Crossed complexes and higher homotopy groupoids as non commutative tools for higher dimensional local-to-global problems*

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Abstract

We outline the main features of the definitions and applications of crossed complexes and cubical ω-groupoids with connections. These give forms of higher homotopy groupoids, and new views of basic algebraic topology and the cohomology of groups, with the ability to obtain some non commutative results and compute some homotopy types in non simply connected situations.

Contents

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Introduction

An aim is to give a survey and explain the origins of results obtained by R. Brown and P.J. Higgins and others over the years 1974-2008, and to point to applications and related areas. These results yield an account of some basic algebraic topology on the border between homology and homotopy; it differs from the standard account through the use of *crossed complexes*, rather than chain complexes, as a fundamental notion. In this way one obtains comparatively quickly[1] not only classical results such as the Brouwer degree and the relative Hurewicz theorem, but also non commutative results on second relative homotopy groups, as well as higher dimensional results involving the fundamental

[1] This comparison is based on the fact that the methods do not require singular homology or simplicial approximation.

group, through its actions and presentations. A basic tool is the fundamental crossed complex ΠX_* of the filtered space X_*, which in the case X_0 is a singleton is fairly classical; applied to the skeletal filtration of a CW-complex X, Π gives a more powerful version of the usual cellular chains of the universal cover of X, because it contains non-Abelian information in dimensions 1 and 2, and has good realisation properties. It also gives a replacement for singular chains by taking X to be the geometric realisation of a singular complex of a space.

One of the major results is a homotopy classification theorem (4.1.9) which generalises a classical theorem of Eilenberg-Mac Lane, though this does require results on geometric realisations of cubical sets.

A replacement for the excision theorem in homology is obtained by using cubical methods to prove a Higher Homotopy van Kampen Theorem (HHvKT)[2] for the fundamental crossed complex functor Π on filtered spaces. This theorem is a higher dimensional version of the van Kampen Theorem (vKT) on the fundamental group of a space with base point, [vKa33][3], which is a classical example of a

non commutative local-to-global theorem,

and was the initial motivation for the work described here. The vKT determines completely the fundamental group $\pi_1(X, x)$ of a space X with base point which is the union of open sets U, V whose intersection is path connected and contains the base point x; the 'local information' is on the morphisms of fundamental groups induced by the inclusions $U \cap V \to U, U \cap V \to V$. The importance of this result reflects the importance of the fundamental group in algebraic topology, algebraic geometry, complex analysis, and many other subjects. Indeed the origin of the fundamental group was in Poincaré's work on monodromy for complex variable theory.

Essential to this use of crossed complexes, particularly for conjecturing and proving local-to-global theorems, is a construction of a *cubical higher homotopy groupoid*, with properties described by *an algebra of cubes*. There are applications to local-to-global problems in homotopy theory which are more powerful than available by purely classical tools, while shedding light on those tools. It is hoped that this account will increase the interest in the possibility of wider applications of these methods and results, since homotopical methods play a key role in many areas.

Background in higher homotopy groups

Topologists in the early part of the 20th century were well aware that:

- the non commutativity of the fundamental group was useful in geometric applications;
- for path connected X there was an isomorphism

$$H_1(X) \cong \pi_1(X, x)^{ab};$$

- the Abelian homology groups $H_n(X)$ existed for all $n \geqslant 0$.

Consequently there was a desire to generalise the non commutative fundamental group to all dimensions.

[2]We originally called this a generalised van Kampen Theorem, but this new term was suggested in 2007 by Jim Stasheff.
[3]An earlier version for simplicial complexes is due to Seifert.

In 1932 Čech submitted a paper on higher homotopy groups $\pi_n(X, x)$ to the ICM at Zurich, but it was quickly proved that these groups were Abelian for $n \geqslant 2$, and on these grounds Čech was persuaded to withdraw his paper, so that only a small paragraph appeared in the Proceedings [Cec32]. We now see the reason for this commutativity as the result (Eckmann-Hilton) that a group internal to the category of groups is just an Abelian group. Thus, since 1932 the vision of a non commutative higher dimensional version of the fundamental group has been generally considered to be a mirage. Before we go back to the vKT, we explain in the next section how nevertheless work on crossed modules did introduce non commutative structures relevant to topology in dimension 2.

Work of Hurewicz, [Hur35], led to a strong development of higher homotopy groups. The fundamental group still came into the picture with its action on the higher homotopy groups, which I once heard J.H.C. Whitehead remark (1957) was especially fascinating for the early workers in homotopy theory. Much of Whitehead's work was intended to extend to higher dimensions the methods of combinatorial group theory of the 1930s – hence the title of his papers: 'Combinatorial homotopy, I, II' [W:CHI, W:CHII]. The first of these two papers has been very influential and is part of the basic structure of algebraic topology. It is the development of work of the second paper which we explain here.

The paper by Whitehead on 'Simple homotopy types' [W:SHT], which deals with higher dimensional analogues of Tietze transformations, has a final section using crossed complexes. We refer to this again later in section 15.

It is hoped also that this survey will be useful background to work on the van Kampen Theorem for diagrams of spaces in [BLo87a], which uses a form of higher homotopy groupoid which is in an important sense much more powerful than that given here, since it encompasses n-adic information; however current expositions are still restricted to the reduced (one base point) case, the proof uses advanced tools of algebraic topology, and the result was suggested by the work exposed here.

1 Crossed modules

In the years 1941-50, Whitehead developed work on crossed modules to represent the structure of the boundary map of the relative homotopy group

$$\pi_2(X, A, x) \to \pi_1(A, x) \tag{1}$$

in which both groups can be non commutative. Here is the definition he found.

A *crossed module* is a morphism of groups $\mu : M \to P$ together with an action $(m, p) \mapsto m^p$ of the group P on the group M satisfying the two axioms

CM1) $\mu(m^p) = p^{-1}(\mu m)p$

CM2) $n^{-1}mn = m^{\mu n}$

for all $m, n \in M, p \in P$.

Standard algebraic examples of crossed modules are:

(i) an inclusion of a normal subgroup, with action given by conjugation;

(ii) the inner automorphism map $\chi : M \to \operatorname{Aut} M$, in which χm is the automorphism $n \mapsto m^{-1}nm$;

(iii) the zero map $M \to P$ where M is a P-module;

(iv) an epimorphism $M \to P$ with kernel contained in the centre of M.

Simple consequences of the axioms for a crossed module $\mu : M \to P$ are:

1.1 $\operatorname{Im}\mu$ *is normal in* P.

1.2 $\operatorname{Ker}\mu$ *is central in* M *and is acted on trivially by* $\operatorname{Im}\mu$*, so that* $\operatorname{Ker}\mu$ *inherits an action of* $M/\operatorname{Im}\mu$.

Another important algebraic construction is the *free crossed* P*-module*

$$\partial : C(\omega) \to P$$

determined by a function $\omega : R \to P$, where P is a group and R is a set. The group $C(\omega)$ is generated by elements $(r, p) \in R \times P$ with the relations

$$(r, p)^{-1}(s, q)^{-1}(r, p)(s, qp^{-1}(\omega r)p);$$

the action is given by $(r, p)^q = (r, pq)$; and the boundary morphism is given by $\partial(r, p) = p^{-1}(\omega r)p$, for all $(r, p), (s, q) \in R \times P$.

A major result of Whitehead was:

Theorem W [W:CHII] *If the space* $X = A \cup \{e^2_r\}_{r\in R}$ *is obtained from* A *by attaching 2-cells by maps* $f_r : (S^1, 1) \to (A, x)$*, then the crossed module of* (1) *is isomorphic to the free crossed* $\pi_1(A, x)$*-module on the classes of the attaching maps of the 2-cells.*

Whitehead's proof, which stretched over three papers, 1941-1949, used transversality and knot theory – an exposition is given in [Bro80]. Mac Lane and Whitehead [MLW50] used this result as part of their proof that crossed modules capture all homotopy 2-types (they used the term '3-types').

The title of the paper in which the first intimation of Theorem W appeared was 'On adding relations to homotopy groups' [Whi41]. This indicates a search for higher dimensional vKTs.

The concept of free crossed module gives a non commutative context for *chains of syzygies*. The latter idea, in the case of modules over polynomial rings, is one of the origins of homological algebra through the notion of *free resolution*. Here is how similar ideas can be applied to groups. Pioneering work here, independent of Whitehead, was by Peiffer [Pei49] and Reidemeister [Rei49]. See [BHu82] for an exposition of these ideas.

Suppose $\mathcal{P} = \langle X \mid \omega\rangle$ is a presentation of a group G, so that X is a set of generators of G and $\omega : R \to F(X)$ is a function, whose image is called the set of relators of the presentation. Then we have an exact sequence

$$1 \xrightarrow{i} N(\omega R) \xrightarrow{\phi} F(X) \longrightarrow G \longrightarrow 1$$

where $N(\omega R)$ is the normal closure in $F(X)$ of the set ωR of relators. The above work of Reidemeister, Peiffer, and Whitehead showed that to obtain the next level of syzygies one should consider the free crossed $F(X)$-module $\partial : C(\omega) \to F(X)$, since this takes into account the operations of $F(X)$ on its normal subgroup $N(\omega R)$. Elements of $C(\omega)$ are a kind of 'formal consequences of the relators', so that the relation between the elements of $C(\omega)$ and those of $N(\omega R)$ is analogous to the relation between the elements of $F(X)$ and those of G. It follows from the rules for a crossed module that the kernel of ∂ is a G-module, called the module of *identities among relations*, and sometimes written $\pi(\mathcal{P})$; there is considerable work on computing it [BHu82, Pri91, HAM93, ElK99, BRS99]. By splicing to ∂ a free G-module resolution of $\pi(\mathcal{P})$ one obtains what is called a *free crossed resolution* of the group G. We explain later (Proposition 15.3) why these resolutions have better realisation properties than the usual resolutions by chain complexes of G-modules. They are relevant to the Schreier extension theory, [BrP96].

This notion of using crossed modules as the first stage of syzygies in fact represents a wider tradition in homological algebra, in the work of Frölich and Lue [Fro61, Lue81].

Crossed modules also occurred in other contexts, notably in representing elements of the cohomology group $H^3(G, M)$ of a group G with coefficients in a G-module M [McL63], and as coefficients in Dedecker's theory of non Abelian cohomology [Ded63]. The notion of free crossed resolution has been exploited by Huebschmann [Hue80, Hue81b, Hue81a] to represent cohomology classes in $H^n(G, M)$ of a group G with coefficients in a G-module M, and also to calculate with these.

The HHvKT can make it easier to compute a crossed module arising from some topological situation, such as an induced crossed module [BWe95, BWe96], or a coproduct crossed module [Bro84], than the cohomology class in $H^3(G, M)$ the crossed module represents. To obtain information on such a cohomology element it is useful to work with a small free crossed resolution of G, and this is one motivation for developing methods for calculating such resolutions. However, it is not so clear what a *calculation* of such a cohomology element would amount to, although it is interesting to know whether the element is non zero, or what is its order. Thus the use of algebraic models of cohomology classes may yield easier computations than the use of cocycles, and this somewhat inverts traditional approaches.

Since crossed modules are algebraic objects generalising groups, it is natural to consider the problem of explicit calculations by extending techniques of computational group theory. Substantial work on this has been done by C.D. Wensley using the program GAP [GAP02, BWe03].

2 The fundamental groupoid on a set of base points

A change in prospects for higher order non commutative invariants was suggested by Higgins' paper [Hig64], and leading to work of the writer published in 1967, [Bro67]. This showed that the van Kampen Theorem could be formulated for the *fundamental groupoid* $\pi_1(X, X_0)$ *on a set* X_0 *of base points*, thus enabling computations in the non-connected case, including those in Van Kampen's original paper [vKa33]. This successful use of groupoids in dimension 1 suggested the question of the use of groupoids in higher homotopy theory, and in particular the question of the existence of *higher homotopy groupoids*.

In order to see how this research programme could progress it is useful to consider the statement and special features of this generalised van Kampen Theorem for the fundamental groupoid. If X_0 is a set, and X is a space, then $\pi_1(X, X_0)$ denotes the fundamental groupoid on the set $X \cap X_0$ of base points. This allows the set X_0 to be chosen in a way appropriate to the geometry. For example, if the circle S^1 is written as the union of two semicircles $E_+ \cup E_-$, then the intersection $\{-1, 1\}$ of the semicircles is not connected, so it is not clear where to take the base point. Instead one takes $X_0 = \{-1, 1\}$, and so has two base points. This flexibility is very important in computations, and this example of S^1 was a motivating example for this development. As another example, you might like to consider the difference between the quotients of the actions of $\mathbb{Z}_2$ on the group $\pi_1(S^1, 1)$ and on the groupoid $\pi_1(S^1, \{-1, 1\})$ where the action is induced by complex conjugation on S^1. Relevant work on *orbit groupoids* has been developed by Higgins and Taylor [HiT81, Tay88], (under useful conditions, the fundamental groupoid of the orbit space is the orbit groupoid of the fundamental groupoid [Bro06, 11.2.3]).

Consideration of a set of base points leads to the theorem:

Theorem 2.1 [Bro67] *Let the space* X *be the union of open sets* U, V *with intersection* W*, and let* X_0 *be a subset of* X *meeting each path component of* U, V, W*. Then*
(C) (connectivity) X_0 *meets each path component of* X *and*
(I) (isomorphism) *the diagram of groupoid morphisms induced by inclusions*

$$\begin{tikzcd} \pi_1(W, X_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(U, X_0) \arrow[d, "j'"] \\ \pi_1(V, X_0) \arrow[r, "i'"'] & \pi_1(X, X_0) \end{tikzcd} \tag{2}$$

is a pushout of groupoids.

From this theorem, one can compute a particular fundamental group $\pi_1(X, x_0)$ using combinatorial information on the graph of intersections of path components of U, V, W, but for this it is useful to develop the algebra of groupoids. Notice two special features of this result.

(i) The computation of the invariant you may want, a fundamental group, is obtained from the computation of a larger structure, and so part of the work is to give methods for computing the smaller structure from the larger one. This usually involves non canonical choices, e.g. that of a maximal tree in a connected graph. The work on applying groupoids to groups gives many examples of this [Hig64, Hig71, Bro06, DiV96].

(ii) The fact that the computation can be done is surprising in two ways: (a) The fundamental group is computed *precisely*, even though the information for it uses input in two dimensions, namely 0 and 1. This is contrary to the experience in homological algebra and algebraic topology, where the interaction of several dimensions involves exact sequences or spectral sequences, which give information only up to extension, and (b) the result is a non commutative invariant, which is usually even more difficult to compute precisely.

The reason for the success seems to be that the fundamental groupoid $\pi_1(X, X_0)$ contains information in dimensions 0 and 1, and so can adequately reflect the geometry of the intersections of the path components of U, V, W and of the morphisms induced by the inclusions of W in U and V.

This suggested the question of whether these methods could be extended successfully to higher dimensions.

Part of the initial evidence for this quest was the intuitions in the proof of this groupoid vKT, which seemed to use three main ideas in order to verify the universal property of a pushout for diagram (2). So suppose given morphisms of groupoids f_U, f_V from $\pi_1(U, X_0), \pi_1(V, X_0)$ to a groupoid G, satisfying $f_U i = f_V j$. We have to construct a morphism $f : \pi_1(X, X_0) \to G$ such that $fi' = f_U, fj' = f_V$ and prove f is unique. We concentrate on the construction.

• One needs a 'deformation', or 'filling', argument: given a path $a : (I, \dot{I}) \to (X, X_0)$ one can write $a = a_1 + \cdots + a_n$ where each a_i maps into U or V, but a_i will not necessarily have end points in X_0. So one has to deform each a_i to a_i' in U, V or W, using the connectivity condition, so that each a_i' has end points in X_0, and $a' = a_1' + \cdots + a_n'$ is well defined. Then one can construct using f_U or f_V an image of each a_i' in G and hence of the composite, called $F(a) \in G$, of these images. Note that we subdivide in X and then put together again in G (this uses the condition $f_U i = f_V j$ to prove that the elements of G are composable), and this part can be summarised as:

• Groupoids provided a convenient *algebraic inverse to subdivision*. Note that the usual exposition in terms only of the fundamental group uses loops, i.e. paths which start and finish at the same point. An appropriate analogy is that if one goes on a train journey from Bangor and back to Bangor, one usually wants to stop off at intermediate stations; this breaking and cotinuing a journey is better described in terms of groupoids rather than groups.

Next one has to prove that $F(a)$ depends only on the class of a in the fundamental groupoid. This involves a homotopy rel end points $h : a \simeq b$, considered as a map $I^2 \to X$; subdivide h as $h = [h_{ij}]$ so that each h_{ij} maps into U, V or W; deform h to $h' = [h_{ij}']$ (keeping in U, V, W) so that each h_{ij}' maps the vertices to X_0 and so determines a commutative square[4] in one of $\pi_1(Q, X_0)$ for $Q = U, V, W$. Move these commutative squares over to G using f_U, f_V and recompose them (this is possible again because of the condition $f_U i = f_V j$), noting that:

• in a groupoid, *any composition of commutative squares is commutative*. Here a 'big' composition of commutative squares is represented by a diagram such as

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd} \tag{3}$$

and one checks that if each individual square is commutative, so also is the boundary square (later called a 2-shell) of the compositions of the boundary edges.

Two opposite sides of the composite commutative square in G so obtained are identities, because h was a homotopy relative to end points, and the other two sides are $F(a)$, $F(b)$. This proves that

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[4]We need the notion of *commutative square* in a category C. This is a quadruple $\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$ of arrows in C, called 'edges' of the square, such that $ab = cd$, i.e. such that these compositions are defined and agree. The commutative squares in C form a *double category* $\Box\mathsf{C}$ in that they compose 'vertically'

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} \circ_1 \begin{pmatrix} & b & \\ a' & & d' \\ & e & \end{pmatrix} = \begin{pmatrix} & c & \\ aa' & & dd' \\ & e & \end{pmatrix}$$

and 'horizontally'

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} \circ_2 \begin{pmatrix} & c' & \\ d & & f \\ & b' & \end{pmatrix} = \begin{pmatrix} & cc' & \\ a & & f \\ & bb' & \end{pmatrix}$$

This notion of $\Box\mathsf{C}$ was defined by C. Ehresmann in papers and in [Ehr83]. Note the obvious geometric conditions for these compositions to be defined. Similarly, one has geometric conditions for a rectangular array $(c_{ij}), 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$, of commutative squares to have a well defined composition, and then their 'multiple composition', written $[c_{ij}]$, is also a commutative square, whose edges are compositions of the 'edges' along the outside boundary of the array. It is easy to give formal definitions of all this.

$F(a) = F(b)$ in G.

Thus the argument can be summarised: a path or homotopy is divided into small pieces, then deformed so that these pieces can be packaged and moved over to G, where they are reassembled. There seems to be an analogy with the processing of an email.

Notable applications of the groupoid theorem were: (i) to give a proof of a formula in van Kampen's paper of the fundamental group of a space which is the union of two connected spaces with non connected intersection, see [Bro06, 8.4.9]; and (ii) to show the topological utility of the construction by Higgins [Hig71] of the groupoid $f_*(G)$ over Y_0 induced from a groupoid G over X_0 by a function $f : X_0 \to Y_0$. (Accounts of these with the notation $U_f(G)$ rather than $f_*(G)$ are given in [Hig71, Bro06].) This latter construction is regarded as a 'change of base', and analogues in higher dimensions yielded generalisations of the Relative Hurewicz Theorem and of Theorem W, using induced modules and crossed modules.

There is another approach to the van Kampen Theorem which goes via the theory of covering spaces, and the equivalence between covering spaces of a reasonable space X and functors $\pi_1(X) \to \mathsf{Set}$ [Bro06]. See for example [DoD79] for an exposition of the relation with traditional Galois theory, and [BoJ01] for a modern account in which *Galois groupoids* make an essential appearance. The paper [BrJ97] gives a general formulation of conditions for the theorem to hold in the case $X_0 = X$ in terms of the map $U \sqcup V \to X$ being an 'effective global descent morphism' (the theorem is given in the generality of lextensive categories). This work has been developed for toposes, [BuL03]. Analogous interpretations for higher dimensional Van Kampen theorems are not known.

The justification of the breaking of a paradigm in changing from groups to groupoids is several fold: the elegance and power of the results; the increased linking with other uses of groupoids [Bro87]; and the opening out of new possibilities in higher dimensions, which allowed for new results and calculations in homotopy theory, and suggested new algebraic constructions. The important and extensive work of Charles Ehresmann in using groupoids in geometric situations (bundles, foliations, germes, ...) should also be stated (see his collected works of which [EH84] is volume 1 and a survey [Bro07]).

3 The search for higher homotopy groupoids

Contemplation of the proof of the groupoid vKT in the last section suggested that a higher dimensional version should exist, though this version amounted to an idea of a proof in search of a theorem. Further evidence was the proof by J.F. Adams of the cellular approximation theorem given in [Bro06]. This type of subdivision argument failed to give algebraic information apparently because of a lack of an appropriate higher homotopy groupoid, i.e. a gadget to capture what might be the underlying 'algebra of cubes'. In the end, the results exactly encapsulated this intuition.

One intuition was that in groupoids we are dealing with a partial algebraic structure[5], in which composition is defined for two arrows if and only if the source of one arrow is the target of the other. This seems to generalise easily to directed squares, in which two such are composable horizontally if and only if the left hand side of one is the right hand side of the other (and similarly vertically).

[5]The study of partial algebraic operations was initiated in [Hig63]. We can now suggest a reasonable definition of 'higher dimensional algebra' as dealing with families of algebraic operations whose domains of definitions are given by geometric conditions.

However the formulation of a theorem in higher dimensions required specification of the *topological data*, the *algebraic data*, and of a functor

$$\Pi : (\text{topological data}) \to (\text{algebraic data})$$

which would allow the expression of these ideas for the proof.

Experiments were made in the years 1967-1973 to define some functor Π from spaces to some kind of double groupoid, using compositions of squares in two directions, but these proved abortive. However considerable progress was made in work with Chris Spencer in 1971-3 on investigating the algebra of double groupoids [BSp76a], and showing a relation to crossed modules. Further evidence was provided when it was found, [BSp76b], that group objects in the category of groupoids (or groupoid objects in the category of groups, either of which are often now called '2-groups') are equivalent to crossed modules, and in particular are not necessarily commutative objects. It turned out this result was known to the Grothendieck school in the 1960s, but not published.

We review next a notion of double category which is not the most general but is appropriate in many cases. It was called an *edge symmetric double category* in [BMo99].

In the first place, a double category, K, consists of a triple of category structures

$$(K_2, K_1, \partial_1^-, \partial_1^+, \circ_1, \varepsilon_1), \quad (K_2, K_1, \partial_2^-, \partial_2^+, \circ_2, \varepsilon_2)$$
$$(K_1, K_0, \partial^-, \partial^+, \circ, \varepsilon)$$

as partly shown in the diagram

$$\begin{tikzcd} K_2 \arrow[r, shift left, "\partial_2^-"] \arrow[r, shift right, "\partial_2^+"'] \arrow[d, shift right, "\partial_1^+"'] \arrow[d, shift left, "\partial_1^-"] & K_1 \arrow[d, shift right, "\partial^+"'] \arrow[d, shift left, "\partial^-"] \\ K_1 \arrow[r, shift left, "\partial^-"] \arrow[r, shift right, "\partial^+"'] & K_0 \end{tikzcd} \tag{4}$$

The elements of K_0, K_1, K_2 will be called respectively *points or objects, edges, squares*. The maps $\partial^\pm$, $\partial_i^\pm$, $i = 1, 2$, will be called *face maps*, the maps $\varepsilon_i : K_1 \longrightarrow K_2$, $i = 1, 2$, resp. $\varepsilon : K_0 \longrightarrow K_1$ will be called *degeneracies*. The boundaries of an edge and of a square are given by the diagrams

$$\partial^- \longrightarrow \partial^+ \qquad \begin{tikzcd} \cdot \arrow[r, "\partial_1^-"] \arrow[d, "\partial_2^-"'] & \cdot \arrow[d, "\partial_2^+"] \\ \cdot \arrow[r, "\partial_1^+"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "2"] \arrow[d, "1"'] & {} \\ {} & \end{tikzcd} \tag{5}$$

The partial compositions, $\circ_1$, resp. $\circ_2$, are referred to as *vertical* resp. *horizontal composition* of squares, are defined under the obvious geometric conditions, and have the obvious boundaries. The axioms for a double category also include the usual relations of a 2-cubical set (for example $\partial^-\partial_2^+ = \partial^+\partial_1^-$), and the *interchange law*. We use matrix notation for compositions as

$$\begin{bmatrix} a \\ c \end{bmatrix} = a \circ_1 c, \quad \begin{bmatrix} a & b \end{bmatrix} = a \circ_2 b,$$

and the crucial interchange law[6] for these two compositions allows one to use matrix notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} [a \quad b] \\ [c \quad d] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} & \begin{bmatrix} b \\ d \end{bmatrix} \end{bmatrix}$$

for double composites of squares whenever each row composite and each column composite is defined. We also allow the multiple composition $[a_{ij}]$ of an array (a_{ij}) whenever for all appropriate i, j we have $\partial_1^+ a_{ij} = \partial_1^- a_{i+1,j}$, $\partial_2^+ a_{ij} = \partial_2^- a_{i,j+1}$. A clear advantage of double categories and cubical methods is this easy expression of multiple compositions which allows for *algebraic inverse to subdivision*, and so applications to local-to-global problems.

The identities with respect to $\circ_1$ (*vertical identities*) are given by ε_1 and will be denoted by $\|$. Similarly, we have *horizontal identities* denoted by $=$. Elements of the form $\varepsilon_1 \varepsilon(a) = \varepsilon_2 \varepsilon(a)$ for $a \in K_0$ are called *double degeneracies* and will be denoted by $\square$.

A *morphism of double categories* $f : K \to L$ consists of a triple of maps $f_i : K_i \to L_i$, $(i = 0, 1, 2)$, respecting the cubical structure, compositions and identities.

Whereas it is easy to describe a commutative square of morphisms in a category, it is not possible with this amount of structure to describe a commutative cube of squares in a double category. We first of all define a cube, or 3-shell, i.e. without any condition of commutativity, in a double category.

Definition 3.1 Let K be a double category. A *cube* (3-*shell*) in K,

$$\alpha = (\alpha_1^-, \alpha_1^+, \alpha_2^-, \alpha_2^+, \alpha_3^-, \alpha_3^+)$$

consists of squares $\alpha_i^\pm \in K_2 \quad (i = 1, 2, 3)$ such that

$$\partial_i^\sigma(\alpha_j^\tau) = \partial_{j-1}^\tau(\alpha_i^\sigma)$$

for $\sigma, \tau = \pm$ and $1 \leqslant i < j \leqslant 3$. ■

It is also convenient to have the corresponding notion of square, or 2-shell, of arrows in a category. The obvious compositions also makes these into a double category.

It is not hard to define three compositions of cubes in a double category so that these cubes form a triple category: this is done in [BKP05], or more generally in Section 5 of [BHi81a]. A key point is that to define the notion of a *commutative cube* we need extra structure on a double category. Thus this step up a dimension is non trivial, as was first observed in the groupoid case in [BHi78a]. The problem is that a cube has six faces, which easily divide into three even and three odd faces. So we cannot say as we might like that 'the cube is commutative if the composition of the even faces equals the composition of the odd faces', since there are no such valid compositions.

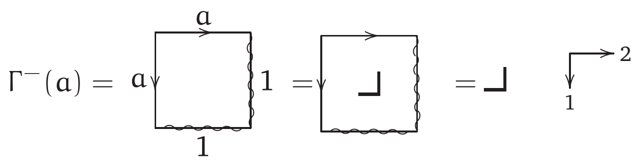
The intuitive reason for the need of a new basic structure in that in a 2-dimensional situation we also need to use the possibility of 'turning an edge clockwise or anticlockwise'. The structure to do this is as follows.

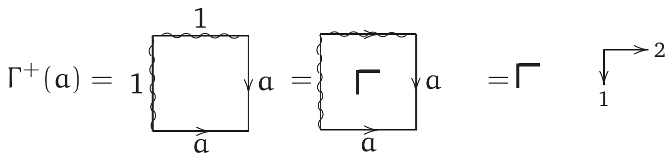
A *connection pair* on a double category K is given by a pair of maps

$$\Gamma^-, \Gamma^+ : K_1 \longrightarrow K_2$$

[6] The interchange implies that a double monoid is simply an Abelian monoid, so partial algebraic operations are essential for the higher dimensional work.

whose edges are given by the following diagrams for $a \in K_1$:





This 'hieroglyphic' notation, which was introduced in [Bro82], is useful for expressing the laws these connections satisfy. The first is a pair of cancellation laws which read

$$\begin{bmatrix}\ulcorner \\ \lrcorner\end{bmatrix} = \,=\,, \qquad [\ulcorner \quad \lrcorner] = |\,|,$$

which can be understood as 'if you turn right and then left, you face the same way', and similarly the other way round. They were introduced in [Spe77]. Note that in this matrix notation we assume that the edges of the connections are such that the composition is defined.

Two other laws relate the connections to the compositions and read

$$\begin{bmatrix}\ulcorner & = \\ |\,| & \ulcorner\end{bmatrix} = \ulcorner, \qquad \begin{bmatrix}\lrcorner & |\,| \\ = & \lrcorner\end{bmatrix} = \lrcorner.$$

These can be interpreted as 'turning left (or right) with your arm outstretched is the same as turning left (or right)'. The term 'connections' and the name 'transport laws' was because these laws were suggested by the laws for path connections in differential geometry, as explained in [BSp76a]. It was proved in [BMo99] that a connection pair on a double category K is equivalent to a 'thin structure', namely a morphism of double categories $\Theta : \square K_1 \to K$ which is the identity on the edges. The proof requires some '2-dimensional rewriting' using the connections.

We can now explain what is a 'commutative cube' in a double category K with connection pair.

Definition 3.2 Suppose given, in a double category with connections K, a cube (3-shell)

$$\alpha = (\alpha_1^-, \alpha_1^+, \alpha_2^-, \alpha_2^+, \alpha_3^-, \alpha_3^+).$$

We define the *composition of the odd faces* of α to be

$$\partial^{\mathrm{odd}}\alpha = \begin{bmatrix}\ulcorner & \alpha_1^- & \lrcorner \\ \alpha_3^- & \alpha_2^+ & =\end{bmatrix} \tag{6}$$

and the *composition of the even faces* of α to be

$$\partial^{\mathrm{even}}\alpha = \begin{bmatrix}= & \alpha_2^- & \alpha_3^+ \\ \ulcorner & \alpha_1^+ & \lrcorner\end{bmatrix} \tag{7}$$

We define α to be *commutative* if it satisfies the Homotopy Commutativity Lemma (HCL), i.e.

$$\partial^{\mathrm{odd}}\alpha = \partial^{\mathrm{even}}\alpha. \tag{HCL}$$

This definition can be regarded as a cubical, categorical (rather than groupoid) form of the Homotopy Addition Lemma (HAL) in dimension 3.

You should draw a 3-shell, label all the edges with letters, and see that this equation makes sense in that the 2-shells of each side of equation (HCL) coincide. Notice however that these 2-shells do not have coincident partitions along the edges: that is the edges of this 2-shell in direction 1 are formed from different compositions of the type $1 \circ a$ and $a \circ 1$. This definition is discussed in more detail in [BKP05], is related to other equivalent definitions, and it is proved that compositions of commutative cubes in the three possible directions are also commutative. These results are extended to all dimensions in [Hig05]; this requires the full structure indicated in section 9 and also the notion of *thin element* indicated in section 12.

The initial discovery of connections arose in [BSp76a] from relating crossed modules to double groupoids. The first example of a double groupoid was the double groupoid $\square G$ of commutative squares in a group G. The first step in generalising this construction was to consider quadruples $\left(\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right)$ of elements of G such that $abn = cd$ for some element n of a subgroup N of G. Experiments quickly showed that for the two compositions of such quadruples to be valid it was necessary and sufficient that N be normal in G. But in this case the element n is determined by the boundary, or 2-shell, a, b, c, d. In homotopy theory we require something more general. So we consider a morphism $\mu : N \to G$ of groups and and consider quintuples $\left(n : \begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right)$ such that $ab\mu(n) = cd$. It then turns out that we get a double groupoid if and only if $\mu : N \to G$ is a crossed module. The next question is which double groupoids arise in this way? It turns out that we need exactly double groupoids with connection pairs, though in this groupoid case we can deduce Γ^- from Γ^+ using inverses in each dimension. This gives the main result of [BSp76a], the equivalence between the category of crossed modules and that of double groupoids with connections and one vertex.

These connections were also used in [BHi78a] to define a 'commutative cube' in a double groupoid with connections using the equation

$$c_1 = \begin{bmatrix} \ulcorner & a_0^{-1} & \urcorner \\ -b_0 & c_0 & b_1 \\ \llcorner & a_1 & \lrcorner \end{bmatrix}$$

representing one face of a cube in terms of the other five and where the other connections $\llcorner, \urcorner$ are obtained from $\lrcorner, \ulcorner$ by using the two inverses in dimension 2. As you might imagine, there are problems in finding a formula in still higher dimensions. In the groupoid case, this is handled by a homotopy addition lemma and thin elements, [BHi81a], but in the category case a formula for just a commutative 4-cube is complicated, see [Gau01].

The blockage of defining a functor Π to double groupoids was resolved after 9 years in 1974 in discussions with Higgins, by considering the Whitehead Theorem W. This showed that a 2-dimensional universal property was available in homotopy theory, which was encouraging; it also suggested that a theory to be any good should recover Theorem W. But this theorem was about *relative* homotopy groups. This suggested studying a relative situation $X_* : X_0 \subseteq X_1 \subseteq X$. On looking for the simplest way to get a homotopy functor from this situation using squares, the ‘obvious’ answer came up: consider maps $(I^2, \partial I^2, \partial\partial I^2) \to (X, X_1, X_0)$, i.e. maps of the square which take the edges into X_1 and the vertices into X_0, and then take homotopy classes of such maps relative to the vertices of I^2 to form a set $\rho_2 X_*$. Of course this set will not inherit a group structure but the surprise is that it does inherit the structure of double groupoid with connections – the proof is not entirely trivial, and is given in [BHi78a] and the expository article [Bro99]. In the case X_0 is a singleton, the equivalence of such double groupoids to crossed modules takes ρX_* to the usual second relative homotopy crossed module.

Thus a search for a *higher homotopy groupoid* was realised in dimension 2. Connes suggests in [Con94] that it has been fashionable for mathematicians to disparage groupoids, and it might be that a lack of attention to this notion was one reason why such a construction had not been found earlier than 40 years after Hurewicz’s papers.

Finding a good homotopy double groupoid led rather quickly, in view of the previous experience, to a substantial account of a 2-dimensional HHvKT [BHi78a]. This recovers Theorem W, and also leads to new calculations in 2-dimensional homotopy theory, and in fact to some new calculations of 2-types. For a recent summary of some results and some new ones, see the paper in the J. Symbolic Computation [BWe03] – publication in this journal illustrates that we are interested in using general methods in order to obtain specific calculations, and ones to which there seems no other route.

Once the 2-dimensional case had been completed in 1975, it was easy to conjecture the form of general results for dimensions > 2. These were proved by 1979 and announcements were made in [BHi78b] with full details in [BHi81a, BHi81b]. However, these results needed a number of new ideas, even just to construct the higher dimensional compositions, and the proof of the HHvKT was quite hard and intricate. Further, for applications, such as to explain how the general Π behaved on homotopies, we also needed a theory of tensor products, found in [BHi87], so that the resulting theory is quite complex. It is also remarkable that ideas of Whitehead in [W:CHII] played a key role in these results.

4 Main results

Major features of the work over the years with Philip Higgins and others can be summarised in the following diagram of categories and functors:

Diagram 4.1

$$\begin{tikzcd}
 & & \text{filtered spaces} \arrow[dl, shift right, "\Pi"'] \arrow[dr, "\rho"] \arrow[dll, bend right, "C_* = \nabla \circ \Pi"'] & \begin{array}{c}\text{filtered}\\ \text{cubical sets}\end{array} \arrow[l, "|\ |"'] \\
\begin{array}{c}\text{operator}\\ \text{chain}\\ \text{complexes}\end{array} \arrow[r, shift right, "\Theta"'] & \begin{array}{c}\text{crossed}\\ \text{complexes}\end{array} \arrow[l, shift right, "\nabla"'] \arrow[ur, shift right, "\mathcal{B}"'] \arrow[rr, shift left, "\lambda"] & & \begin{array}{c}\text{cubical}\\ \omega\text{-groupoids}\\ \text{with connections}\end{array} \arrow[ll, shift left, "\gamma"] \arrow[u, "U_*"']
\end{tikzcd}$$

in which

4.1.1 the categories FTop of filtered spaces, ω-Gpd of cubical ω-groupoids with connections, and Crs of crossed complexes are monoidal closed, and have a notion of homotopy using $\otimes$ and a unit interval object;

4.1.2 ρ, Π are homotopical functors (that is they are defined in terms of homotopy classes of certain maps), and preserve homotopies;

4.1.3 λ, γ are inverse adjoint equivalences of monoidal closed categories;

4.1.4 there is a natural equivalence $\gamma\rho \simeq \Pi$, so that either ρ or Π can be used as appropriate;

4.1.5 ρ, Π preserve certain colimits and certain tensor products;

4.1.6 the category of chain complexes with a groupoid of operators is monoidal closed, ∇ preserves the monoid structures, and is left adjoint to Θ;

4.1.7 by definition, the *cubical filtered classifying space* is $\mathcal{B}^{\square} = |\,| \circ U_*$ where U_* is the forgetful functor to filtered cubical sets[7] using the filtration of an ω-groupoid by skeleta, and $|\,|$ is geometric realisation of a cubical set;

4.1.8 there is a natural equivalence $\Pi \circ \mathcal{B}^{\square} \simeq 1$;

4.1.9 if C is a crossed complex and its cubical classifying space is defined as $B^{\square}C = (\mathcal{B}^{\square}C)_\infty$, then for a CW-complex X, and using homotopy as in 4.1.1 for crossed complexes, there is a natural bijection of sets of homotopy classes

$$[X, B^{\square}C] \cong [\Pi X_*, C].$$

Recent applications of the simplicial version of the classifying space are in [Bro08b, PoT07, FMP07].

Here a *filtered space* consists of a (compactly generated) space X_∞ and an increasing sequence of subspaces

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\infty.$$

With the obvious morphisms, this gives the category FTop. The tensor product in this category is the usual

$$(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q.$$

The closed structure is easy to construct from the law

$$\mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)).$$

An advantage of this monoidal closed structure is that it allows an enrichment of the category FTop over either crossed complexes or ω-Gpd using Π or ρ applied to $\mathsf{FTOP}(Y_*, Z_*)$.

The structure of *crossed complex* is suggested by the canonical example, the *fundamental crossed complex* ΠX_* of the filtered space X_*. So it is given by a diagram

[7] Cubical sets are defined, analogously to simplicial sets, as functors $K : \square^{op} \to \mathsf{Set}$ where $\square$ is the 'box' category with objects I^n and morphisms the compositions of inclusions of faces and of the various projections $I^n \to I^r$ for $n > r$. The *geometric realisation* $|K|$ of such a cubical set is obtained by quotienting the disjoint union of the sets $K(I^n) \times I^n$ by the relations defined by the morphisms of $\square$. For more details, see [Jar06], and for variations on the category $\square$ to include for example connections, see [GrM03]. See also section 9.

Diagram 4.2

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "t"] & C_{n-1} \arrow[r] \arrow[d, "t"] & \cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
& C_0 & C_0 & & C_0 & C_0
\end{tikzcd}$$

in which in this example C_1 is the fundamental groupoid $\pi_1(X_1, X_0)$ of X_1 on the 'set of base points' $C_0 = X_0$, while for $n \geqslant 2$, C_n is the family of relative homotopy groups $\{C_n(x)\} = \{\pi_n(X_n, X_{n-1}, x) \mid x \in X_0\}$. The boundary maps are those standard in homotopy theory. There is for $n \geqslant 2$ an action of the groupoid C_1 on C_n (and of C_1 on the groups $C_1(x)$, $x \in X_0$ by conjugation), the boundary morphisms are operator morphisms, $\delta_{n-1}\delta_n = 0$, $n \geqslant 3$, and the additional axioms are satisfied that

4.3 $b^{-1}cb = c^{\delta_2 b}$, $b, c \in C_2$, *so that* $\delta_2 : C_2 \to C_1$ *is a crossed module (of groupoids);*

4.4 *if* $c \in C_2$ *then* $\delta_2 c$ *acts trivially on* C_n *for* $n \geqslant 3$*;*

4.5 *each group* $C_n(x)$ *is Abelian for* $n \geqslant 3$*, and so the family* C_n *is a* C_1*-module.*

Clearly we obtain a category Crs of crossed complexes; this category is not so familiar and so we give arguments for using it in the next section.

As algebraic examples of crossed complexes we have: $C = \mathbb{C}(G, n)$ where G is a group, commutative if $n \geqslant 2$, and C is G in dimension n and trivial elsewhere; $C = \mathbb{C}(G, 1 : M, n)$, where G is a group, M is a G-module, $n \geqslant 2$, and C is G in dimension 1, M in dimension n, trivial elsewhere, and zero boundary if $n = 2$; C is a crossed module (of groups) in dimensions 1 and 2 and trivial elsewhere.

A crossed complex C has a fundamental groupoid $\pi_1 C = C_1 / \operatorname{Im} \delta_2$, and also for $n \geqslant 2$ a family $\{H_n(C, p) | p \in C_0\}$ of homology groups.

5 Why crossed complexes?

• They generalise groupoids and crossed modules to all dimensions. Note that the natural context for second relative homotopy groups is crossed modules of groupoids, rather than groups.

• They are good for modelling *CW*-complexes.

• Free crossed resolutions enable calculations with small *CW*-complexes and *CW*-maps, see section 15.

• Crossed complexes give a kind of 'linear model' of homotopy types which includes all 2-types. Thus although they are not the most general model by any means (they do not contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. This is how a general n-adic Hurewicz Theorem was found [BLo87b].

• They are convenient for *calculation*, and the functor Π is classical, involving relative homotopy groups. We explain some results in this form later.

• They are close to chain complexes with a group(oid) of operators, and related to some classical homological algebra (e.g. *chains of syzygies*). In fact if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly non commutative version of the singular chains of a space.

• The monoidal structure is suggestive of further developments (e.g. *crossed differential algebras*) see [BaT97, BaBr93]. It is used in [BGi89] to give an algebraic model of homotopy 3-types, and to discuss automorphisms of crossed modules.

• Crossed complexes have a good homotopy theory, with a *cylinder object, and homotopy colimits*, [BGo89]. The homotopy classification result 4.1.9 generalises a classical theorem of Eilenberg-Mac Lane. Applications of (the simplicial version) are given in for example [FM07, FMP07, PoT07].

• They have an interesting relation with the Moore complex of simplicial groups and of simplicial groupoids (see section 18).

6 Why cubical ω-groupoids with connections?

The definition of these objects is more difficult to give, but will be indicated in section 9. Here we explain why these structures are a kind of engine giving the power behind the theory.

• The functor ρ gives a form of *higher homotopy groupoid*, thus confirming the visions of the early topologists.

• They are equivalent to crossed complexes.

• They have a clear *monoidal closed structure*, and a notion of homotopy, from which one can deduce those on crossed complexes, using the equivalence of categories.

• It is easy to relate the functor ρ to tensor products, but quite difficult to do this directly for Π.

• Cubical methods, unlike globular or simplicial methods, allow for a simple *algebraic inverse to subdivision*, which is crucial for our local-to-global theorems.

• The additional structure of 'connections', and the equivalence with crossed complexes, allows for the sophisticated notion of *commutative cube*, and the proof that *multiple compositions of commutative cubes are commutative*. The last fact is a key component of the proof of the HHvKT.

• They yield a construction of a *(cubical) classifying space* $B^{\square}C = (\mathcal{B}^{\square}C)_{\infty}$ of a crossed complex C, which generalises (cubical) versions of Eilenberg-Mac Lane spaces, including the local coefficient case. This has convenient relation to homotopies.

• There is a current *resurgence of the use of cubes* in for example combinatorics, algebraic topology, and concurrency. There is a Dold-Kan type theorem for cubical Abelian groups with connections [BrH03].

7 The equivalence of categories

Let Crs, ω-Gpd denote respectively the categories of crossed complexes and ω-groupoids: we use the latter term as an abbreviation of 'cubical ω-groupoids with connections'. A major part of the work consists in defining these categories and proving their equivalence, which thus gives an example of two algebraically defined categories whose equivalence is non trivial. It is even more subtle than that because the functors $\gamma : \mathsf{Crs} \to \omega\text{--}\mathsf{Gpd}$, $\lambda : \omega\text{--}\mathsf{Gpd} \to \mathsf{Crs}$ are not hard to define, and it is easy to prove $\gamma\lambda \simeq 1$. The hard part is to prove $\lambda\gamma \simeq 1$, which shows that an ω-groupoid G may be reconstructed from the crossed complex $\gamma(G)$ it contains. The proof involves using the connections to construct a 'folding map' $\Phi : G_n \to G_n$, with image $\gamma(G)_n$, and establishing its major properties, including the relations with the compositions. This gives an algebraic form of some old intuitions of several ways of defining relative homotopy groups, for example using cubes or cells.

On the way we establish properties of *thin elements*, as those which fold down to 1, and show that G satisfies a strong Kan extension condition, namely that every box has a unique thin filler. This result plays a key role in the proof of the HHvKT for ρ, since it is used to show an independence of choice. That part of the proof goes by showing that the two choices can be seen, since we start with a homotopy, as given by the two ends $\partial^{\pm}_{n+1}x$ of an $(n+1)$-cube x. It is then shown by induction, using the method of construction and the above result, that x is degenerate in direction $n+1$. Hence the two ends in that direction coincide.

Properties of the folding map are used also in showing that ΠX_* is actually included in ρX_*; in relating two types of thinness for elements of ρX_*; and in proving a *homotopy addition lemma* in ρX_*.

Any ω-Gpd G has an underlying cubical set UG. If C is a crossed complex, then the cubical set $U(\lambda C)$ is called the *cubical nerve* $N^{\square}C$ of C. It is a conclusion of the theory that we can also obtain $N^{\square}C$ as

$$(N^{\square}C)_n = \mathsf{Crs}(\Pi I^n_*, C)$$

where I^n_* is the usual geometric cube with its standard skeletal filtration. The (cubical) geometric realisation $|N^{\square}C|$ is also called the *cubical classifying space* $B^{\square}C$ of the crossed complex C. The filtration C^* of C by skeleta gives a filtration $B^{\square}C^*$ of $B^{\square}C$ and there is (as in 4.1.6) a natural isomorphism $\Pi(B^{\square}C^*) \cong C$. Thus the properties of a crossed complex are those that are universally satisfied by ΠX_*. These proofs use the equivalence of the homotopy categories of Kan[8] cubical sets and of CW-complexes. We originally took this from the Warwick Masters thesis of S. Hintze, but it is now available with different proofs from Antolini [Ant96] and Jardine [Jar06].

As said above, by taking particular values for C, the classifying space $B^{\square}C$ gives cubical versions of Eilenberg-Mac Lane spaces $K(G,n)$, including the case $n=1$ and G non commutative. If C is essentially a crossed module, then $B^{\square}C$ is called the *cubical classifying space* of the crossed module, and in fact realises the k-invariant of the crossed module.

Another useful result is that if K is a cubical set, then $\rho(|K|_*)$ may be identified with $\rho(K)$, the *free* ω-Gpd *on the cubical set* K, where here $|K|_*$ is the usual filtration by skeleta. On the other hand, our proof that $\Pi(|K|_*)$ is the free crossed complex on the non-degenerate cubes of K uses the generalised HHvKT of the next section.

It is also possible to give simplicial and globular versions of some of the above results, because the category of crossed complexes is equivalent also to those of simplicial T-complexes [Ash88] and of globular ∞-groupoids [BHi81c]. In fact the published paper on the classifying space of a crossed complex [BHi91] is given in simplicial terms, in order to link more easily with well known theories.

[8]The notion of Kan cubical set K is also called a cofibrant cubical set. It is an extension condition that any partial r-box in K is the partial boundary of an element of K_r. See for example [Jar06], but the idea goes back to the first papers by D. Kan [Kan55, Kan55].

8 First main aim of the work: Higher Homotopy van Kampen Theorems

These theorems give *non commutative tools for higher dimensional local-to-global problems* yielding a variety of new, often non commutative, calculations, which *prove* (i.e. test) the theory. We now explain these theorems in a way which strengthens the relation with descent, since that was a theme of the conference at which the talk was given on which this survey is based.

We suppose given an open cover $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of X. This cover defines a map

$$q : E = \bigsqcup_{\lambda\in\Lambda} U^\lambda \to X$$

and so we can form an augmented simplicial space

$$\check{C}(q) : \ \cdots E \times_X E \times_X E \rightrightarrows\!\!\!\to E \times_X E \rightrightarrows E \xrightarrow{q} X$$

where the higher dimensional terms involve disjoint unions of multiple intersections U^ν of the U^λ.

We now suppose given a filtered space X_*, a cover $\mathcal{U}$ as above of $X = X_\infty$, and so an augmented simplicial filtered space $\check{C}(q_*)$ involving multiple intersections U^ν_* of the induced filtered spaces.

We still need a connectivity condition.

Definition 8.1 A filtered space X_* is *connected* if and only if the induced maps $\pi_0 X_0 \to \pi_0 X_n$ are surjective and $\pi_n(X_r, X_n, \nu) = 0$ for all $n > 0, r > n$ and $\nu \in X_0$.

Theorem 8.2 (MAIN RESULT (HHvKT)) *If U^ν_* is connected for all finite intersections U^ν of the elements of the open cover, then*
(C) (connectivity) X_* *is connected, and*
(I) (isomorphism) *the following diagram as part of* $\rho(\check{C}(q_*))$

$$\rho(E_* \times_{X_*} E_*) \rightrightarrows \rho E_* \xrightarrow{\rho(q_*)} \rho X_*. \qquad (c\rho)$$

is a coequaliser diagram. Hence the following diagram of crossed complexes

$$\Pi(E_* \times_{X_*} E_*) \rightrightarrows \Pi E_* \xrightarrow{\Pi(q_*)} \Pi X_*. \qquad (c\Pi)$$

is also a coequaliser diagram.

So we get calculations of the fundamental crossed complex ΠX_*.

It should be emphasised that to get to and apply this theorem takes just the two papers [BHi81a, BHi81b] totalling 58 pages. With this we deduce in the first instance:

- the usual vKT for the fundamental groupoid on a set of base points;
- the Brouwer degree theorem ($\pi_n S^n = \mathbb{Z}$);
- the relative Hurewicz theorem;

- Whitehead's theorem that $\pi_n(X \cup \{e^2_\lambda\}, X)$ is a free crossed module;
- an excision result, more general than the previous two, on $\pi_n(A \cup B, A, x)$ as an induced module (crossed module if $n = 2$) when $(A, A \cap B)$ is $(n-1)$-connected.

The assumptions required of the reader are quite small, just some familiarity with CW-complexes. This contrasts with some expositions of basic homotopy theory, where the proof of say the relative Hurewicz theorem requires knowledge of singular homology theory. Of course it is surprising to get this last theorem without homology, but this is because it is seen as a statement on the morphism of relative homotopy groups

$$\pi_n(X, A, x) \to \pi_n(X \cup CA, CA, x) \cong \pi_n(X \cup CA, x)$$

and is obtained, like our proof of Theorem W, as a special case of an excision result. The reason for this success is that *we use algebraic structures which model the underlying processes of the geometry* more closely than those in common use. These algebraic structures and their relations are quite intricate, as befits the complications of homotopy theory, so the theory is tight knit.

Note also that these results cope well with the action of the fundamental group on higher homotopy groups.

The calculational use of the HHvKT for ΠX_* is enhanced by the relation of Π with tensor products (see section 15 for more details).

9 The fundamental cubical ω–groupoid ρX_* of a filtered space X_*

Here are the basic elements of the construction.

I^n_*: the n-cube with its skeletal filtration.

Set $R_n X_* = \mathsf{FTop}(I^n_*, X_*)$. This is a *cubical set with compositions, connections, and inversions*.

For $i = 1, \ldots, n$ there are standard:

face maps $\partial^\pm_i : R_n X_* \to R_{n-1} X_*$;

degeneracy maps $\varepsilon_i : R_{n-1} X_* \to R_n X_*$

connections $\Gamma^\pm_i : R_{n-1} X_* \to R_n X_*$

compositions $a \circ_i b$ defined for $a, b \in R_n X_*$ such that $\partial^+_i a = \partial^-_i b$

inversions $-_i : R_n \to R_n$.

The connections are induced by $\gamma^\alpha_i : I^n \to I^{n-1}$ defined using the monoid structures $\max, \min : I^2 \to I$. They are essential for many reasons, e.g. to discuss the notion of *commutative cube*.

These operations have certain algebraic properties which are easily derived from the geometry and which we do not itemise here – see for example [AABS02]. These were listed first in the Bangor thesis of Al-Agl [AAl89]. (In the paper [BHi81a] the only basic connections needed are the Γ^+_i, from which the Γ^-_i are derived using the inverses of the groupoid structures.)

Now it is natural and convenient to define $f \equiv g$ for $f, g : I^n_* \to X_*$ to mean f is homotopic to g through filtered maps an relative to the vertices of I^n. This gives a quotient map

$$p : R_n X_* \to \rho_n X_* = (R_n X_* / \equiv).$$

The following results are proved in [BHi81b].

9.1 *The compositions on RX_* are inherited by ρX_* to give ρX_* the structure of cubical multiple groupoid with connections.*

9.2 *The map $p : RX_* \to \rho X_*$ is a Kan fibration of cubical sets.*

The proofs of both results use methods of collapsing which are indicated in the next section. The second result is almost unbelievable. Its proof has to give a systematic method of deforming a cube with the right faces 'up to homotopy' into a cube with exactly the right faces, using the given homotopies. In both cases, the assumption that the relation $\equiv$ uses homotopies relative to the vertices is essential to start the induction. (In fact the paper [BHi81b] does not use homotopy relative to the vertices, but imposes an extra condition J_0, that each loop in X_0 is contractible X_1, which again starts the induction. This condition is awkward in applications, for example to function spaces. A full exposition of the whole story is in preparation, [BHS09].)

An essential ingredient in the proof of the HHvKT is the notion of *multiple composition*. We have discussed this already in dimension 2, with a suggestive picture in the diagram (3). In dimension n, the aim is to give algebraic expression to the idea of a cube I^n being subdivided by hyperplanes parallel to the faces into many small cubes, a subdivision with a long history in mathematics.

Let $(m) = (m_1, \ldots, m_n)$ be an n-tuple of positive integers and

$$\phi_{(m)} : I^n \to [0, m_1] \times \cdots \times [0, m_n]$$

be the map $(x_1, \ldots, x_n) \mapsto (m_1 x_1, \ldots, m_n x_n)$. Then a *subdivision of type* (m) of a map $\alpha : I^n \to X$ is a factorisation $\alpha = \alpha' \circ \phi_{(m)}$; its *parts* are the cubes $\alpha_{(r)}$ where $(r) = (r_1, \ldots, r_n)$ is an n-tuple of integers with $1 \leqslant r_i \leqslant m_i$, $i = 1, \ldots, n$, and where $\alpha_{(r)} : I^n \to X$ is given by

$$(x_1, \ldots, x_n) \mapsto \alpha'(x_1 + r_1 - 1, \ldots, x_n + r_n - 1).$$

We then say that α is the *composite* of the cubes $\alpha_{(r)}$ and write $\alpha = [\alpha_{(r)}]$. The *domain* of $\alpha_{(r)}$ is then the set $\{(x_1, \ldots, x_n) \in I^n : r_i - 1 \leqslant x_i \leqslant r_i, 1 \leqslant i \leqslant n\}$. This ability to express 'algebraic inverse to subdivision' is one benefit of using cubical methods.

Similarly, in a cubical set with compositions satisfying the interchange law we can define the multiple composition $[\alpha_{(r)}]$ of a multiple array $(\alpha_{(r)})$ provided the obviously necessary multiple incidence relations of the individual $\alpha_{(r)}$ to their neighbours are satisfied.

Here is an application which is essential in many proofs, and which seems hard to prove without the techniques involved in 9.2.

Theorem 9.3 (Lifting multiple compositions) *Let $[\alpha_{(r)}]$ be a multiple composition in $\rho_n X_*$. Then representatives $a_{(r)}$ of the $\alpha_{(r)}$ may be chosen so that the multiple composition $[a_{(r)}]$ is well defined in $R_n X_*$.*

Proof: The multiple composition $[\alpha_{(r)}]$ determines a cubical map

$$A : K \to \rho X_*$$

where the cubical set K corresponds to a representation of the multiple composition by a subdivision of the geometric cube, so that top cells $c_{(r)}$ of K are mapped by A to $\alpha_{(r)}$.

Consider the diagram, in which $*$ is a corner vertex of K,

$$\begin{tikzcd} * \arrow[r] \arrow[d] & RX_* \arrow[d, "p"] \\ K \arrow[r, "A"'] \arrow[ur, dashed, "A'"] & \rho X_* \end{tikzcd} .$$

Then K collapses to $*$, written $K \searrow *$. (As an example, see how the subdivision in the diagram (3) may be collapsed row by row to a point.) By the fibration result, A lifts to A', which represents $[a_{(r)}]$, as required.

So we have to explain collapsing.

10 Collapsing

We use a basic notion of collapsing and expanding due to J.H.C. Whitehead, [W:SHT].

Let $C \subseteq B$ be subcomplexes of I^n. We say C is an *elementary collapse* of B, $B \searrow^e C$, if for some $s \geqslant 1$ there is an s-cell a of B and $(s-1)$-face b of a, the *free face*, such that

$$B = C \cup a, \qquad C \cap a = \dot{a} \setminus b$$

(where $\dot{a}$ denotes the union of the proper faces of a).

We say B_1 *collapses* to B_r, written $B_1 \searrow B_r$, if there is a sequence

$$B_1 \searrow^e B_2 \searrow^e \cdots \searrow^e B_r$$

of elementary collapses.

If C is a subcomplex of B then

$$B \times I \searrow (B \times \{0\} \cup C \times I)$$

(this is proved by induction on dimension of $B \setminus C$).

Further, I^n collapses to any one of its vertices (this may be proved by induction on n using the first example). These collapsing techniques allows the construction of the extensions of filtered maps and filtered homotopies that are crucial for proving 9.1, that ρX_* does obtain the structure of multiple groupoid.

However, more subtle collapsing techniques using partial boxes are required to prove the fibration theorem 9.2, as partly explained in the next section.

11 Partial boxes

Let C be an r-cell in the n-cube I^n. Two $(r-1)$-faces of C are called *opposite* if they do not meet.

A partial box in C is a subcomplex B of C generated by one $(r-1)$-face b of C (called a *base* of B) and a number, possibly zero, of other $(r-1)$-faces of C none of which is opposite to b.

The partial box is a box if its $(r-1)$-cells consist of all but one of the $(r-1)$-faces of C.

The proof of the fibration theorem uses a filter homotopy extension property and the following:

Proposition 11.1 (Key Proposition) *Let B, B' be partial boxes in an r-cell C of I^n such that $B' \subseteq B$. Then there is a chain*

$$B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1 = B'$$

such that

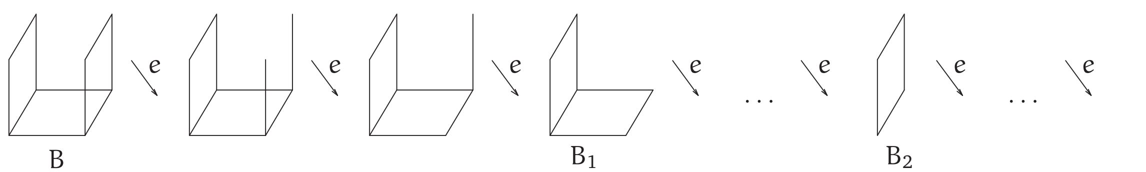
(i) *each B_i is a partial box in C;*

(ii) $B_{i+1} = B_i \cup a_i$ *where a_i is an $(r-1)$-cell of C not in B_i;*

(iii) $a_i \cap B_i$ *is a partial box in a_i.*

The proof is quite neat, and follows the pictures. Induction up such a chain of partial boxes is one of the steps in the proof of the fibration theorem 9.2. The proposition implies that an inclusion of partial boxes is what is known as an anodyne extension, [Jar06].

Here is an example of a sequence of collapsings of a partial box B, which illustrate some choices in forming a collapse $B \searrow \mathbf{0}$ through two other partial boxes B_1, B_2.



The proof of the fibration theorem gives a program for carrying out the deformations needed to do the lifting. In some sense, it implies computing a multiple composition can be done using collapsing as the guide.

Methods of collapsing generalise methods of trees in dimension 1. The above use of partial boxes is related to methods of Kan in [Kan55].

12 Thin elements

Another key concept is that of *thin element* $\alpha \in \rho_n X_*$ for $n \geqslant 2$. The proofs here use strongly results of [BHi81a].

We say α is *geometrically thin* if it has a *deficient* representative, i.e. an $a : I^n_* \to X_*$ such that $a(I^n) \subseteq X_{n-1}$.

We say α is *algebraically thin* if it is a multiple composition of degenerate elements or those coming from repeated (including 0) negatives of connections. Clearly any multiple composition of algebraically thin elements is thin.

Theorem 12.1 (i) *Algebraically thin is equivalent to geometrically thin.*

(ii) *In a cubical ω–groupoid with connections, any box has a unique thin filler.*

Proof The proof of the forward implication in (i) uses lifting of multiple compositions, in a stronger form than stated above.

The proofs of (ii) and the backward implication in (i) use the full force of the algebraic relation between ω–groupoids and crossed complexes. □

These results allow one to replace arguments with commutative cubes by arguments with thin elements.

13 Sketch proof of the HHvKT

The proof goes by verifying the required universal property. Let $\mathcal{U}$ be an open cover of X as in Theorem 8.2.

We go back to the following diagram whose top row is part of $\rho(\check{C}(q_*))$

$$\begin{tikzcd} \rho(E_* \times_{X_*} E_*) \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & \rho(E_*) \arrow[r, "\rho(q_*)"] \arrow[dr, "f"'] & \rho X_* \arrow[d, dashed, "f'"] \\ & & G \end{tikzcd} \qquad (c\rho)$$

To prove this top row is a coequaliser diagram, we suppose given a morphism $f : \rho(E_*) \to G$ of cubical ω-groupoids with connection such that $f \circ \partial_0 = f \circ \partial_1$, and prove that there is a unique morphism $f' : \rho X_* \to G$ such that $f' \circ \rho(q_*) = f$.

To define $f'(\alpha)$ for $\alpha \in \rho X_*$, you subdivide a representative a of α to give $a = [a_{(r)}]$ so that each $a_{(r)}$ lies in an element $U^{(r)}$ of $\mathcal{U}$; use the connectivity conditions and this subdivision to deform a into $b = [b_{(r)}]$ so that

$$b_{(r)} \in R(U_*^{(r)})$$

and so obtain

$$\beta_{(r)} \in \rho(U_*^{(r)}).$$

The elements

$$f\beta_{(r)} \in G$$

may be composed in G (by the conditions on f), to give an element

$$\theta(\alpha) = [f\beta_{(r)}] \in G.$$

So the proof of the universal property has to use an *algebraic inverse to subdivision*. Again an analogy here is with sending an email: the element you start with is subdivided, deformed so that each part is correctly labelled, the separate parts are sent, and then recombined.

The proof that $\theta(\alpha)$ is independent of the choices made uses crucially properties of thin elements. The key point is: *a filter homotopy* $h : \alpha \equiv \alpha'$ *in* $R_n X_*$ *gives a deficient element of* $R_{n+1}X_*$.

The method is to do the subdivision and deformation argument on such a homotopy, push the little bits in some

$$\rho_{n+1}(U_*^\lambda)$$

(now thin) over to G, combine them and get a thin element

$$\tau \in G_{n+1}$$

all of whose faces not involving the direction $(n + 1)$ are thin *because* h *was given to be a filter homotopy*. An inductive argument on unique thin fillers of boxes then shows that τ *is degenerate in direction* $(n + 1)$, so that the two ends in direction $(n + 1)$ are the same.

This ends a rough sketch of the proof of the HHvKT for ρ.

Note that the theory of these forms of multiple groupoids is designed to make this last argument work. We replace a formula for saying a cube h has commutative boundary by a statement that h is thin. It would be very difficult to replace the above argument, on the composition of thin elements, by a higher dimensional manipulation of formulae such as that given in section 3 for a commutative 3-cube.

Further, the proof does not require knowledge of the existence of all coequalisers, not does it give a recipe for constructing these in specific examples.

14 Tensor products and homotopies

The construction of the monoidal closed structure on the category ω-Gpd is based on rather formal properties of cubical sets, and the fact that for the cubical set $\mathbb{I}^n$ we have $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$. The details are given in [BHi87]. The equivalence of categories implies then that the category Crs is also monoidal closed, with a natural isomorphism

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C)).$$

Here the elements of the 'internal hom' crossed complex $\mathsf{CRS}(B, C)$ are in dimension: $n = 0$ the morphisms $B \to C$; in dimension $n = 1$ the *left homotopies of morphisms*; and in higher dimensions are forms of higher homotopies. The precise description of these is obtained of course by tracing out in detail the equivalence of categories. It should be emphasised that certain choices are made in constructing this equivalence, and these choices are reflected in the final formulae that are obtained.

An important result is that if X_*, Y_* are filtered spaces, then there is a natural transformation

$$\begin{aligned} \eta \;:\; \rho X_* \otimes \rho Y_* &\to \rho(X_* \otimes Y_*) \\ [a] \otimes [b] \;&\mapsto [a \otimes b] \end{aligned}$$

where if $a : I_*^m \to X_*$, $b : I_*^n \to Y_*$ then $a \otimes b : I_*^{m+n} \to X_* \otimes Y_*$. It not hard to see, in this cubical setting, that η is well defined. It can also be shown using previous results that η is an isomorphism if X_*, Y_* are the geometric realisations of cubical sets with the usual skeletal filtration.

The equivalence of categories now gives a natural transformation of crossed complexes

$$\eta' \;:\; \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*). \tag{8}$$

It would be hard to construct this directly. It is proved in [BHi91] that η' is an isomorphism if X_*, Y_* are the skeletal filtrations of CW-complexes. The proof uses the HHvKT, and the fact that $A \otimes -$ on crossed complexes has a right adjoint and so preserves colimits. It is proved in [BaBr93] that η is an isomorphism if X_*, Y_* are cofibred, connected filtered spaces. This applies in particular to the useful case of the filtration $B^{\square}C^*$ of the classifying space of a crossed complex.

It turns out that the defining rules for the tensor product of crossed complexes which follows from the above construction are obtained as follows. We first define a bimorphism of crossed complexes.

Definition 14.1 A *bimorphism* $\theta : (A, B) \to C$ of crossed complexes is a family of maps $\theta : A_m \times B_n \to C_{m+n}$ satisfying the following conditions, where $a \in A_m, b \in B_n, a_1 \in A_1, b_1 \in B_1$ (temporarily using additive notation throughout the definition):

(i)

$$\beta(\theta(a, b)) \;=\; \theta(\beta a, \beta b) \;\text{ for all }\; a \in A, b \in B\,.$$

(ii)

$$\begin{aligned} \theta(a, b^{b_1}) \;&=\; \theta(a, b)^{\theta(\beta a, b_1)} \;\text{ if }\; m \geqslant 0, n \geqslant 2\,, \\ \theta(a^{a_1}, b) \;&=\; \theta(a, b)^{\theta(a_1, \beta b)} \;\text{ if }\; m \geqslant 2, n \geqslant 0\,. \end{aligned}$$

(iii)

$$\begin{aligned} \theta(a, b + b') \;&=\; \begin{cases} \theta(a, b) + \theta(a, b') & \text{if } m = 0, n \geqslant 1 \text{ or } m \geqslant 1, n \geqslant 2\,, \\ \theta(a, b)^{\theta(\beta a, b')} + \theta(a, b') & \text{if } m \geqslant 1, n = 1\,, \end{cases} \\ \theta(a + a', b) \;&=\; \begin{cases} \theta(a, b) + \theta(a', b) & \text{if } m \geqslant 1, n = 0 \;\text{ or } m \geqslant 2, n \geqslant 1\,, \\ \theta(a', b) + \theta(a, b)^{\theta(a', \beta b)} & \text{if } m = 1, n \geqslant 1\,. \end{cases} \end{aligned}$$

(iv)

$$\delta_{m+n}(\theta(a, b)) \;=\; \begin{cases} \theta(\delta_m a, b) + (-)^m \theta(a, \delta_n b) & \text{if } m \geqslant 2, n \geqslant 2\,, \\ -\theta(a, \delta_n b) - \theta(\beta a, b) + \theta(\alpha a, b)^{\theta(a, \beta b)} & \text{if } m = 1, n \geqslant 2\,, \\ (-)^{m+1}\theta(a, \beta b) + (-)^m \theta(a, \alpha b)^{\theta(\beta a, b)} + \theta(\delta_m a, b) & \text{if } m \geqslant 2, n = 1\,, \\ -\theta(\beta a, b) - \theta(a, \alpha b) + \theta(\alpha a, b) + \theta(a, \beta b) & \text{if } m = n = 1\,. \end{cases}$$

(v)

$$\delta_{m+n}(\theta(a,b)) \;=\; \begin{cases} \theta(a,\delta_n b) & \text{if } m=0, n\geqslant 2\,, \\ \theta(\delta_m a, b) & \text{if } m\geqslant 2, n=0\,. \end{cases}$$

(vi)

$$\begin{aligned} \alpha(\theta(a,b)) &= \theta(a,\alpha b) \quad \text{and} \quad \beta(\theta(a,b)) = \theta(a,\beta b) \quad \text{if } m=0, n=1\,, \\ \alpha(\theta(a,b)) &= \theta(\alpha a, b) \quad \text{and} \quad \beta(\theta(a,b)) = \theta(\beta a, b) \quad \text{if } m=1, n=0\,. \end{aligned}$$

The *tensor product* of crossed complexes A, B is given by the universal bimorphism $(A,B) \to A \otimes B$, $(a,b) \mapsto a \otimes b$. The rules for the tensor product are obtained by replacing $\theta(a,b)$ by $a \otimes b$ in the above formulae.

The conventions for these formulae for the tensor product arise from the derivation of the tensor product via the category of cubical ω-groupoids with connections, and the formulae are forced by our conventions for the equivalence of the two categories [BHi81a, BHi87].

The complexity of these formulae is directly related to the complexities of the cell structure of the product $E^m \times E^n$ where the n-cell E^n has cell structure e^0 if $n=0$, $e^0_\pm \cup e^1$ if $n=1$, and $e^0 \cup e^{n-1} \cup e^n$ if $n \geqslant 2$.

It is proved in [BHi87] that the bifunctor $-\otimes-$ is symmetric and that if a_0 is a vertex of A then the morphism $B \to A \otimes B,\ b \to a_0 \otimes b$, is injective.

There is a standard groupoid model I of the unit interval, namely the indiscrete groupoid on two objects $0, 1$. This is easily extended trivially to either a crossed complex or an ω-Gpd. So using $\otimes$ we can define a 'cylinder object' $\mathsf{I} \otimes -$ in these categories and so a homotopy theory, [BGo89].

15 Free crossed complexes and free crossed resolutions

Let C be a crossed complex. A *free basis* B_* for C consists of the following:

B_0 is set which we take to be C_0;

B_1 is a graph with source and target maps $s, t : B_1 \to B_0$ and C_1 is the free groupoid on the graph B_1: that is B_1 is a subgraph of C_1 and any graph morphism $B_1 \to G$ to a groupoid G extends uniquely to a groupoid morphism $C_1 \to G$;

B_n is, for $n \geqslant 2$, a totally disconnected subgraph of C_n with target map $t : B_n \to B_0$; for $n = 2$, C_2 is the free crossed C_1-module on B_2 while for $n > 2$, C_n is the free $(\pi_1 C)$-module on B_n.

It may be proved using the HHvKT that if X_* is a CW-complex with the skeletal filtration, then ΠX_* is the free crossed complex on the characteristic maps of the cells of X_*. It is proved in [BHi91] that the tensor product of free crossed complexes is free.

A *free crossed resolution* F_* of a groupoid G is a free crossed complex which is aspherical together with an isomorphism $\phi : \pi_1(F_*) \to G$. Analogues of standard methods of homological algebra show that free crossed resolutions of a group are unique up to homotopy equivalence.

In order to apply this result to free crossed resolutions, we need to replace free crossed resolutions by CW-complexes. A fundamental result for this is the following, which goes back to Whitehead [W:SHT] and Wall [Wal66], and which is discussed further by Baues in [Bau89, Chapter VI, §7]:

Theorem 15.1 *Let X_* be a CW-filtered space, and let $\phi : \Pi X_* \to C$ be a homotopy equivalence to a free crossed complex with a preferred free basis. Then there is a CW-filtered space Y_*, and an isomorphism $\Pi Y_* \cong C$ of crossed complexes with preferred basis, such that ϕ is realised by a homotopy equivalence $X_* \to Y_*$.*

In fact, as pointed out by Baues, Wall states his result in terms of chain complexes, but the crossed complex formulation seems more natural, and avoids questions of realisability in dimension 2, which are unsolved for chain complexes.

Corollary 15.2 *If A is a free crossed resolution of a group G, then A is realised as free crossed complex with preferred basis by some CW-filtered space Y_*.*

Proof We only have to note that the group G has a classifying CW-space BG whose fundamental crossed complex $\Pi(BG)$ is homotopy equivalent to A. □

Baues also points out in [Bau89, p.657] an extension of these results which we can apply to the realisation of morphisms of free crossed resolutions. A new proof of this extension is given by Faria Martins in [FM07a], using methods of Ashley [Ash88].

Proposition 15.3 *Let $X = K(G, 1)$, $Y = K(H, 1)$ be CW-models of Eilenberg - Mac Lane spaces and let $h : \Pi X_* \to \Pi(Y_*)$ be a morphism of their fundamental crossed complexes with the preferred bases given by skeletal filtrations. Then $h = \Pi(g)$ for some cellular $g : X \to Y$.*

Proof Certainly h is homotopic to $\Pi(f)$ for some $f : X \to Y$ since the set of pointed homotopy classes $X \to Y$ is bijective with the morphisms of groups $A \to B$. The result follows from [Bau89, p.657,(**)] ('if f is Π-realisable, then each element in the homotopy class of f is Π-realisable'). □

These results are exploited in [Moo01, BMPW02] to calculate free crossed resolutions of the fundamental groupoid of a graph of groups.

An algorithmic approach to the calculation of free crossed resolutions for groups is given in [BRS99], by constructing partial contracting homotopies for the universal cover at the same time as constructing this universal cover inductively. This has been implemented in GAP4 by Heyworth and Wensley [HWe06].

16 Classifying spaces and the homotopy classification of maps

The formal relations of cubical sets and of cubical ω-groupoids with connections and the relation of Kan cubical sets with topological spaces, allow the proof of a homotopy classification theorem:

Theorem 16.1 *If K is a cubical set, and G is an ω-groupoid, then there is a natural bijection of sets of homotopy classes*

$$[|K|, |UG|] \cong [\rho(|K|_*), G],$$

where on the left hand side we work in the category of spaces, and on the right in ω-groupoids.

Here $|K|_*$ is the filtration by skeleta of the geometric realisation of the cubical set.

We explained earlier how to define a cubical classifying space say $B^{\square}(C)$ of a crossed complex C as $B^{\square}(C) = |UN^{\square}C| = |U\lambda C|$. The properties already stated now give the homotopy classification theorem 4.1.9.

It is shown in [BHi81b] that for a CW-complex Y there is a map $p : Y \to B^{\square}\Pi Y_*$ whose homotopy fibre is n-connected if Y is connected and $\pi_i Y = 0$ for $2 \leqslant i \leqslant n-1$. It follows that if also X is a connected CW-complex with $\dim X \leqslant n$, then p induces a bijection

$$[X, Y] \to [X, B\Pi Y_*].$$

So under these circumstances we get a bijection

$$[X, Y] \to [\Pi X_*, \Pi Y_*]. \tag{9}$$

This result, due to Whitehead [W:CHII], translates a topological homotopy classification problem to an algebraic one. We explain below how this result can be translated to a result on chain complexes with operators.

It is also possible to define a simplicial nerve $N^{\Delta}(C)$ of a crossed complex C by

$$N^{\Delta}(C)_n = \mathsf{Crs}(\Pi(\Delta^n), C).$$

The *simplicial classifying space* of C is then defined using the simplicial geometric realisation

$$B^{\Delta}(C) = |N^{\Delta}(C)|.$$

The properties of this simplicial classifying space are developed in [BHi91], and in particular an analogue of 4.1.9 is proved.

The simplicial nerve and an adjointness

$$\mathsf{Crs}(\Pi(L), C) \cong \mathsf{Simp}(L, N^{\Delta}(C))$$

are used in [BGPT97, BGPT01] for an equivariant homotopy theory of crossed complexes and their classifying spaces. Important ingredients in this are notions of coherence and an Eilenberg-Zilber type theorem for crossed complexes proved in Tonks' Bangor thesis [Ton93, Ton03]. See also [BSi07].

Labesse in [Lab99] defines a *crossed set*. In fact a crossed set is exactly a crossed module (of groupoids) $\delta : C \to X \rtimes G$ where G is a group acting on the set X, and $X \rtimes G$ is the associated actor groupoid; thus the simplicial construction from a crossed set described by Larry Breen in [Lab99] is exactly the simplicial nerve of the crossed module, regarded as a crossed complex. Hence the cohomology with coefficients in a crossed set used in [Lab99] is a special case of cohomology with coefficients in a crossed complex, dealt with in [BHi91]. (We are grateful to Breen for pointing this out to us in 1999.)

17 Relation with chain complexes with a groupoid of operators

Chain complexes with a group of operators are a well known tool in algebraic topology, where they arise naturally as the chain complex $C_*\widetilde{X}_*$ of cellular chains of the universal cover $\widetilde{X}_*$ of a reduced CW-complex X_*. The group of operators here is the fundamental group of the space X.

J.H.C. Whitehead in [W:CHII] gave an interesting relation between his free crossed complexes (he called them 'homotopy systems') and such chain complexes. We refer later to his important homotopy classification results in this area. Here we explain the relation with the Fox free differential calculus [Fox53].

Let $\mu : M \to P$ be a crossed module of groups, and let $G = \mathrm{Coker}\mu$. Then there is an associated diagram

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "h_2"'] & P \arrow[r, "\phi"] \arrow[d, "h_1"] & G \arrow[d, "h_0"] \\ M^{ab} \arrow[r, "\partial_2"'] & D_\phi \arrow[r, "\partial_1"'] & \mathbb{Z}[G] \end{tikzcd} \tag{10}$$

in which the second row consists of (right) G-modules and module morphisms. Here h_2 is simply the Abelian isation map; $h_1 : P \to D_\phi$ is the universal ϕ-derivation, that is it satisfies $h_1(pq) = h_1(p)^{\phi q} + h_1(q)$, for all $p, q \in P$, and is universal for this property; and h_0 is the usual derivation $g \mapsto g - 1$. Whitehead in his Lemma 7 of [W:CHII] gives this diagram in the case P is a free group, when he takes D_ϕ to be the free G-module on the same generators as the free generators of P. Our formulation, which uses the derived module due to Crowell [Cro71], includes his case. It is remarkable that diagram (10) is a commutative diagram in which the vertical maps are operator morphisms, and that the bottom row is defined by this property. The proof in [BHi90] follows essentially Whitehead's proof. The bottom row is exact: this follows from results in [Cro71], and is a reflection of a classical fact on group cohomology, namely the relation between central extensions and the Ext functor, see [McL63]. In the case the crossed module is the crossed module $\delta : C(\omega) \to F(X)$ derived from a presentation of a group, then $C(\omega)^{ab}$ is isomorphic to the free G-module on R, D_ϕ is the free G-module on X, and it is immediate from the above that ∂_2 is the usual derivative $(\partial r/\partial x)$ of Fox's free differential calculus [Fox53]. Thus Whitehead's results anticipate those of Fox.

It is also proved in [W:CHII] that if the restriction $M \to \mu(M)$ of μ has a section which is a morphism but not necessarily a P-map, then h_2 maps $\mathrm{Ker}\,\mu$ isomorphically to $\mathrm{Ker}\,\partial_2$. This allows calculation of the module of identities among relations by using module methods, and this is commonly exploited, see for example [ElK99] and the references there.

Whitehead introduced the categories CW of reduced CW-complexes, HS of homotopy systems, and FCC of free chain complexes with a group of operators, together with functors

$$\mathsf{CW} \xrightarrow{\Pi} \mathsf{HS} \xrightarrow{C} \mathsf{FCC}.$$

In each of these categories he introduced notions of homotopy and he proved that C induces an equivalence of the homotopy category of HS with a subcategory of the homotopy category of FCC. Further, $C\Pi X_*$ is isomorphic to the chain complex $C_*\widetilde{X}_*$ of cellular chains of the universal cover of X, so that under these circumstances there is a bijection of sets of homotopy classes

$$[\Pi X_*, \Pi Y_*] \to [C_*\widetilde{X}_*, C_*\widetilde{Y}_*]. \tag{11}$$

This with the bijection (9) can be interpreted as an operator version of the Hopf classification theorem. It is surprisingly little known. It includes results of Olum [Olu53] published later, and it enables quite useful calculations to be done easily, such as the homotopy classification of maps from a surface to the projective plane [Ell88], and other cases. Thus we see once again that this general theory leads to specific calculations.

All these results are generalised in [BHi90] to the non free case and to the non reduced case, which requires a groupoid of operators, thus giving functors

$$\mathsf{FTop} \xrightarrow{\Pi} \mathsf{Crs} \xrightarrow{\nabla} \mathsf{Chain}.$$

(The paper [BHi90] uses the notation Δ for this ∇.) One utility of the generalisation to groupoids is that the functor ∇ then has a right adjoint, and so preserves colimits. An example of this preservation is given in [BHi90, Example 2.10]. The construction of the right adjoint Θ to ∇ builds on a number of constructions used earlier in homological algebra.

The definitions of the categories under consideration in order to obtain a generalisation of the bijection (11) has to be quite careful, since it works in the groupoid case, and not all morphisms of the chain complex are realisable.

This analysis of the relations between these two categories is used in [BHi91] to give an account of cohomology with local coefficients.

It is also proved in [BHi90] that the functor ∇ preserves tensor products, where the tensor in the category Chain is a generalisation to modules over groupoids of the usual tensor for chain complexes of modules of groups. Since the tensor product is described explicitly in dimensions $\leqslant 2$ in [BHi87], and $(\nabla C)_n = C_n$ for $n \geqslant 3$, this preservation yields a complete description of the tensor product of crossed complexes.

18 Crossed complexes and simplicial groups and groupoids

The Moore complex NG of a simplicial group G is not in general a (reduced) crossed complex. Let D_nG be the subgroup of G_n generated by degenerate elements. Ashley showed in his thesis [Ash88] that NG is a crossed complex if and only if $(NG)_n \cap (DG)_n = \{1\}$ for all $n \geqslant 1$.

Ehlers and Porter in [EhP97, EhP99] show that there is a functor C from simplicial groupoids to crossed complexes in which $C(G)_n$ is obtained from $N(G)_n$ by factoring out

$$(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1}),$$

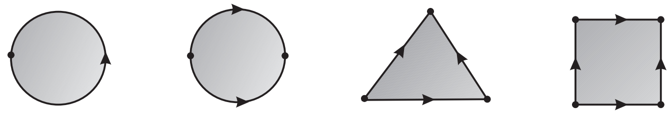
where the Moore complex is defined so that its differential comes from the last simplicial face operator.

This is one part of an investigation into the Moore complex of a simplicial group, of which the most general investigation is by Carrasco and Cegarra in [CaC91].

An important observation in [Por93] is that if $N \lhd G$ is an inclusion of a normal simplicial subgroup of a simplicial group, then the induced morphism on components $\pi_0(N) \to \pi_0(G)$ obtains the structure of crossed module. This is directly analogous to the fact that if $F \to E \to B$ is a fibration sequence then the induced morphism of fundamental groups $\pi_1(F, x) \to \pi_1(E, x)$ also obtains the structure of crossed module. This last fact is relevant to algebraic K-theory, where for a ring R the homotopy fibration sequence is taken to be $F \to B(GL(R)) \to B(GL(R))^+$.

19 Other homotopy multiple groupoids

A natural question is whether there are other useful forms of higher homotopy groupoids. It is because the geometry of convex sets is so much more complicated in dimensions > 1 than in dimension 1 that new complications emerge for the theories of higher order group theory and of higher homotopy groupoids. We have different geometries for example those of disks, globes, simplices, cubes, as shown in dimension 2 in the following diagram.



The cellular decomposition for an n-disk is $D^n = e^0 \cup e^{n-1} \cup e^n$, and that for globes is

$$G^n = e^0_\pm \cup e^1_\pm \cup \cdots \cup e^{n-1}_\pm \cup e^n.$$

The higher dimensional group(oid) theory reflecting the n-disks is that of crossed complexes, and that for the n-globes is called *globular ω-groupoids*.

A common notion of higher dimensional category is that of n-category, which generalise the 2-categories studied in the late 1960s. A 2-category C is a category enriched in categories, in the sense that each hom set $\mathsf{C}(x, y)$ is given the structure of category, and there are appropriate axioms. This gives inductively the notion of an n-category as a category enriched in $(n-1)$-categories. This is called a 'globular' approach to higher categories. The notion of n-category for all n was axiomatised in [BHi81c] and called an ∞–category; the underlying geometry of a family of sets $S_n, n \geqslant 0$ with operations

$$D^\alpha_i : S_n \to S_i, E_i : S_i \to S_n, \alpha = 0, 1; i = 1, \ldots, n-1$$

was there axiomatised. This was later called a 'globular set' [Str00], and the term ω-category was used instead of the earlier ∞-category. Difficulties of the globular approach are to define multiple compositions, and also monoidal closed structures, although these are clear in the cubical approach. A globular higher homotopy groupoid of a filtered space has been constructed in [Bro08a], deduced from cubical results.

Although the proof of the HHvKT outlined earlier does seem to require cubical methods, there is still a question of the place of globular and simplicial methods in this area. A simplicial analogue of the equivalence of categories is given in [Ash88, NTi89], using Dakin's notion of *simplicial* T-*complex*, [Dak76]. However it is difficult to describe in detail the notion of tensor product of such structures, or to formulate a proof of the HHvKT theorem in that context. There is a tendency to replace the term T-complex from all this earlier work such as [BHi77, Ash88] by *complicial set*, [Ver08].

It is easy to define a homotopy globular *set* $\rho^{\bigcirc} X_*$ of a filtered space X_* but it is not quite so clear how to prove directly that the expected compositions are well defined. However there is a natural graded map

$$i : \rho^{\bigcirc} X_* \to \rho X_* \tag{12}$$

and applying the folding map of [AAl89, AABS02] analogously to methods in [BHi81b] allows one to prove that i of (12) is injective. It follows that the compositions on ρX_* are inherited by $\rho^{\bigcirc} X_*$ to make the latter a globular ω-groupoid. The details are in [Bro08a].

Loday in 1982 [Lod82] defined the fundamental cat^n-group of an n-cube of spaces (a cat^n-group may be defined as an n-fold category internal to the category of groups), and showed that cat^n-groups model all reduced weak homotopy $(n+1)$-types. Joint work [BLo87a] formulated and proved a

HHvKT for the cat^n-group functor from n-cubes of spaces. This allows new local to global calculations of certain homotopy n-types [Bro92], and also an n-adic Hurewicz theorem, [BLo87b]. This work obtains more powerful results than the purely linear theory of crossed complexes. It yields a group-theoretic description of the first non-vanishing homotopy group of a certain $(n+1)$-ad of spaces, and so several formulae for the homotopy and homology groups of specific spaces; [ElM08] gives new applications. Porter in [Por93] gives an interpretation of Loday's results using methods of simplicial groups. There is clearly a lot to do in this area. See [CELP02] for relations of cat^n-groups with homological algebra.

Recently some absolute homotopy 2-groupoids and double groupoids have been defined, see [BHKP02] and the references there, while [BrJ04] applies generalised Galois theory to give a new homotopy double groupoid of a map, generalising previous work of [BHi78a]. It is significant that crossed modules have been used in a differential topology situation by Mackaay and Picken [MaP02]. Reinterpretations of these ideas in terms of double groupoids are started in [BGl93].

It seems reasonable to suggest that in the most general case double groupoids are still somewhat mysterious objects. The paper [AN06] gives a kind of classification of them.

20 Conclusion and questions

• The emphasis on filtered spaces rather than the absolute case is open to question. But no useful definition of a higher homotopy groupoid of a space with clear uses has been proposed. The work of Loday makes use of n-cubes of filtered spaces (see [Lod82, Ste86]).

• *Mirroring the geometry by the algebra* is crucial for conjecturing and proving universal properties.

• *Thin elements* are crucial for modelling a concept not so easy to define or handle algebraically, that of commutative cubes. See also [Hig05, Ste06].

• The cubical methods summarised in section 9 have also been applied in concurrency theory, see for example [GaG03, FRG06].

• *HHvKT theorems* give, when they apply, exact information even in non commutative situations. The implications of this for homological algebra could be important.

• One construction inspired eventually by this work, the *non Abelian tensor product of groups*, has a bibliography of 90 papers since it was defined with Loday in [BLo87a].

• Globular methods do fit into this scheme. They have not so far yielded new calculations in homotopy theory, see [Bro08a], but have been applied to directed homotopy theory, [GaG03]. Globular methods are the main tool in approaches to weak category theory, see for example [Lei04, Str00], although the potential of cubical methods in that area is hinted at in [Ste06].

• For computations we really need strict structures (although we do want to compute invariants of homotopy colimits).

• No work seems to have been done on Poincaré duality, i.e. on finding special qualities of the fundamental crossed complex of the skeletal filtration of a combinatorial manifold. However the book by Sharko, [Sha93, Chapter VI], does use crossed complexes for investigating Morse functions on a manifold.

• In homotopy theory, identifications in low dimensions can affect high dimensional homotopy. So we need structure in a range of dimensions to model homotopical identifications algebraically. The idea of identifications in low dimensions is reflected in the algebra by 'induced constructions'.

• In this way we calculate some crossed modules modelling homotopy 2-types, whereas the corresponding k-invariant is often difficult to calculate.

• The use of crossed complexes in Čech theory is a current project with Jim Glazebrook and Tim Porter.

• **Question:** Are there applications of higher homotopy groupoids in other contexts where the fundamental groupoid is currently used, such as algebraic geometry?

• **Question:** There are uses of double groupoids in differential geometry, for example in Poisson geometry, and in 2-dimensional holonomy [BrI03]. Is there a non Abelian De Rham theory, using an analogue of crossed complexes?

• **Question:** Is there a truly non commutative integration theory based on limits of multiple compositions of elements of multiple groupoids, in analogy to the way length is defined using limits of sequences of line segments?

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