Crossed modules and the homotopy 2-type of a free loop space

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Abstract

The question was asked by Niranjan Ramachandran: how to describe the fundamental groupoid of LX, the free loop space of a space X? We show how this depends on the homotopy 2-type of X by assuming X to be the classifying space of a crossed module over a group, and then describe completely a crossed module over a groupoid determining the homotopy 2-type of LX; that is we describe crossed modules representing the 2-type of each component of LX. The method requires detailed information on the monoidal closed structure on the category of crossed complexes.[1]

1 Introduction

It is well known that for a connected CW-complex X with fundamental group G the set of components of the free loop space LX of X is bijective with the set of conjugacy classes of the group G, and that the fundamental groups of LX fit into a family of exact sequences derived from the fibration $LX \to X$ obtained by evaluation at the base point.

Our aim is to describe the homotopy 2-type of LX, the free loop space on X, when X is a connected CW-complex, in terms of the 2-type of X. Weak homotopy 2-types are described by crossed modules (over groupoids), defined in [BH81b] as follows.

A *crossed module* $\mathcal{M}$ is a morphism $\delta : M \to P$ of groupoids which is the identity on objects such that M is just a disjoint union of groups $M(x), x \in P_0$, together with an action of P on

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M written $(m,p) \mapsto m^p$, $m \in M(x), p : x \to y$ with $m^p \in M(y)$ satisfying the usual rules for an action. We find it convenient to use (non-commutative) additive notation for composition so if $p : x \to y, q : y \to z$ then $p + q : x \to z$, and $(m+n)^p = m^p + n^p, (m^p)^q = m^{p+q}, m^0 = m$. Further we have the two crossed module rules for all $p \in P, m, n \in M$:

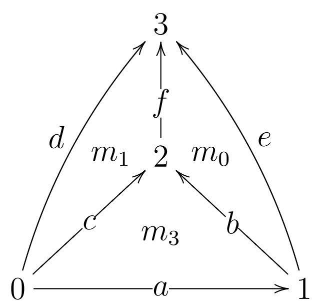
CM1) $\delta(m^p) = -p + \delta m + p$;

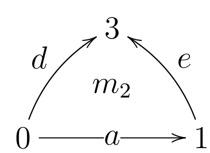
CM2) $-n + m + n = m^{\delta n}$;

whenever defined. This is a *crossed module of groups* if P_0 is a singleton.

A crossed module $\mathcal{M}$ as above has a simplicial nerve $K = N^\Delta \mathcal{M}$ which in low dimensions is described as follows:

- $K_0 = P_0$;
- $K_1 = P$;
- K_2 consists of quadruples $\sigma = (m; c, a, b)$ where $m \in M, a, b, c \in P$ and $\delta m = -c + a + b$ is well defined;
- K_3 consists of quadruples $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ where $\sigma_i \in K_2$ and the σ_i make up the faces of a 3-simplex, as shown in the following diagrams:





$\delta m_2 = -d + a + e$

providing we have the rules

$$\mu m_0 = -e + b + f, \quad \mu m_1 = -d + c + f,$$
$$\mu m_2 = -d + a + e, \quad \mu m_3 = -c + a + b,$$

together with the rule

$$(m_3)^f - m_0 - m_2 + m_1 = 0.$$

You may like to verify that these rules are consistent.

A crossed module is the dimension 2 case of a *crossed complex*, the definition of which in the single vertex case goes back to Blakers in [Bla48], there called a 'group system', and in the many vertex case is in [BH81b]. The definition of the nerve of a crossed complex C in the one vertex case is also in [Bla48], and in the general case is in [Ash88, BH91]. An alternative description of K is that K_n consists of the crossed complex morphisms $\Pi\Delta^n_* \to \mathcal{M}$ where $\Pi\Delta^n_*$ is the fundamental crossed complex of the n-simplex, with its skeletal filtration, and $\mathcal{M}$ is also considered as a crossed complex trivial in dimensions > 2. This shows the analogy with the Dold-Kan theorem for chain complexes and simplicial abelian groups, [Dol58].

We thus define the *classifying space $B\mathcal{M}$ of $\mathcal{M}$* to be the geometric realisation $|N^{\Delta}\mathcal{M}|$, a special case of the definition in [BH91]. It follows that an $a \in P(x)$ for some $x \in P_0$ determines a 1-simplex in $X = B\mathcal{M}$ which is a loop and so a map $a' : S^1 \to B\mathcal{M}$, i.e. $a' \in LX$.

The chief properties of $X = B\mathcal{M}$ are that $\pi_0(X) \cong \pi_0(P)$ and for each $x \in P_0$

$$\pi_i(X, x) \cong \begin{cases} \mathrm{Cok}(\delta : M(x) \to P(x)) & \text{if } i = 1, \\ \mathrm{Ker}(\delta : M(x) \to P(x)) & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases}$$

Further if Y is a CW-complex, then there is a crossed module $\mathcal{M}$ and a map $Y \to B\mathcal{M}$ inducing isomorphisms of π_0, π_1, π_2. For an exposition of some basic facts on crossed modules and crossed complexes in relation to homotopy theory, see for example [Bro99]. There are other versions of the classifying space, for example the cubical version given in [BHS10], and one for crossed module of groups using the equivalence of these with groupoid objects in groups, see for example [Lod82, BS08]. However the latter have not been shown to lead to the homotopy classification Theorem 1.6 below.

Our main result is:

Theorem 1.1 *Let $\mathcal{M}$ be the crossed module of groups $\delta : M \to P$ and let $X = B\mathcal{M}$ be the classifying space of $\mathcal{M}$. Then the components of LX, the free loop space on X, are determined by equivalence classes of elements $a \in P$ where a, b are equivalent if and only if there are elements $m \in M, p \in P$ such that*

$$b = p + a + \delta m - p.$$

Further the homotopy 2-type of a component of LX given by $a \in P$ is determined by the crossed module of groups $L\mathcal{M}[a] = (\delta_a : M \to P(a))$ where

(i) *$P(a)$ is the group of elements $(m, p) \in M \times P$ such that $\delta m = [a, p]$, with composition $(n, q) + (m, p) = (m + n^p, q + p)$;*

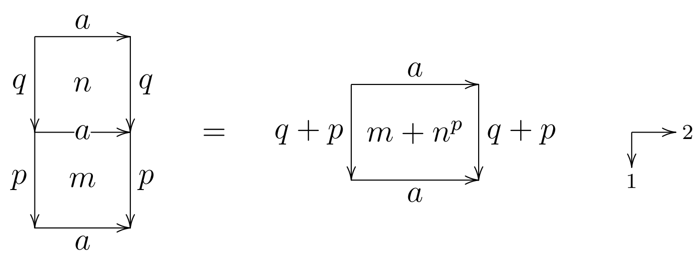
(ii) *$\delta_a(m) = (-m^a + m, \delta m)$, for $m \in M$;*

(iii) *the action of $P(a)$ on M is given by $n^{(m,p)} = n^p$ for $n \in M, (m,p) \in P(a)$.*

In particular $\pi_1(LX, a)$ is isomorphic to $\operatorname{Cok} \delta_a$, and $\pi_2(LX, a) \cong \pi_2(X, *)^{\bar{a}}$, the elements of $\pi_2(X, *)$ fixed under the action of $\bar{a}$, the class of a in $G = \pi_1(X, *)$.

We give a detailed proof that $L\mathcal{M}[a]$ is a crossed module in Appendix A.

Remark 1.2 The composition in (i) can be seen geometrically in the following diagram:



□

The following examples are due to C.D. Wensley.

Example 1.3 $\delta = 0 : M \to P$, so that M is a P-module. Then $P(a)$ is the set of (m,p) s.t. $[a,p] = 0$, i.e. $p \in C_a(P)$, and so is $M \rtimes C_a(P)$. ($P = G$ the fundamental group, as $\delta = 0$). But $\delta_a(m) = (-m^a + m, 0)$. So $\pi_1(L\mathcal{M}, a) = (M/[a, M]) \rtimes C_a(P)$. □

Example 1.4 If $a \in Z(P)$, the center of P, then $[a,p] = 0$ for all p. (For example, P might be abelian.) Hence $P(a) = \pi \rtimes P$. Then $\pi_1(L\mathcal{M}, a) = (\pi \rtimes P)/\{(-m^a + m, \delta m) \mid m \in M\}$.

It is not clear to me that even in this case the exact sequence splits. (??) □

It is also possible to give a less explicit description of $\pi_1(LX, a)$ as part of an exact sequence:

Theorem 1.5 *Under the circumstances of Theorem 1.1, if we set $\pi = \operatorname{Ker} \delta = \pi_2(X), G = \operatorname{Cok} \delta = \pi_1(X)$, with the standard module action of G on π, then the fundamental group $\pi_1(LX, a)$ in the component given by $a \in P$ is part of an exact sequence:*

$$0 \to \pi^{\bar{a}} \to \pi \to \pi/\{\bar{a}\} \to \pi_1(LX, a) \to C_{\bar{a}}(G) \to 1 \qquad (1)$$

where: $\pi/\{\alpha\}$ denotes π with the action of α killed; and $C_\alpha(G)$ denotes the centraliser of the element $\alpha \in G$.

The proof of Theorem 1.1, which will be given in Section 2, is essentially an exercise in the use of the following classification theorem [BH91, Theorem A]:

Theorem 1.6 *Let Y be a CW-complex with its skeletal filtration Y_* and let C be a crossed complex, with its classifying space written BC. Then there is a natural weak homotopy equivalence*

$$B(\mathsf{CRS}(\Pi Y_*, C)) \to (BC)^Y.$$

In the statement of this theorem we use the internal hom $\mathsf{CRS}(-,-)$ in the category Crs of crossed complexes: this internal hom is described explicitly in [BH87], in order to set up the exponential law

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C))$$

for crossed complexes A, B, C, i.e. to give a monoidal closed structure on the category Crs. Note that $\mathsf{CRS}(B,C)_0 = \mathsf{Crs}(B,C)$, $\mathsf{CRS}(B,C)_1$ gives the homotopies of morphisms, and $\mathsf{CRS}(B,C)_n$ for $n \geqslant 2$ gives the higher homotopies.

2 Proofs

We deduce Theorem 1.1 from the following Theorem.

Theorem 2.1 *Let $X = B\mathcal{M}$, where $\mathcal{M}$ is the crossed module of groups $\delta : M \to P$. Then the homotopy 2-type of LX, the free loop space of X, is described by the crossed module over groupoids $L\mathcal{M}$ where*

(i) $(L\mathcal{M})_0 = P$;

(ii) $(L\mathcal{M})_1 = M \times P \times P$ *with source and target given by*

$$s(m,p,a) = p + a + \delta m - p, \quad t(m,p,a) = a$$

for $a, p \in P, m \in M$;

(iii) *the composition of such triples is given by*

$$(n,q,b) + (m,p,a) = (m + n^p, q + p, a)$$

which of course is defined under the condition that

$$b = p + a + \delta m - p$$

or, equivalently, $b^p = a + \delta m$;

(iv) *if* $a \in P$ *then* $(L\mathcal{M})_2(a)$ *consists of pairs* (m, a) *for all* $m \in M$, *with addition and boundary*

$$(m,a) + (n,a) = (m+n, a), \qquad \delta(m,a) = (\delta m, -m^a + m, a);$$

(v) *the action of* $(L\mathcal{M})_1$ *on* $(L\mathcal{M})_2$ *is given by:* $(n,b)^{(m,p,a)}$ *is defined if and only if* $b^p = a+\delta m$ *and then its value is* (n^p, a).

Proof In Theorem 1.6 we set $Y = S^1$ with its standard cell structure $e^0 \cup e^1$, and can write $\Pi Y_* \cong \mathbb{K}(\mathbb{Z}, 1)$ where the latter is the crossed complex with a base point z_0 and a free generator z in dimension 1, and otherwise trivial. Thus morphisms of crossed complexes from $\mathbb{K}(\mathbb{Z}, 1)$, and homotopies and higher homotopies of such morphisms, are completely determined by their values on z_0 and on z.

A crossed module over a group or groupoid is also regarded as a crossed complex trivial in dimensions > 2.

All the formulae required to prove Theorem 2.1 follow from those for the internal hom CRS on the category Crs given in [BH87, Proposition 3.14] or [BHS10, §7.1.vii, §9.3].

We set $L\mathcal{M} = \mathsf{CRS}(\mathbb{K}(\mathbb{Z}, 1), \mathcal{M})$.

Since $\mathbb{K}(\mathbb{Z}, 1)$ is a free crossed complex with one generator z in dimension 1, the elements $a \in P$ are bijective with the morphisms $f : \mathbb{K}(\mathbb{Z}, 1) \to \mathcal{M}$, and we write this bijection as $a \mapsto \hat{a}$, where $a = \hat{a}(z)$. Also the homotopies and higher homotopies from $\mathbb{K}(\mathbb{Z}, 1) \to \mathcal{M}$ are determined by their values on z and on the element z_0 of $\mathbb{K}(\mathbb{Z}, 1)$ in dimension 0. Thus a 1-homotopy $(h, \hat{a}) : \hat{b} \simeq \hat{a}$ is such that h lifts dimension by 1, and is given by elements $p = h(z_0) \in P, m = h(z) \in M$ and so $(h, \hat{a})$ is given by a triple (p, m, a). The condition that this triple gives a homotopy $\hat{b} \simeq \hat{a}$ translates to

$$b = p + a + \delta m - p$$

or, equivalently, $a + \delta m = b^p$. It follows easily that $\hat{b}, \hat{a}$ belong to the same component of $L\mathcal{M}$ if and only if b, a give conjugate elements in the quotient group $\pi_1(\mathcal{M})$. (The use of such general homotopies was initiated in [Whi49].)

The composition of such homotopies $\hat{c} \simeq \hat{b} \simeq \hat{a}$ is given by:

$$(n, q, b) + (m, p, a) = (m + n^p, q + p, a)$$

which of course is defined if and only if

$$b^p = a + \delta m.$$

A 2-homotopy $(H, \hat{a})$ of $\hat{a}$ is such that H lifts dimension by 2 and so is given by an element $H(z_0) \in M$. There are rules giving the composition, actions, and boundaries of such 1- and 2-homotopies.

In particular the action of a 1-homotopy $(h, f^+) : f^- \simeq f^+$ on a 2-homotopy (H, f^-) gives a 2-homotopy (H^h, f^+) where $H^h(c) = H(c)^{h(tc)}$. Here we take $c = z_0$ so that we obtain the action $(n, b)^{(m,p,a)} = n^p$.

All these formulae follow from those given in [BH87, Proposition 3.14] or [BHS10, §9.3].

A 2-homotopy $(H, \hat{a})$ is given by $a = \hat{a}(z)$ and $m = H(z_0) \in M$. We then have to work out $\delta_2(H)$. We find that

$$\begin{aligned} \delta_2(H)(x) &= \begin{cases} \delta H(z_0) & \text{if } x = z_0, \\ -H(sz)^{\hat{a}(z)} + H(tz) + \delta H(z) & \text{if } x = z, \end{cases} \\ &= \begin{cases} \delta m & \text{if } x = z_0, \\ -m^a + m & \text{if } x = z. \end{cases} \end{aligned}$$

This completes the proof of Theorem 2.1. □

The proof of Theorem 1.1 now follows by restricting the crossed module of groupoids given in Theorem 2.1 to $L\mathcal{M}(a)$, the crossed module of groups over the object $a \in (L\mathcal{M})_1 = P$. Then we have an isomorphism $\theta : L\mathcal{M}(a) \to L\mathcal{M}[a]$ given by $\theta_0(a) = *,\ \theta_1(m, p, a) = (m, p),\ \theta_2(m, a) = m$.

For the next result we need the notion of fibration of crossed modules of groupoids which is a special case of fibrations of crossed complexes as defined in [How79] and applied in [Bro08].

Theorem 2.2 *In the situation of Theorem 2.1, there is a fibration $L\mathcal{M} \to \mathcal{M}$ of crossed modules of groupoids. Hence if*

$$\pi = \pi_2(X) \cong \operatorname{Ker} \delta, \quad G = \pi_1(X) \cong \operatorname{Cok} \delta$$

then for each $a \in P$ there is an exact sequence

$$0 \to \pi^{\bar{a}} \to \pi \to (\pi)/\{\bar{a}\} \to \pi_1(LX, a') \to C_{\bar{a}}(G) \to 1 \tag{2}$$

where: $\bar{a}$ denotes the image of a' in G; $\pi/\{\alpha\}$ denotes π with the action of α killed; and $C_\alpha(G)$ denotes the centraliser of the element $\alpha \in G$.

Proof We define the fibration $\psi : L\mathcal{M} \to \mathcal{M}$ by the inclusion $i : \{z_0\} \to \mathbb{K}(\mathbb{Z}, 1)$ and the identification $\mathsf{CRS}(\{z_0\}, \mathcal{M}) \cong \mathcal{M}$, where here $\{z_0\}$ denotes also the trivial crossed complex on the point z_0. Then ψ is a fibration since i is a cofibration, see [BG89]. The exact description of ψ in terms given earlier is that

$$\begin{aligned} \psi_0(a) &= *, \quad a \in P, \\ \psi_1(m, p, a) &= p, \quad (m, p, a) \in M \times P \times P, \\ \psi_2(n, a) &= n, \quad (n, a) \in M \times P. \end{aligned}$$

To say that ψ is a fibration of crossed modules over groupoids is to say that: (i) it is a morphism; (ii) (ψ_1, ψ_0) is a fibration of groupoids, [Bro70, And78]; and (iii) ψ_2 is piecewise surjective.

Let $\mathcal{F}$ denote the fibre of ψ. Then

$$\mathcal{F}_0 = P, \quad \mathcal{F}_1 = \{0\} \times M \times P, \quad \mathcal{F}_2 = \{0\} \times P.$$

The exact sequence of the fibration for a given base point $a \in \mathcal{F}_0 = P$ is

$$0 \to \pi_2(\mathcal{F}, a) \to \pi_2(L\mathcal{M}, a) \to \pi_2(\mathcal{M}, *) \xrightarrow{\partial}$$
$$\to \pi_1(\mathcal{F}, a) \to \pi_1(L\mathcal{M}, a) \to \pi_1(\mathcal{M}, *) \xrightarrow{\partial} \pi_0(\mathcal{F}) \to \pi_0(L\mathcal{M}) \to *.$$

Under the obvious identifications, this leads to the exact sequence of Theorem 1.5. □

Remark 2.3 These results and methods should be related to the description in [Bro87, §6] of the homotopy type of the function space $(BG)^Y$ where G is an abstract group and Y is a CW-complex, and which gives a result of Gottlieb in [Got69]. □

Remark 2.4 Here is a methodological point. The category Crs of crossed complexes is equivalent to that of ∞-groupoids, as in [BH81a], where these ∞-groupoids are now commonly called 'strict globular ω-groupoids'. However the internal hom in the latter category is bound to be more complicated than that for crossed complexes, because the cell structure of the standard n-globe, $n > 1$,

$$E^n = e^0_\pm \cup e^1_\pm \cup \cdots \cup e^{n-1}_\pm \cup e^n$$

is more complicated than that for the standard cell for which

$$E^n = e^0 \cup e^{n-1} \cup e^n, n > 1.$$

Also we obtain a precise answer using filtered spaces and strict structures, whereas the current fashion is to go for weak structures as yielding more homotopy n-types for $n > 2$. In fact many results on crossed complexes are obtained using cubical methods. □

A Verification of crossed module rules

We now verify the crossed module rules for the structure $L\mathcal{M}[a] = (M \xrightarrow{\delta_a} P(a)$ defined in Theorem 1.1 from a crossed module of groups $\mathcal{M} = (M \xrightarrow{\delta} P)$ and $a \in P$ as follows:

$$\begin{aligned} P(a) &= \{(m,p) \in M \times P \mid \delta m = -[a,p] = -a - p + a + p\}. \\ \delta_a m &= (-m^a + m, \delta m). \\ (n,q) + (m,p) &= (m + n^p, q + p). \\ n^{(m,p)} &= n^p. \end{aligned}$$

Proposition A.1 *If $\delta : M \to P$ is a crossed module of groups, and $a \in P$, then $L\mathcal{M}[a]$ as defined above is also a crossed module of groups.*

Proof It is easy to check that $\delta(-m^a + m) = [a, \delta m]$, so that $\delta_a(m) \in P(a)$.

We next show that δ_a is a morphism:

$$\begin{aligned}
\delta_a(n) + \delta_a(m) &= (-n^a + n, \delta n) + (-m^a + m, \delta m) \\
&= (-m^a + m + (-n^a + n)^{\delta m}, \delta n + \delta m) \\
&= (-m^a - n^a + n + m, \delta n + \delta m) \\
&= \delta_a(n + m).
\end{aligned}$$

Now we verify the first crossed module rule. Let $(m, p) \in P(a), n \in M$:

$$\begin{aligned}
-(m, p) + \delta_a n + (m, p) &= (-m^{-p}, -p) + (-n^a + n, \delta n) + (m, p) \\
&= (-n^a + n + (-m^{-p})^{\delta n}, -p + \delta n) + (m, p) \\
&= (-n^a - m^{-p} + n, -p + \delta n) + (m, p) \\
&= (m + (-n^a - m^{-p} + n)^p, -p + \delta n + p) \\
&= (m - n^{a+p} - m + n^p, \delta(n^p)) \\
&= (-n^{a+p-\delta m} + n^p, \delta(n^p)) \\
&= (-n^{p+a} + n^p, \delta(n^p)) \qquad \text{since } \delta m = [a, p] \\
&= \delta_a(n^p).
\end{aligned}$$

Now we verify the second crossed module rule:

$$\begin{aligned}
m^{\delta_a n} &= m^{(-n^a + n,\, \delta n)} \\
&= m^{\delta n} \\
&= -n + m + n. \qquad \square
\end{aligned}$$

In effect, this illustrates that verifying the crossed complex rules for the internal hom $\mathsf{CRS}(C, D)$ is possible but tedious, and that is it is easier to say it follows from the general construction in terms of ω-groupoids and the equivalence of categories, as in [BH87]. On the other hand, this direct proof 'proves', in the old sense of 'tests', the general theory.

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