.*

Proof. The previous results give the constructions on modules and crossed modules. The functoriality of these constructions give the construction of the boundary maps, and the axioms for all these follow. ■

We will also need for later applications (Proposition 9.11) the relation of a connected crossed complex to the full, reduced (single vertex) crossed complex it contains, analogous to the well known relation of a connected groupoid to any of its vertex groups.

Recall that a *codiscrete groupoid* T is one on which $T(x, y)$ is a singleton for all objects $x, y \in T_0$. This is called a *tree groupoid* in [Bro06]. Similarly, a *codiscrete crossed complex* T is one in which the groupoid T_1 is codiscrete and which is trivial in higher dimensions.

We follow similar conventions for crossed complexes as for groupoids in [Bro06]. Thus if D and E are crossed complexes, and $S = D_0 \cap E_0$ then by the *free product* $D * E$ we mean the crossed complex given by the pushout of crossed complexes

$$\begin{tikzcd} S \arrow[r] \arrow[d] & E \arrow[d, "j"] \\ D \arrow[r, "i"'] & D * E \end{tikzcd} \qquad (4)$$

where the set S is identified with the discrete crossed complex which is S in dimension 0 and trivial in higher dimensions. The following result is analogous to and indeed includes standard facts for groupoids (cf. [Bro06, 6.7.3, 8.1.5]).

8.6. Proposition. *Let C be a connected crossed complex, let $x_0 \in C_0$ and let T be codiscrete wide subcrossed complex of C. Let $C(x_0)$ be the subcrossed complex of C at the base point x_0. Then the natural morphism*

$$\varphi : C(x_0) * T \to C$$

determined by the inclusions is an isomorphism, and T determines a strong deformation retraction

$$r : C \to C(x_0).$$

Further, if $f : C \to D$ is a morphism of crossed complexes which is the identity on $C_0 \to D_0$ then we can find a retraction $s : D \to D(x_0)$ giving rise to a pushout square

$$\begin{tikzcd} C \arrow[r, "r"] \arrow[d, "f"'] & C(x_0) \arrow[d, "f'"] \\ D \arrow[r, "s"'] & D(x_0) \end{tikzcd} \tag{5}$$

in which f' is the restriction of f.

Proof Let $i : C(x_0) \to C$, $j : T \to C$ be the inclusions. We verify the universal property of the free product. Let $\alpha : C(x_0) \to E, \beta : T \to E$ be morphisms of crossed complexes agreeing on x_0. Suppose $g : C \to E$ satisfies $gi = \alpha, gj = \beta$. Then g is determined on C_0. Let $c \in C_1(x, y)$. Then

$$c = (\tau x)((\tau x)^{-1}c(\tau y))(\tau y)^{-1} \tag{*}$$

and so

$$\begin{aligned} gc &= g(\tau x)g((\tau x)^{-1}c(\tau y))g(\tau y)^{-1} \\ &= \beta(\tau x)\alpha((\tau x)^{-1}c(\tau y))\beta(\tau y)^{-1}. \end{aligned}$$

If $c \in C_n(x), n \geqslant 2$, then

$$c = (c^{\tau x})^{(\tau x)^{-1}} \tag{**}$$

and so

$$g(c) = \alpha(c^{\tau x})^{\beta(\tau x)^{-1}}.$$

This proves uniqueness of g, and conversely one checks that this formula defines a morphism g as required.

In effect, equations (*) and (**) give for the elements of C normal forms in terms of elements of $C(x_0)$ and of T.

This isomorphism and the constant map $T \to \{x_0\}$ determine the strong deformation retraction $r : C \to C(x_0)$.

The retraction s is defined by the elements $f\tau(x), x \in C_0$, and then the diagram (5) is a pushout since it is a retract of the pushout square

$$\begin{tikzcd} C \arrow[r, "1"] \arrow[d, "f"'] & C \arrow[d, "f"] \\ D \arrow[r, "1"'] & D \end{tikzcd}$$ □

9. Homotopical excision and induced constructions

We now interpret the HHvKT (Theorem 8.2) when the filtrations have essentially just two stages.

9.1. Definition. By a *based pair* $X_\bullet = (X, X_1; X_0)$ of spaces we mean a pair (X, X_1) of spaces together with a set $X_0 \subseteq X_1$ of *base points*. For such a based pair and $n \geqslant 2$ we have an associated filtered space $X_\bullet^{[n]}$ which is X_0 in dimension 0, X_1 in dimensions $1 \leqslant i < n$ and X for dimensions $i \geqslant n$. We write $\Pi_n X_\bullet$ for the crossed complex $\Pi X_\bullet^{[n]}$. This crossed complex is trivial in all dimensions $\neq 1, n$, and in dimension n is the family of relative homotopy groups $\pi_n(X, X_1, x_0)$ for $x_0 \in X_0$, considered as a module (crossed module if $n = 2$) over the fundamental groupoid $\pi_1(X_1, X_0)$. Colimits of such crossed complexes are equivalent to colimits of the corresponding module or crossed module. ■

9.2. Remark. This definition allows a reformulation of Theorem 8.2 in terms of based pairs, which we will use in proving the application to Homotopical Excision. These methods were used in [BH78] for the single pointed case.

It is not easy to see for this based version of Theorem 8.2 a direct proof in terms of modules or crossed modules, since one needs the intermediate structure between 0 and n to use the connectivity conditions. ■

We now concentrate on excision, since this gives rise to cocartesian morphisms and so induced modules and crossed modules. The following is a generalisation to many base points of [BH81b, Theorem E].

9.3. Theorem. [Homotopy Excision Theorem] *Let the topological space X be the union of the interiors of sets U, V, and let $W = U \cap V$. Let $n \geqslant 2$. Let $W_0 \subseteq U_0 \subseteq U$ be such that the based pair $(V, W; W_0)$ is $(n-1)$-connected and U_0 meets each path component of U. Then $(X, U; U_0)$ is $(n-1)$-connected, and the morphism of modules (crossed if $n = 2$)*

$$\pi_n(V, W; W_0) \to \pi_n(X, U; U_0)$$

induced by inclusions is cocartesian over the morphism of fundamental groupoids

$$\pi_1(W, W_0) \to \pi_1(U, U_0)$$

induced by inclusion.

Proof. We deduce this excision theorem from the pushout theorem 8.2, applied to the filtrations deduced from the based pair $(X, U; U_0)$ and the following diagram of morphisms:

$$\begin{tikzcd} \Pi_n(W, W; W_0) \arrow[r] \arrow[d] & \Pi_n(U, U; U_0) \arrow[d] \\ \Pi_n(V, W; W_0) \arrow[r] & \Pi_n(X, U; U_0) \end{tikzcd} \tag{6}$$

This is a pushout of modules if $n \geqslant 3$ and of crossed modules if $n = 2$, by Theorem 8.2, applied to the associated filtrations $X_\bullet^{[n]}$ of the based pairs. However

$$\Pi_n(W, W; W_0) = (0, \pi_1(W, W_0)), \qquad \Pi_n(U, U; U_0) = (0, \pi_1(U, U_0)).$$

So the theorem follows from Theorem 4.2 and our discussion of the examples of induced modules and crossed modules. ■

9.4. Corollary. [Homotopical excision for an adjunction] *Let $i : W \to V$ be a closed cofibration and $f : W \to U$ a map. Let W_0 be a subset of W meeting each path component of W and V, and let U_0 be a subset of U meeting reach path component of U and such that $f(W_0) \subseteq U_0$. Suppose that the based pair (V, W) is $(n-1)$-connected. Let $X = U \cup_f V$. Then the based pair (X, U) is $(n-1)$-connected and the induced morphism of modules (crossed if $n = 2$)*

$$\pi_n(V, W; W_0) \to \pi_n(X, U; U_0)$$

is cocartesian over the induced morphism of fundamental groupoids

$$\pi_1(W, W_0) \to \pi_1(U, U_0).$$

Proof. This follows from Theorem 9.3 using mapping cylinders in a similar manner to the proof of a corresponding result for the fundamental groupoid [Bro06, 9.1.2]. That is, we form the mapping cylinder $Y = M(f) \cup W$. The closed cofibration assumption ensures that the projection from Y to $X = U \cup_f V$ is a homotopy equivalence. ■

9.5. Corollary. [Attaching n-cells] *Let the space Y be obtained from the space X by attaching n-cells, $n \geqslant 2$, at a set of base points A of X, so that $Y = X \cup_{f_\lambda} e^n_\lambda, \lambda \in \Lambda$, where $f_\lambda : (S^{n-1}, 0) \to (X, A)$. Then $\pi_n(Y, X; A)$ is isomorphic to the free $\pi_1(X, A)$-module (crossed if $n = 2$) on the characteristic maps of the n-cells.*

9.6. Remark. The previous corollary for $n = 2$ was a theorem of J.H.C. Whitehead. An account of Whitehead's proof is given in [Bro80]. There are several other proofs in the literature but none give the more general homotopical excision result, theorem 9.3. Part of our argument is that any restriction to a single base point and so to presentations of groups rather than of groupoids inhibits the natural expression of the theorems and proofs. Corollary 9.4 also deals conveniently with identifications in a discrete subspace, where the morphism of fundamental groupoids is described in the form $G \to U_u G$ in [Bro06, p. 343]. ■

9.7. Example. We can now explain the example in the Introduction. That S^n is $(n-1)$-connected and $\pi_n(S^n, 0) \cong \mathbb{Z}$ follows by induction in the usual way from the homotopical excision theorem and the calculation $\pi_1(S^1, 0) \cong \mathbb{Z}$ by the groupoid van Kampen theorem. Applying the HET to writing $S^n \vee [0, 2]$ as a union of S^n and $[0, 2]$ we get that $\pi_n(S^n \vee [0, 2], [0, 2]; \{0, 1, 2\})$ is the free $\pi_1([0, 2], \{0, 1, 2\})$-module on one generator. Again applying the HET but now identifying $0, 1, 2$ to 0 we get that $\pi_n(S^n \vee S^1 \vee S^1, 0)$ is the free $\pi_1(S^1 \vee S^1, 0)$-module on one generator. We leave the reader to examine the example of $Y \vee K(G, 1)$. ■

9.8. Corollary. [Relative Hurewicz Theorem] [BH81b, Example 6] *Let $A \to X$ be a closed cofibration and suppose A is path connected and (X, A) is $(n-1)$-connected. Then $X \cup CA$ is $(n-1)$-connected and $\pi_n(X \cup CA, x)$ is isomorphic to $\pi_n(X, A, x)$ factored by the action of $\pi_1(A, x)$.*

We now point out that a generalisation of a famous result of Hopf, [Hop42], is a corollary of the relative Hurewicz theorem. The following for $n = 2$ is part of Hopf's result. The algebraic description of $H_2(G)$ which he gives for G a group is shown in [BH78] to follow from the HHvKT.

9.9. Proposition. [Hopf's theorem] *Let (V, A) be a pair of pointed spaces such that:*

(i) $\pi_i(A) = 0$ *for* $1 < i < n$*;*

(ii) $\pi_i(V) = 0$ *for* $1 < i \leqslant n$*;*

(iii) *the inclusion $A \to V$ induces an isomorphism on fundamental groups.*

Then the pair (V, A) is n-connected, and the inclusion $A \to V$ induces an epimorphism $H_nA \to H_nV$ whose kernel consists of spherical elements, i.e. of the image of π_nA under the Hurewicz morphism $\omega_n : \pi_n(A) \to H_n(A)$.

Proof. That (V, A) is n-connected follows immediately from the homotopy exact sequence of the pair (V, A) up to $\pi_n(V)$. We now consider the next part of the exact homotopy sequence and its relation to the homology exact sequence as shown in the commutative diagram:

$$\begin{tikzcd}
\pi_{n+1}(V, A) \arrow[r, "\partial"] \arrow[d, "\omega_{n+1}"'] & \pi_n(A) \arrow[r] \arrow[d, "\omega_n"'] & \pi_n(V) \arrow[r] \arrow[d] & \pi_n(V, A) \arrow[d] \\
H_{n+1}(V, A) \arrow[r, "\partial'"'] & H_n(A) \arrow[r, "i_*"'] & H_n(V) \arrow[r] & H_n(V, A)
\end{tikzcd}$$

The Relative Hurewicz Theorem implies that $H_n(V, A) = 0$, and that ω_{n+1} is surjective. Also ∂ in the top row is surjective, since $\pi_n(V) = 0$. It follows easily that the sequence $\pi_n(A) \to H_n(A) \to H_n(V) \to 0$ is exact. ■

There is a nice generalisation of Corollary 9.5, which so far has been proved only as a deduction from a HHvKT. This is given in [BH81b] for the single base point case.

9.10. Corollary. [Attaching cones] *Let A be a space and let S be a set consisting of one point in each path component of A. By CA, the cone on A, we mean the union of cones on each path component of A. Let $f : A \to X$ be a map, and let S' be the image of S by f. Then $\pi_2(X \cup CA, X; S')$ is isomorphic to the $\pi_1(X, S')$-crossed module induced from the identity crossed module $\pi_1(A, S) \to \pi_1(A, S)$ by the induced morphism $f_* : \pi_1(A, S) \to \pi_1(X, S')$.*

The paper [BW03] uses this result to give explicit calculations for the crossed modules representing the homotopy 2-types of certain mapping cones.

We now explain the relevance to free crossed modules of Proposition 8.6, leaving the module and other cases to the reader.

9.11. Proposition. *Let X be a path connected space with base point a, and let $Y = X \cup_{f_\lambda} \{e^2_\lambda\}$ be obtained by attaching cells by means of pointed maps $f_\lambda : (S^1, 0) \to (X, a_\lambda)$, determining elements $x_\lambda \in \pi_1(X, a_\lambda)$, $\lambda \in \Lambda$. Let $A = \{a\} \cup \{a_\lambda \mid \lambda \in \Lambda\}$. Let T be a tree groupoid in $\pi_1(X, A)$ determining a retraction $r : \pi_1(X, A) \to \pi_1(X, a)$. Then $\pi_2(Y, X, a)$ is isomorphic to the free crossed $\pi_1(X, a)$-module on the elements $r(x_\lambda)$.*

Proof. We consider the following diagram:

$$\begin{tikzcd}
(0 \to \bigsqcup_\lambda \mathbb{Z}) \arrow[r] \arrow[d] & (0 \to P) \arrow[r, "r"] \arrow[d] & (0 \to P(a)) \arrow[d] \\
(\bigsqcup_\lambda \mathbb{Z} \to \bigsqcup_\lambda \mathbb{Z}) \arrow[r] & (C(\Lambda) \to P) \arrow[r, "s"'] & (F \to P(a))
\end{tikzcd} \tag{7}$$

The left hand square is the pushout defining the free crossed module $C(\Lambda) \to P$ as an induced crossed module. The right hand square is the special case of crossed modules of the retraction of Proposition 8.6, and so is also a pushout. Hence the composite square is a pushout. Hence the crossed module $F \to P(a)$ is the free crossed module as described. ∎

10. Crossed squares and triad homotopy groups

In this section we give a brief sketch of the theory of triad homotopy groups, including the exact sequence relating them to homotopical excision, and show that the third triad group forms part of a *crossed square* which, as an algebraic structure with links over several dimensions, in this case dimensions 1,2,3, fits our criteria for a HHvKT. Finally we indicate a bifibration from crossed squares, so leading to the notion of *induced crossed square*, which is relevant to a triadic Hurewicz theorem in dimension 3.

A *triad of spaces* $(X : A, B; x)$ consists of a pointed space (X, x) and two pointed subspaces $(A, x), (B, x)$. Then $\pi_n(X : A, B; x)$ is defined for $n \geqslant 2$ as the set of homotopy classes of maps

$$(I^n : \partial^-_1 I^n, \partial^-_2 I^n; J^{n-1}_{1,2}) \to (X : A, B; x)$$

where $J^{n-1}_{1,2}$ denotes the union of the faces of I^n other than $\partial_1^- I^n, \partial_2^- I^n$. For $n \geqslant 3$ and using the direction 3, say, this set obtains a group structure, which is Abelian for $n \geqslant 4$. Further there is an exact sequence

$$\to \pi_{n+1}(X : A, B; x) \to \pi_n(A, C, x) \xrightarrow{\varepsilon} \pi_n(X, B, x) \to \pi_n(X : A, B; x) \to \tag{8}$$

where $C = A \cap B$, and ε is the excision map. It was the fact that this exact sequence measures the failure of excision that was the main interest of the triad groups. However they do not shed light on the above Homotopical Excision Theorem 9.3.

The third triad homotopy group fits into a diagram of possibly non-Abelian groups

$$\Pi(X; A, B, x) := \begin{tikzcd} \pi_3(X; A, B, x) \arrow[r] \arrow[d] & \pi_2(B, C, x) \arrow[d] \\ \pi_2(A, C, x) \arrow[r] & \pi_1(C, x) \end{tikzcd} \tag{9}$$

in which $\pi_1(C, x)$ operates on the other groups and there is also a function

$$\pi_2(A, C, x) \times \pi_2(B, C, x) \to \pi_3(X : A, B; x)$$

known as the generalised Whitehead product. The diagram (9) has structure and properties which are known as those of a *crossed square*, [GWL81, BL87a], explained below, and so this gives a homotopical functor

$$\Pi : (\text{based triads}) \to (\text{crossed squares}). \tag{10}$$

A *crossed square* is a commutative diagram of morphisms of groups

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd} \tag{11}$$

together with left actions of P on L, M, N and a function $h : M \times N \to L$ satisfying a number of axioms which we do not give in full here. Suffice it to say that the morphisms in the square preserve the action of P, which acts on itself by conjugation; M, N act on each other and on L via P; $\lambda, \lambda', \mu, \nu$ and $\mu\lambda$ are crossed modules; and h satisfies axioms reminiscent of commutator rules, summarised by saying it is a *biderivation*. Morphisms of crossed squares are defined in the obvious way, giving a category XSq of crossed squares.

Let XMod^2 be the category of pairs of crossed modules $\mu : M \to P, \nu : N \to P$ (with P and μ, ν variable), and with the obvious notion of morphism. There is a forgetful functor $\Phi : \mathsf{XSq} \to \mathsf{XMod}^2$. This functor has a right adjoint D which completes the pair $\mu : M \to P, \nu : N \to P$ with $L = M \times_P N$ and λ, λ' given by the projections and $h : M \times N \to L$ given by $h(m, n) = ({}^n m m^{-1}, n\, {}^m n^{-1}), m \in M, n \in N$. More interestingly, it has a left adjoint which to the above pair of crossed P-modules yields the 'universal crossed square'

$$\begin{tikzcd} M \otimes N \arrow[r] \arrow[d] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd} \tag{12}$$

where $M \otimes N$, as defined in [BL87a], is the nonabelian tensor product of groups which act on each other.

Then Φ is a fibration of categories and also a cofibration. Thus we have a notion of *induced crossed square*, as follows: given a crossed square $\begin{pmatrix} L & N \\ M & P \end{pmatrix}$ and morphisms of crossed modules $(\alpha, \gamma) : (M \to P) \to (R \to Q), (\beta, \gamma) : (N \to P) \to (S \to Q)$ we get an *induced crossed square* $\begin{pmatrix} T & S \\ R & Q \end{pmatrix}$ which according to Proposition 4.2 is given by a pushout in the category of crossed squares of the form

$$\begin{tikzcd} \begin{pmatrix} M \otimes N & N \\ M & P \end{pmatrix} \arrow[r, "\begin{pmatrix} \alpha \otimes \beta & \beta \\ \alpha & \gamma \end{pmatrix}"] \arrow[d, "\begin{pmatrix} u & 1 \\ 1 & 1 \end{pmatrix}"'] & \begin{pmatrix} R \otimes S & S \\ R & Q \end{pmatrix} \arrow[d, "\begin{pmatrix} v & 1 \\ 1 & 1 \end{pmatrix}"] \\ \begin{pmatrix} L & N \\ M & P \end{pmatrix} \arrow[r, "\begin{pmatrix} \delta & \beta \\ \alpha & \gamma \end{pmatrix}"'] & \begin{pmatrix} T & S \\ R & Q \end{pmatrix} \end{tikzcd}$$

This gives another view of a presentation of the induced crossed square in [BL87b, Proposition 5.5], and which is applied to free crossed squares in [Ell93] for homotopy type calculations.

The functor Π is exploited in [BL87a] for an HHvKT implying some calculations of the non-Abelian group $\pi_3(X\colon A, B; x)$[4]. The applications are developed in [BL87b, Bro89] for a triadic Hurewicz Theorem, and for the notion of free crossed square, both based on 'induced crossed squares'.

In fact the HHvKT works in all dimensions and in the more general setting of n-cubes of spaces, although not in a many base point situation. For a recent application, see [EM08].

[4] Earlier results had used homological methods to obtain some Abelian values for higher order triad groups, [BlM53], or for n-ads, [BaW56].

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