FUNCTION SPACES AND PRODUCT TOPOLOGIES

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Introduction

In a previous paper (4) I defined ten product topologies on $X \times Y$. In this paper five of these products are applied to problems on function spaces. All spaces will be Hausdorff spaces.

The exponential law for function spaces with the compact-open topology is discussed in § 1. The main result (Theorem 1.6) is that the spaces $X^{Z \mathbf{x} Y}$ and $(X^Y)^Z$ are homeomorphic for all X, Y, Z (in this paper $Z \mathbf{x} Y$ will denote the product $Z \times_S Y$ defined in (4), and $Z \times Y$ will denote the usual, cartesian, product). Hence the exponential law holds for $Z \times Y$ if $Z \times Y = Z \mathbf{x} Y$, and this contains and explains many known results. We deduce also some new results. For example we prove that the answer is 'no' to Dr. S. Wylie's question: are the spaces $(X^Y)^Z$ and $(X^Z)^Y$ naturally homeomorphic?

In § 2 we discuss the law

$$(X \times Y)^Z = X^Z \times Y^Z.$$

This fails in general for products other than the cartesian.

§ 3 is the most important section. It advertises the category of Hausdorff spaces and functions continuous on compact subspaces (here called *k-continuous functions*).

In § 4 the exponential law of § 3 is generalized to the category of M-ads.

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1. The exponential law

In this section X^Y will mean the space of continuous functions $Y \to X$ with the compact-open topology (8); X^Y is a Hausdorff space [(8) Theorem 7.4].

In this paper we write $Z \times Y$ for the cartesian product, $Z \times_W Y$ for the weak product, and $Z \mathbf{x} Y$ for the product $Z \times_S Y$ of (4). We shall

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use also the products $Z \times_T Y$ and $Z \times_Q Y$ of (4). The only properties of $Z \mathbf{x} Y$ needed for the proof of Theorem 1.6 are the following:

[1.1] *A function* f: $Z \mathbf{x} Y \to X$ *is continuous if and only if*

$$f \mid \{z\} \times Y, \qquad f \mid Z \times B$$

are continuous for each z *of* Z *and compact subset* B *of* Y.

[1.2] *The natural map*

$$(Z \mathbf{x} Y) \mathbf{x} X \to Z \mathbf{x} (Y \mathbf{x} X)$$

is a homeomorphism.

[1.1] is immediate from the definition of the product $Z \mathbf{x} Y$ [(4) § 3] and in fact characterizes this product. [1.2], which is a special case of Theorem 4.7 of (4), allows us to write $Z \mathbf{x} Y \mathbf{x} X$ for either of the spaces $(Z \mathbf{x} Y) \mathbf{x} X$, $Z \mathbf{x} (Y \mathbf{x} X)$.

The *evaluation map* ϵ: $X^Y \times Y \to X$ is defined by

$$\epsilon(f, y) = f(y) \quad (f \in X^Y, y \in Y).$$

LEMMA 1.3. *The evaluation map is continuous as a function*

$$\epsilon: X^Y \mathbf{x} Y \to X.$$

Proof. Any f of X^Y is continuous, and so $\epsilon \mid \{f\} \times Y$ is continuous. If $B \subseteq Y$ is compact, then $\epsilon \mid X^Y \times B$ is continuous [(8) Theorem 7.5]. Hence ϵ is continuous on $X^Y \mathbf{x} Y$ by [1.1].

We next show that the *exponential map*

$$\mu: X^{Z \mathbf{x} Y} \to (X^Y)^Z$$

is well defined. Let $f \in X^{Z \mathbf{x} Y}$ and let $z \in Z$; the formula

$$\mu(f)(z)(y) = f(z, y) \quad (y \in Y)$$

defines a function $\mu(f)(z)$: $Y \to X$.

LEMMA 1.4. $\mu(f)(z)$ *is continuous.*

Proof. This function coincides with the composition

$$Y \xrightarrow{i} \{z\} \times Y \xrightarrow{f'} X$$

where $i(y) = (z, y)$, $y \in Y$, and f' is the restriction of f. Obviously i is continuous, and f' is continuous by [1.1]. The lemma follows.

By this lemma $\mu(f): Z \to X^Y$ is well defined.

LEMMA 1.5. $\mu(f)$ *is continuous.*

The proof is given at the end of this section. By this last lemma μ: $X^{Z \mathbf{x} Y} \to (X^Y)^Z$ is well defined.

THEOREM 1.6. *μ is a homeomorphism.*

Proof. The proof uses a standard technique.

By Lemma 1.3 the evaluation map

$$\epsilon\colon X^{Z \mathbin{\mathbf{x}} Y} \mathbin{\mathbf{x}} Z \mathbin{\mathbf{x}} Y \to X$$

is continuous. Hence, from Lemma 1.5, the maps

$$\mu' = \mu(\epsilon)\colon X^{Z \mathbin{\mathbf{x}} Y} \mathbin{\mathbf{x}} Z \to X^{Y},$$

$$\mu'' = \mu(\mu')\colon X^{Z \mathbin{\mathbf{x}} Y} \to (X^{Y})^{Z}$$

are continuous. But it is trivial to verify that $\mu'' = \mu$. Hence μ is continuous.

Again, by Lemma 1.3, the composition $h = \epsilon(\epsilon \mathbin{\mathbf{x}} 1)$ is continuous, where h maps $\quad (X^{Y})^{Z} \mathbin{\mathbf{x}} Z \mathbin{\mathbf{x}} Y \to X^{Y} \mathbin{\mathbf{x}} Y \to X.$

By Lemma 1.5, $\quad \nu = \mu(h)\colon (X^{Y})^{Z} \to X^{Z \mathbin{\mathbf{x}} Y}$

is continuous. It is trivial to verify that ν is an inverse to μ. So μ is a homeomorphism.

LEMMA 1.7. *The identity maps*

$$Z \times_{W} Y \to Z \times_{Q} Y \to Z \mathbin{\mathbf{x}} Y \to Z \times_{T} Y \to Z \times Y$$

are continuous, and induce embeddings

$$X^{Z \times Y} \to X^{Z \times_{T} Y} \to X^{Z \mathbin{\mathbf{x}} Y} \to X^{Z \times_{Q} Y} \to X^{Z \times_{W} Y}.$$

Proof. The first part follows from § 3 of (**4**). The second part follows from the first and the fact that all the products have the same compact subsets [(**4**) Proposition 3.3].

COROLLARY 1.8. *The exponential map for the cartesian product*

$$\mu_{C}\colon X^{Z \times Y} \to (X^{Y})^{Z}$$

is well defined and a homeomorphism into. Given Z and Y, μ_C is onto for all X if and only if $Z \times Y = Z \mathbin{\mathbf{x}} Y$.

Proof. The first part [which is due to Jackson (**7**)] follows immediately from Theorem 1.6 and Lemma 1.7.

Obviously μ_C is onto if $Z \times Y = Z \mathbin{\mathbf{x}} Y$. Suppose conversely that $Z \times Y \neq Z \mathbin{\mathbf{x}} Y$, i.e. that the identity $Z \times Y \to Z \mathbin{\mathbf{x}} Y$ is not continuous. Let $X = Z \mathbin{\mathbf{x}} Y$. Then $\mu(1) \in (X^{Y})^{Z}$ cannot lie in the image of μ_C, and so μ_C cannot be onto.

COROLLARY 1.9. *μ_C is onto if* (i) *Y is locally compact, or* (ii) *Z and Y satisfy the first axiom of countability, or* (iii) *Z and Y are CW-complexes such that $Z \times Y$ is a CW-complex, or* (iv) *Z is locally compact and Y is a k-space.*

Proof. By results of (**4**), $Z \times Y = Z \boldsymbol{\times} Y$ in each of these cases.

Cases (i) and (ii) are due to Fox (**5**), and (iii) is due to Barcus–Barratt (**3**); case (iv) is due to Morita (**10**), as is the following result, which we deduce immediately from Theorem 1.6 and [(**4**) Theorem 4.4].

COROLLARY 1.10. *Let Z and Y be k-spaces. Then the exponential map for the weak product*

$$\mu_W\colon X^{Z\times_W Y} \to (X^Y)^Z$$

is defined and is a homeomorphism. Also the spaces $(X^Y)^Z$ and $(X^Z)^Y$ are naturally homeomorphic.

For our next results we need the following theorem, whose proof is given in §§ 5, 6 of (**4**):

THEOREM 1.11. *There are spaces Z, Y such that the natural maps*

$$Z \times_T Y \to Z \boldsymbol{\times} Y, \qquad Y \boldsymbol{\times} Z \to Z \boldsymbol{\times} Y,$$
$$Y \boldsymbol{\times} Z \to Y \times_Q Z$$

are not continuous.

In the example given, Z is the one-point compactification of a countable discrete space, and $Y = R^J$, where R is the real line and J is an uncountable discrete space.†

COROLLARY 1.12. *In the category of all spaces there is no natural map*

$$(X^Y)^Z \to (X^Z)^Y.$$

Proof. Suppose that there is a natural map $\phi\colon (X^Y)^Z \to (Z^Z)^Y$. Let $X = Z \boldsymbol{\times} Y$, where Z and Y are as in Theorem 1.11. Then

$$(\mu^{-1}\phi\mu)(1)\colon Y \boldsymbol{\times} Z \to Z \boldsymbol{\times} Y$$

is the natural map, which is however not continuous.

COROLLARY 1.13. *There are spaces X, Y for which the evaluation map $\epsilon_T\colon X^Y \times_T Y \to X$ is not continuous: that is, there is a compact subset $A \subseteq X^Y$ such that $\epsilon \mid A \times Y$ is not continuous.*

Proof. The second statement is equivalent to the first since $X^Y \times_T Y$ has the weak topology with respect to $\{A \times Y,\ X^Y \times B\}$ (A, B compact), and ϵ is always continuous on $X^Y \times B$ [(**8**) Theorem 7.5].

By Theorem 1.6 and Lemma 1.7, the exponential map

$$\mu_T\colon X^{Z\times_T Y} \to (X^Y)^Z$$

is well defined. It is easy to see that μ_T is onto if $\epsilon_T\colon X^Y \times_T Y \to X$ is continuous.

† A simpler example is to take $Z = I$ (the unit interval) and Y the unit interval retopologized as in Lemma 5.5 of (**11**).

Let $X = Z \mathbf{\times} Y$, where Z, Y are as in Theorem 1.11. Then

$$\mu(1) \in (X^Y)^Z$$

cannot lie in the image of μ_T. Thus μ_T cannot be onto and ϵ_T cannot be continuous.

COROLLARY 1.14. *There are spaces Z, Y, X and a function*

$$f\colon Z \times Y \to X$$

such that

(i) *f is continuous on $\{z\} \times Y$, $Z \times \{y\}$ for all z of Z, y of Y,*
(ii) *f is continuous on all compact subsets of $Z \times Y$,*
(iii) *$\mu(f)\colon Z \to X^Y$ is well-defined but not continuous.*

Proof. Since $Z \times_Q Y$ has the weak topology with respect to

$$\{C, \{z\} \times Y,\ Z \times \{y\}\} \quad (C \text{ compact, } z \in Z,\ y \in Y),$$

(i) and (ii) are together equivalent to the continuity of $f\colon Z \times_Q Y \to X$. Given that f is continuous, we can prove as in Lemma 1.4 that

$$\mu_Q(f)\colon Z \to X^Y$$

is well defined.

Let Z, Y be the Y, Z respectively of Theorem 1.11, so that the identity $Z \mathbf{\times} Y \to Z \times_Q Y$ is not continuous. Let $X = Z \times_Q Y$, and let

$$f = 1\colon Z \times_Q Y \to X.$$

Then $\mu^{-1}\mu_Q(f)$ is not continuous, and so $\mu_Q(f)$ cannot be continuous.

Remark 1.15. The exponential law is valid also for the product $Z \times_D Y$ of (**4**) and the topology of pointwise convergence on X^Y ($Z \times_D Y$ has the weak topology with respect to

$$\{\{z\} \times Y,\ Z \times \{y\}\colon z \in Z,\ y \in Y\}).$$

The proof is similar to, but simpler than that of Theorem 1.6. I do not know of other function space and product topologies for which the exponential law is valid.

Proof of Lemma 1.5. The following proof is closely related to the usual proof that μ_C is well-defined [cf. (**5**), (**2**)].

Let $g = \mu(f)\colon Z \to X^Y$. Let $W = M(B, U)$ be a sub-basic set for the compact-open topology on X^Y: thus $B \subseteq Y$ is compact, $U \subseteq X$ is open, and

$$W = \{h \in X^Y\colon h(B) \subseteq U\}.$$

We prove $g^{-1}(W)$ is open in Z.

Since B is compact,

$$k = f \mid Z \times B\colon Z \times B \to X$$

is continuous [1.1], and therefore $U' = k^{-1}(U)$ is open in $Z \times B$.

Let $z \in g^{-1}(W)$. Then $\{z\} \times B \subseteq U'$. By a standard argument on compact sets [(8) Theorem 5.12] there is an open set $V \subseteq Z$ such that $z \in V$ and $V \times B \subseteq U'$. This implies that $z \in V \subseteq g^{-1}(W)$, and so that $g^{-1}(W)$ is open.

2. Diagonal maps

For most of this section X^Y will denote simply the *set* of continuous functions $Y \to X$.

The *diagonal map* $d: X \to X \times X$ is defined by $d(x) = (x, x)$ $(x \in X)$. Then d is continuous, and we are interested in conditions for d to be continuous when $X \times X$ has topologies other than the cartesian. It is convenient to give some simple, and probably well-known, facts in full generality.

By a *natural product of spaces* is meant for each X, Y a space $X \times_\Sigma Y$, with underlying set $X \times Y$, and such that:

(i) the projections $p_1: X \times_\Sigma Y \to X$, $p_2: X \times_\Sigma Y \to Y$ are continuous,

(ii) if $f: X \to X'$, $g: Y \to Y'$ are continuous, then the product map $f \times_\Sigma g: X \times_\Sigma Y \to X' \times_\Sigma Y'$ is continuous.

For such a product the natural injection

$$\rho_\Sigma: (X \times_\Sigma Y)^Z \to X^Z \times Y^Z$$

is defined by

$$\rho_\Sigma(h) = (p_1 h, p_2 h), \qquad h \in (X \times_\Sigma Y)^Z.$$

PROPOSITION 2.1. *Given Z, ρ_Σ is onto for all X, Y if and only if the diagonal map $d_\Sigma: Z \to Z \times_\Sigma Z$ is continuous.*

Proof. If d_Σ is continuous, then the formula

$$\tau_\Sigma(f, g) = (f \times_\Sigma g) d_\Sigma \quad (f \in X^Z,\ g \in Y^Z)$$

defines a function $\tau_\Sigma: X^Z \times Y^Z \to (X \times_\Sigma Y)^Z$.

It is easily seen that τ_Σ is an inverse to ρ_Σ; hence ρ_Σ is onto.

Conversely, if ρ_Σ is onto, then it has an inverse τ_Σ. Let $X = Y = Z$ and let $1: Z \to Z$ be the identity map. Then

$$\tau_\Sigma(1, 1) = d_\Sigma,$$

and so d_Σ is continuous.

PROPOSITION 2.2. *Let X, Y be spaces such that $X \times_\Sigma Y \neq X \times Y$, and let $Z = X \times Y$. The diagonal map $d_\Sigma: Z \to Z \times_\Sigma Z$ is not continuous.*

Proof. The cartesian product has the smallest topology such that the projections are continuous. Therefore $1: X \times_\Sigma Y \to X \times Y$ is continuous, and so $X \times_\Sigma Y \neq X \times Y$ implies that $1: X \times Y \to X \times_\Sigma Y$ is not continuous.

The map $\rho: (X\times Y)^Z \to X^Z \times Y^Z$ is a bijection. Let $Z = X\times Y$; then $\rho(1)$ cannot lie in the image of ρ_Σ. So ρ_Σ is not onto, and $d_\Sigma: Z \to Z\times_\Sigma Z$ is not continuous.

The question now arises: when is $d: Z \to Z\times Z$ continuous for the product topologies defined in (4)? This seems a difficult question except for the products $\times_W$, $\times_R$, $\times_P$, $\times_{R^*}$, $\times_{P^*}$, when d is continuous if and only if Z is a k-space. This fact follows easily from the proposition:

PROPOSITION 2.3. *The following statements are equivalent*:

(i) *Z is a k-space,*

(ii) *$d_W: Z \to Z\times_W Z$ is continuous,*

(iii) *$d_R: Z \to Z\times_R Z$ is continuous.*

Proof. We prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

By § 2 of (4) there is a functor k from spaces to k-spaces such that $X\times_W Y = k(X\times Y)$ and Z is a k-space if and only if $Z = kZ$. Now $d: Z \to Z\times Z$ is continuous, and therefore

$$k(d): kZ \to Z\times_W Z$$

is continuous. Hence, if Z is a k-space, $d_W: Z \to Z\times_W Z$ is continuous. Thus (i) ⇒ (ii).

The identity $Z\times_W Z \to Z\times_R Z$ is continuous; therefore (ii) ⇒ (iii).

Let $d_R: Z \to Z\times_R Z$ be continuous, and let $A \subseteq Z$ be closed in kZ. Then for each compact subset $C \subseteq Z$, $A\cap C$ is closed in C, and therefore

$$(Z\times A)\cap(Z\times C) = Z\times(A\cap C)$$

is closed in $Z\times C$. Hence by definition of $Z\times_R Z$, $Z\times A$ is closed in $Z\times_R Z$. Therefore $A = d_R^{-1}(Z\times A)$ is closed in Z. So $Z = kZ$.

Remark 2.4. Let the function spaces be given the compact-open topology. Then, as is well known, $\rho: (X\times Y)^Z \to X^Z\times Y^Z$ is a homeomorphism. However

$$\rho_\Sigma: (X\times_\Sigma Y)^Z \to X^Z\times_\Sigma Y^Z$$

is in general not even continuous.

For example, let $\times_\Sigma = \times_T$, let $X = Y$ be a k-space such that $X\times_T X \neq X\times X$ [cf. (4) Proposition 5.3], and let Z have exactly two elements. Since W^Z is then homeomorphic to $W\times W$ for any W, ρ_T induces a map

$$\rho': (X\times_T X)\times(X\times_T X) \to (X\times X)\times_T(X\times X)$$

such that $\rho'(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2, x_4) \quad (x_i \in X).$

In the commutative diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{d'} & (X \times_T X) \times (X \times_T X) \\ & {}_{d_T}\searrow & \downarrow{\scriptstyle \rho'} \\ & & (X \times X) \times_T (X \times X) \end{array}$$

$d' = d_T \times d_T$. Then d' is continuous [Proposition 2.3 and Lemma 1.7], but d_T is not continuous [Proposition 2.2]. Therefore ρ', and hence also ρ_T, is not continuous. These results show that we cannot obtain a convenient category of spaces simply by changing the product topology.

The considerations of this section were suggested by remarks of the referee.

3. k-continuous functions

The results of §§ 1, 2 show that the category of Hausdorff spaces and continuous maps does not have all the formal properties one would like. Specifically, there is no product topology and function-space topology such that

$$X^{Z \times Y} = (X^Y)^Z, \qquad (X \times Y)^Z = X^Z \times Y^Z$$

for all X, Y, Z. (Another difficulty, discussed in (4), is that the cartesian product of identification maps is not, in general, an identification map.)

In this section we show that these difficulties disappear in the category $\mathscr{K}$ of Hausdorff spaces and functions continuous on compact subspaces. These functions have been called 'weakly continuous'; we suggest instead the term 'k-continuous'. It is possible to argue on intuitive grounds that these k-continuous functions are to be preferred to the continuous functions; in any case, there is no doubt about their advantages in practice.

A function $f\colon X \to Y$ is k-continuous if and only if $f\colon kX \to kY$ is continuous (where kX is X with the weak topology with respect to compact subspaces). So instead of generalizing from continuous functions to k-continuous functions we could instead replace each space X by kX. This would entail replacing the usual subspace, product, and function-space topologies by the corresponding weak topologies. But it is more convenient to use the category $\mathscr{K}$, and nothing is lost by this since the functor k sending X to kX defines an equivalence between $\mathscr{K}$ and the category $\mathscr{K}'$ of k-spaces and continuous functions.†

† It may be convenient to replace some spaces X by kX: for example, to use the weak product for the product of CW-complexes.

We now define some useful concepts in $\mathscr{K}$. A map $f: X \to Y$ in $\mathscr{K}$ is a *k-homeomorphism* if f is an equivalence in $\mathscr{K}$, which means that f is a k-continuous bijection with k-continuous inverse. A map $f: X \to Y$ is a *k-identification map* if for all Z and all functions $g: Y \to Z$, g is k-continuous if gf is k-continuous. It is easily proved that $f: X \to Y$ is a k-identification map if and only if $k(f): kX \to kY$ is an identification map.

The ten products defined in (4) are all equivalent in the category $\mathscr{K}$, and so it does not matter which one we use. We therefore make the obvious choice, and use the cartesian product. This product is a functor in $\mathscr{K}$ since if f, g are k-continuous, then $k(f\times g) = (kf)\times_W (kg)$ [(4) Proposition 2.8] is continuous, and so $f\times g$ is k-continuous.

PROPOSITION 3.1. *If f, g are k-identification maps, then $f\times g$ is a k-identification map.*

Proof. Since kf, kg are identification maps, so is $kf\times_W kg$ [(4) Corollary 4.9]. But $kf\times_W kg = k(f\times g)$ [(4) Proposition 2.8]. Hence $f\times g$ is a k-identification map.

In this and the next section X^Y will denote the space of k-continuous functions $Y \to X$ with the compact-open topology. The usual function space of continuous functions $Y \to X$ will be denoted by $F(Y, X)$; since Y and kY have the same compact subspaces, we have the equality of spaces

$$X^Y = F(kY, X). \tag{3.2}$$

THEOREM 3.3. *The exponential map*

$$\mu: X^{Z\times Y} \to (X^Y)^Z$$

is defined and is a homeomorphism.

Proof. By Corollary 1.10 the exponential map

$$\mu_W: F(kZ\times_W kY, X) \to F(kZ, F(kY, X))$$

is a homeomorphism. The theorem follows from (3.2) and the fact that $kZ\times_W kY = k(Z\times Y)$.

We now show that X^Y is a functor in X and in Y. Let $f: W \to X$, $g: Y \to Z$ be k-continuous functions. By composition, these functions induce functions

$$f^Y: W^Y \to X^Y, \qquad X^g: X^Z \to X^Y.$$

PROPOSITION 3.4. *f^Y is k-continuous, and X^g is continuous.*

Proof. Let f' be the composition

$$W^Y\times Y \xrightarrow{\epsilon} W \xrightarrow{f} X.$$

Then f' is k-continuous, and hence so is $f^Y = \mu(f'): W^Y \to X^Y$.

A similar argument shows that X^g is k-continuous. But the stronger result that X^g is continuous is easily proved using the definition of the compact-open topology.

We now suppose that $g: Y \to Z$ is k-continuous and onto. Then $X^g: X^Z \to X^Y$ is continuous and one-to-one.

PROPOSITION 3.5. *If g is a proper map, then X^g is a homeomorphism into.*† *If g is a k-identification map, then X^g is a k-homeomorphism into.*

Proof. Let H be the image of $h = X^g$, and let $j: H \to X^Z$ be the inverse of h.

We suppose first that g is a proper map, i.e. that

$$g^{-1}(\text{compact}) = \text{compact}.$$

Let
$$M(C, U) = \{u \in X^Z : u(C) \subseteq U\}$$
be a sub-basic set for the compact-open topology on X^Z. Then

$$\begin{aligned} j^{-1}(M(C, U)) &= \{ug \in X^Y : u \in X^Z,\ u(C) \subseteq U\} \\ &= \{ug \in X^Y : u \in X^Z,\ ugg^{-1}(C) \subseteq U\} \\ &= \{v \in H : vg^{-1}(C) \subseteq U\}, \end{aligned}$$

which is a sub-basic set for the compact-open topology on H. Thus j is continuous, and h is a homeomorphism into.

We now suppose that g is a k-identification map. In the following diagram, $1 \times g$ is a k-identification map (Proposition 3.1),

$$\begin{array}{ccc} H \times Y & \xrightarrow{\ \epsilon\ } & X \\ {\scriptstyle 1\times g}\big\downarrow & \nearrow{\scriptstyle j'} & \\ H \times Z & & \end{array}$$

ϵ is k-continuous and $\epsilon(1 \times g)^{-1}$ is single-valued. Hence there is a unique map $j': H \times Z \to X$ such that $j'(1 \times g) = \epsilon$, and j' is k-continuous. But $j = \mu(j')$; so j is k-continuous, and h is a k-homeomorphism into.

The following useful propositions are easily proved using (3.2):

PROPOSITION 3.6. *The natural map*

$$\rho: (X \times Y)^Z \to X^Z \times Y^Z$$

is a homeomorphism.

PROPOSITION 3.7. *Let $Z \cup Y$ be the disjoint union of spaces Z and Y The natural map*

$$\sigma: X^{Z \cup Y} \to X^Z \times X^Y$$

is a homeomorphism.

(It is clear that $k(Z \cup Y) = kZ \cup kY$.)

† W. F. Newns has pointed out that this holds if g satisfies the condition: for each compact set $C \subseteq Z$ there is a compact set $B \subseteq Y$ such that $g(B) = C$.

We now discuss homotopies in $\mathcal{K}$. Let $f, g\colon Y \to X$ be maps in $\mathcal{K}$; a homotopy $F\colon f \simeq g$ is defined as usual except that $F\colon Y \times I \to X$ need only be k-continuous. From this definition of homotopy we obtain notions of domination, homotopy equivalence, deformation retract, etc.

PROPOSITION 3.8. *Let F be a homotopy $f \simeq g\colon Y \to X$. For any space Z, F induces homotopies $f^Z \simeq g^Z\colon Y^Z \to X^Z$ and $Z^f \simeq Z^g\colon Z^X \to Z^Y$.*

Proof. The composition F'

$$Y^Z \times I \times Z \xrightarrow{1 \times T} Y^Z \times Z \times I \xrightarrow{\epsilon \times 1} Y \times I \xrightarrow{F} X$$

is k-continuous (T is the twisting map), and it is easily checked that $\mu(F')\colon Y^Z \times I \to X^Z$ is a homotopy $f^Z \simeq g^Z$.

The composition F''

$$Z^X \times I \times Y \xrightarrow{1 \times T} Z^X \times Y \times I \xrightarrow{1 \times F} Z^X \times X \xrightarrow{\epsilon} Z$$

is k-continuous, and it is easily checked that

$$\mu(F'')\colon Z^X \times I \to Z^Y$$

is a homotopy $Z^f \simeq Z^g$.

COROLLARY 3.9. *In the category $\mathcal{K}$ the homotopy type of X^Y is a homotopy invariant of X, and of Y.*

For other applications of this kind of technique, we refer the reader to (**11**). It is also possible to generalize many theorems in homotopy theory from locally compact spaces to arbitrary spaces, or from countable CW-complexes to arbitrary CW-complexes, by use of these methods.

4. M-ads

In this section we generalize the theorem and propositions 3.3, 3.6, 3.7 to the category of M-ads (**9**), where M is an arbitrary indexing set. The theory of M-ads has something in common with carrier theory (**12**).

The motivation of the definitions which follow is in part obtained from the theory of spaces with base point. L, M, N will denote indexing sets, possibly empty. An 'M-ad' $\mathbf{X} = (X; M)$ will consist of a space X and a family $\{X_m\}_{m \in M}$ of subspaces of X; $(X; \phi)$ will mean the same as X. If m is an integer, then an m-ad will mean an M-ad in which $M = \{1, 2, \ldots, m-1\}$.

The M-ads are the objects of the category of M-ads, in which a map $f\colon \mathbf{X} \to \mathbf{Y}$ of M-ads $\mathbf{X}$, $\mathbf{Y}$ will be a k-continuous function $f\colon X \to Y$ such that $f(X_m) \subseteq Y_m$ ($m \in M$).

The *restriction* $\mathbf{X} \mid A$ of an M-ad $\mathbf{X}$ to a subspace $A \subseteq X$ is the M-ad consisting of $X \cap A$ and the family $\{X_m \cap A\}_{m \in M}$. The *disjoint union* $\mathbf{X} \cup \mathbf{Y}$ of M-ads $\mathbf{X}$, $\mathbf{Y}$ is the M-ad consisting of the disjoint union $X \cup Y$ and the family $\{X_m \cup Y_m\}_{m \in M}$. The *product* $\mathbf{X} \# \mathbf{Y}$ of two M-ads is the M-ad consisting of $X \times Y$ and the family $\{X_m \times Y_m\}_{m \in M}$. The *smashed product* $\mathbf{X} \divideontimes \mathbf{Y}$ of two M-ads is the M-ad consisting of $X \times Y$ and the family $\{X_m \times Y \cup X \times Y_m\}_{m \in M}$. If $M = \phi$ then $X \divideontimes Y = X \# Y = X \times Y$.

The category of spaces with base point is a subcategory of the category of 2-ads. However, the smashed product as defined here of spaces with base point $\mathbf{X}$, $\mathbf{Y}$ is a 2-ad but is not a space with base point. The usual smashed product is of course obtained by shrinking the indexed subspace of $\mathbf{X} \divideontimes \mathbf{Y}$ to a point.

The *function space* $\mathbf{X}^{\mathbf{Y}}$, where $\mathbf{X}$, $\mathbf{Y}$ are M-ads, is the space of maps of M-ads $\mathbf{Y} \to \mathbf{X}$ with the compact-open topology. The *M-ad of maps* $\mathbf{Y} \to \mathbf{X}$ is the M-ad $\mathbf{Y} \pitchfork \mathbf{X}$ whose total space is $\mathbf{X}^{\mathbf{Y}}$ and whose family of subspaces is

$$\{(\mathbf{X} \mid X_m)^{\mathbf{Y}}\}_{m \in M}.$$

Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be M-ads.

THEOREM 4.1. *The exponential map induces an isomorphism of M-ads*

$$\mu\colon (\mathbf{Z} \divideontimes \mathbf{Y}) \pitchfork \mathbf{X} \to \mathbf{Z} \pitchfork (\mathbf{Y} \pitchfork \mathbf{X}).$$

Proof. From the formula

$$\mu(f)(z)(y) = f(z, y) \quad (f \in X^{Z \times Y},\ z \in Z,\ y \in Y),$$

it is easily seen that the exponential map $X^{Z \times Y} \to (X^Y)^Z$ induces a homeomorphism

$$\mu\colon \mathbf{X}^{\mathbf{Z} \mathbf{x} \mathbf{Y}} \to (\mathbf{Y} \pitchfork \mathbf{X})^{\mathbf{Z}}. \tag{4.2}$$

The space $\mathbf{X}^{\mathbf{Z} \mathbf{x} \mathbf{Y}}$ has indexed subspaces

$$(\mathbf{X} \mid X_m)^{\mathbf{Z} \mathbf{x} \mathbf{Y}} \quad (m \in M)$$

and $(\mathbf{Y} \pitchfork \mathbf{X})^{\mathbf{Z}}$ has indexed subspaces

$$\{(\mathbf{Y} \pitchfork \mathbf{X}) \mid (\mathbf{X} \mid X_m)^{\mathbf{Y}}\}^{\mathbf{Z}} = \{\mathbf{Y} \pitchfork (\mathbf{X} \mid X_m)\}^{\mathbf{Z}} \quad (m \in M).$$

So (4.2) implies that μ maps the indexed subspaces of $\mathbf{X}^{\mathbf{Z} \mathbf{x} \mathbf{Y}}$ onto the correct images. This completes the proof.

We now use the natural homeomorphisms ρ, σ of Propositions 3.6, 3.7. Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be M-ads.

THEOREM 4.3. *ρ and σ induce isomorphisms of M-ads*

$$\rho\colon \mathbf{Z} \pitchfork (\mathbf{X} \# \mathbf{Y}) \to (\mathbf{Z} \pitchfork \mathbf{X}) \# (\mathbf{Z} \pitchfork \mathbf{Y}),$$

$$\sigma\colon (\mathbf{Z} \cup \mathbf{Y}) \pitchfork \mathbf{X} \to (\mathbf{Z} \pitchfork \mathbf{X}) \# (\mathbf{Y} \pitchfork \mathbf{X}).$$

The proof is simple and is left to the reader.

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