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Towards a 2-dimensional notion of holonomy

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Abstract

Previous work (Pradines, C. R. Acad. Sci. Paris 263 (1966) 907; Aof and Brown, Topology Appl. 47 (1992) 97) has given a setting for a holonomy Lie groupoid of a locally Lie groupoid. Here we develop analogous 2-dimensional notions starting from a locally Lie crossed module of groupoids. This involves replacing the Ehresmann notion of a local smooth coadmissible section of a groupoid by a local smooth coadmissible homotopy (or free derivation) for the crossed module case. The development also has to use corresponding notions for certain types of double groupoids. This leads to a holonomy Lie groupoid rather than double groupoid, but one which involves the 2-dimensional information.

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0. Introduction

An intuitive notion of holonomy is that it is concerned with *iterations of local procedures* and it is detected when such an iteration returns to its starting point but with a 'change of phase'. An aim of theoretical developments is to provide appropriate algebraic settings for this intuition. Major areas in which holonomy occurs are in foliation theory and in differential geometry, and it is with ideas arising out of the first area with which we are mainly concerned.

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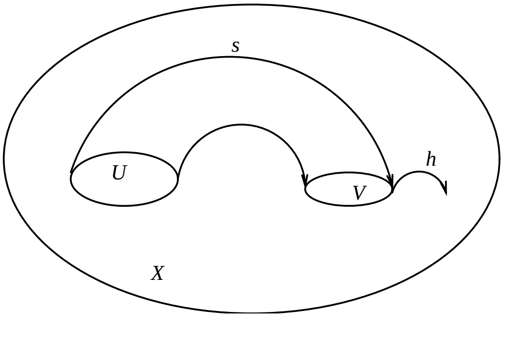
The leaves of a foliation on a manifold M are path connected subsets of M forming a partition of M, and so an equivalence relation R, say, on M. This equivalence relation is usually not a submanifold of M—for example in the classical foliation of the Möbius Band, R is of dimension 3 but has self-intersections. However, the local structure describing the foliation determines a subset W of R such that W is a smooth manifold. Further, R can be regarded as a groupoid with the usual multiplication $(x, y)(y, z) = (x, z)$ for $(x, y), (y, z) \in R$. The pair (R, W) becomes what is known as a *locally Lie groupoid*—that is, the groupoid operations are as smooth as they can be on W, granted that they are not totally defined on W. This concept was first formulated by Pradines [22] and called 'un morceau d'un groupoïde différentiables'. This description of (R, W) for a foliation was due to Kubarski [18] and independently to Brown and Mucuk [12], though with slightly differing conditions.

It is classical that if G is a group and W is a subset of G containing 1 and W has a topology, then under reasonable conditions on the pair (G, W) the topology of W can be transported around G to give a base for a topology so that G becomes a topological group in which W is an open subspace. We say that a *locally Lie group is extendible*. This is not generally true for locally Lie groupoids—and the above (R, W) provides a counterexample. A basic reason for this is that while for a topological group G left multiplication L_g by an element g maps open sets to open sets, this is no longer true for general topological groupoids, since the domain of L_g is usually not open in G.

Ehresmann realised that the failure of this useful property of left multiplication by an element could be remedied by replacing the element g by a local smooth coadmissible section through g (for technical reasons we are replacing 'admissible' by 'coadmissible'). This is a continuous function $s : U \to G$ for some open subset of the base space X and such that $\beta s = 1_U$, and αs maps U homeomorphically to an open subset of X (here α, β are source and target maps of G). Then L_s is the partial map on G given by $h \mapsto s(\alpha h) + h$, and L_s does indeed map open sets of G to open sets.



The local section s of these is *through* an element $g \in G$ if $s(\beta g) = g$. We adopt the idea that s is a kind of thickening or localisation of g, that is it extends g to a local neighbourhood.

Thus, one part of the legacy of Charles Ehresmann [14] is the realisation that from the point of view of the topology of a Lie groupoid G the interest is less in the elements but more in the local coadmissible sections and their actions on G. In the case of the locally Lie groupoid (G, W), the local $\mathscr{C}^r$ coadmissible sections with values in W can be regarded as 'local procedures'. They have a multiplication due to Ehresmann, and germs of their iterates form a groupoid $J^r(G, W)$. Pradines realised that a quotient groupoid of $J^r(G, W)$ does obtain a Lie groupoid structure with W as an open subspace, and is the minimal such overgroupoid of G. This is reasonably called the *holonomy groupoid* of (G, W) and written $Hol(G, W)$. It encapsulates many of the usual intuitions behind the holonomy groupoid of a foliation (as described for example in [17,21,27]) but in much greater generality. The major lines of Pradines' construction were explained to the first author in the period 1981–1987 and indicated in [2] and were published in full detail by Aof and Brown [1] with the generous agreement of Pradines.

It is this intuition which we would like to extend to dimension 2, using one of several notions of 2-dimensional groupoid.

A key idea as to how this might work is that a coadmissible section $s : X \to G$ of a groupoid G on X can also be regarded as a homotopy $s : \Delta(s) \simeq 1$ from an automorphism $\Delta(s)$ of G to the identity on G. Thus, the construction of $Hol(G, W)$ can be expected to be realised in situations where we have a notion of homotopy. In dimension 2, this is available in the most well worked out way for *crossed modules of groupoids* and this is one of the major areas in which we work. These objects are in fact equivalent to *edge symmetric double groupoids with connection*, and also to 2-*groupoids*. We find it possible to work with the first two of these structures and these give the framework for our constructions.

We work in a kind of inductive and non symmetric format: that is we suppose given a Lie structure in one of the two dimensions and use a localisation at the next dimension—precise definitions are given later.

Applications are expected to be in situations with multiple geometric structures, such as foliated bundles. Experience in this area of 'higher dimensional group theory' has suggested that it is necessary to build first a strong feel for the appropriate algebra. For example, the notion of double groupoid was considered by the first author in 1967; double groupoids with connection were found by Brown and Spencer in 1971, but the homotopical notion of the homotopy double groupoid of a pair of spaces, and its application to a 2-dimensional Van Kampen Theorem for the crossed module $\pi_2(X, A) \to \pi_1(A)$ was not found by Brown and Higgins till 1974, and published only in 1978. Once the algebra was developed and linked with the geometry, then quite novel geometrical results were obtained.

By following this paradigm, we intend to come nearer to 2-dimensional extensions of the notions of transport along a path. This should give ideas of, for example, transport over a surface, and pave the way for further extensions to all dimensions. It is hoped that this will lead to a deeper understanding of higher dimensional constructions and operations in differential topology. It is for these reasons, that we still title this paper with the word 'towards'.

This work is also intended to be a continuation of other work applying double groupoids in differential topology such as Mackenzie [20], Brown–Mackenzie [9]. It also allows for possible extensions to the holonomy of 'local subdouble groupoids,' analogously to [7].

In the first section, we outline our plan of the construction.

1. Plan of the work

For a 2-dimensional version, there are a number of possible choices for analogues of groupoids, for example double groupoids, 2-groupoids, crossed modules of groupoids. We are not able at this stage to give a version of holonomy for the most general locally Lie double groupoids. It seems reasonable therefore to restrict attention to those forms of double groupoids whose algebra is better understood, and we therefore considered the possibility of a theory for one of the equivalent categories

$$(CrsMod) \sim (2 - Grpd) \sim (DGrpd!),$$

which denote, respectively, the categories of crossed modules over groupoids, 2-groupoids and 'edge symmetric double groupoids with connection'. For consideration of homotopies the outside seem more convenient. There are more general versions of double groupoids (see [9]) whose consideration we leave to further work.

The steps that we take are as follows:

(i) We need to formulate the notion of a locally Lie structure on a double groupoid $\mathscr{D}(\mathscr{C})$ which corresponds to a crossed module $\mathscr{C} = (C, G, \delta)$ with base space X. For this reason, here (G, X) is supposed a Lie groupoid and a smooth manifold structure on a set W such that $X \subseteq W \subseteq C$. Then $(\mathscr{D}(\mathscr{C}), W^G)$ can given as a locally Lie groupoid over G, where W^G is the subset of $\mathscr{D}(\mathscr{C})$ given by

$$\left\{ \begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix} : w_1 \in W,\ \ a, b, c \in G,\ \ \beta(b) = \alpha(a), \beta(a) = \beta(c) = \beta(w_1), \right.$$
$$\left. d = b + a + \delta(w_1) - c \right\}.$$

(ii) Next, we are replacing local coadmissible sections of a groupoid by local linear coadmissible sections of an edge symmetric double groupoid. We define a product on the set of all local linear coadmissible sections. This easily leads to a 2-dimensional version of $\Gamma(\mathscr{D}(\mathscr{C}))$ of $\Gamma(G)$, again an inverse semigroup.

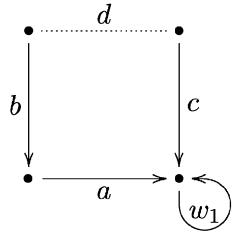
(iii) Now we form germs $[s]_a$ of s, where $a \in G$, $s \in \Gamma(\mathscr{D}(\mathscr{C}))$. The set of these germs forms a groupoid $J(\mathscr{D}(\mathscr{C}))$ over G.

(iv) A key matter for decision is that of the final map ψ and its values on $[s]_a$, since the formation of the holonomy groupoid will involve $Ker\,\psi$.

(v) A further question is that of deciding the meaning of the generalisation to dimension 2 of the term 'enough local linear coadmissible sections'. This requires further discussion.

Recall from [1] that, in the groupoid case, we ask that for any $a \in G$ there is a local coadmissible section s such that $\beta a \in \mathsf{D}(s)$ (where $\mathsf{D}(s)$ is the domain of s) and $s\beta a = a$. Under certain conditions, we require s to be smooth and such that αs is a diffeomorphism of open sets. The intuition here is that $a \in G$ can be regarded as a deformation of βa, and s gives a 'thickening' of this deformation.

In dimension 2, we therefore suppose given $a \in G(x, y)$ and $b \in G(z, x)$, $c \in G(w, y)$ and $w_1 \in C(y)$.



where $d = b + a + \delta(w_1) - c$. Then a local linear coadmissible section $s = (s_0, s_1)$ will be 'through $w = \begin{pmatrix} w_1 : b & \begin{matrix} d \\ \\ a \end{matrix} & c \end{pmatrix}$' if $s_0 x = b$, $s_0 y = c$ and $s_1 a = w_1$. Our 'final map' ψ will be a morphism from $J(\mathscr{D}(\mathscr{C}))$ to a groupoid. This groupoid $\mathscr{D}(\mathscr{C})$ will be one of the groupoid structures of the double groupoid associated to the crossed module $\mathscr{C} = (C, G, \delta)$. We write

$$\psi([s]_a) = s(a) = \begin{pmatrix} s_1(a) : s_0(x) & \begin{matrix} f_1(a) \\ \\ a \end{matrix} & s_0(y) \end{pmatrix},$$

so that the value of ψ on $[s]_a$ does use all the information given by $s = (s_0, s_1)$ at the arrow $a \in G$. This explains why our theory develops crossed modules and double groupoids in parallel.

2. Crossed modules and edge symmetric double groupoids with connection

In a previous paper [8], we explored the idea that a natural generalisation to crossed modules of the notion of coadmissible section for groupoids is that of *coadmissible homotopy*. This arises naturally from the work of Brown–Higgins [6] on homotopies for crossed complexes over groupoids.

We recall the definition of crossed modules of groupoids. The original reference is Brown–Higgins [5], but see also [8].

The source and target maps of a groupoid G are written α, β, respectively. If G is *totally intransitive*, i.e. if $\alpha = \beta$, then we usually use the notation β. The composition in a groupoid G of elements a, b with $\beta a = \alpha b$ will be written additively, as $a + b$. The main reason for this is the convenience for dealing with combinations of inverses and actions.

Definition 2.1. Let G, C be groupoids over the same object set and let C be totally intransitive. Then an *action* of G on C is given by a partially defined function

$$C \times G \circ\!\!\longrightarrow C$$

written $(c, a) \mapsto c^a$, which satisfies

(i) c^a is defined if and only if $\beta(c) = \alpha(a)$, and then $\beta(c^a) = \beta(a)$,
(ii) $(c_1 + c_2)^a = c_1^a + c_2^a$,
(iii) $c_1^{a+b} = (c_1^a)^b$ and $c_1^{e_x} = c_1$

for all $c_1, c_2 \in C(x, x)$, $a \in G(x, y)$, $b \in G(y, z)$.

Definition 2.2. A *crossed module of groupoids* [5] consists of a morphism $\delta : C \to G$ of groupoids C and G which is the identity on the object sets such that C is totally intransitive, together with an action of G on C which satisfies

(i) $\delta(c^a) = -a + \delta c + a$,
(ii) $c^{\delta c_1} = -c_1 + c + c_1$,

for $c, c_1 \in C(x, x)$, $a \in G(x, y)$.

Definition 2.3 (Brown and İçen [8]). Let $\mathscr{C} = (C, G, \delta)$ a crossed module with the base space X. A *free derivation* s is a pair of maps $s_0 : X \to G$, $s_1 : G \to C$ which satisfy the following:

$$\beta(s_0 x) = x, \quad x \in X,$$

$$\beta(s_1 a) = \beta(a), \quad a \in G,$$

$$s_1(a + b) = s_1(a)^b + s_1(b), \quad a, b \in G.$$

Let $\mathsf{FDer}(\mathscr{C})$ be the set of free derivations of $\mathscr{C}$.

We proved in a previous paper [8] that if s is a free derivation of the crossed module $\mathscr{C} = (C, G, \delta)$ over groupoids, then the formulae

$$f_0(x) = \alpha s_0(x), \quad x \in X,$$

$$f_1(a) = s_0(\alpha a) + a + \delta s_1(a) - s_0(\beta a), \quad a \in G,$$

$$f_2(c) = (c + s_1 \delta c)^{-s_0 \beta(c)}, \quad c \in C$$

define an endomorphism $\Delta(s) = (f_0, f_1, f_2)$ of $\mathscr{C}$.

Also $\mathsf{FDer}(\mathscr{C})$ has a monoid structure with the following multiplication [8]:

$$(s * t)_\varepsilon(z) = \begin{cases} t_1(z) + (s_1 g_1(z))^{t_0(\beta z)}, & \varepsilon = 1, \ z \in G(x, y), \\ (s_0 g_0(z)) + t_0(z), & \varepsilon = 0, \ z \in X, \end{cases}$$

where $g = (g_0, g_1, g_2) = \Delta(t)$. This multiplication for $\varepsilon = 0$ give us Ehresmann's multiplication of coadmissible sections [14], and for $\varepsilon = 1$ gives the generalisation to free derivations of the multiplication of derivations introduced by Whitehead [25].

Let $\mathsf{FDer}^*(\mathscr{C})$ denote the group of invertible elements of this monoid. Then each element of $\mathsf{FDer}^*(\mathscr{C})$ is also called a *coadmissible homotopy*.

Theorem 2.4. *Let* $s \in \mathsf{FDer}(\mathscr{C})$ *and let* $f = \Delta(s)$. *Then the following conditions are equivalent*:

(i) $s \in \mathsf{FDer}^*(\mathscr{C})$,
(ii) $f_1 \in \mathsf{Aut}(G)$,
(iii) $f_2 \in \mathsf{Aut}(C)$.

The proof is given in [8].

We shall also deal with double groupoids, especially edge symmetric double groupoids. A double groupoid $\mathscr{D}$ is a groupoid object internal to the category of groupoids. It may also be represented as consisting of four groupoid structures

$$(D, H, \alpha_1, \beta_1, +_1, \varepsilon_1) \quad (D, V, \alpha_2, \beta_2, +_2, \varepsilon_2),$$

$$(V, X, \alpha, \beta, +, \varepsilon) \quad (H, X, \alpha, \beta, +, \varepsilon)$$

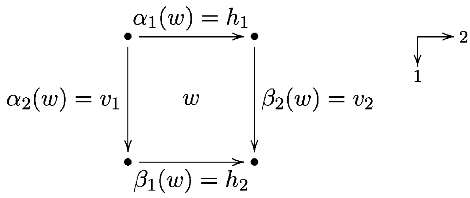
as partially shown in the diagram

$$\begin{array}{ccc} D & \rightrightarrows & V \\ \downdownarrows & & \downdownarrows \\ H & \rightrightarrows & X \end{array}.$$

Here H, V are called the *horizontal* and *vertical edge groupoids*. The functions written as α, β are the source and target maps of the groupoids, and the ε denote the functions giving the identity (zero) elements. Thus $\mathscr{D}$ has two groupoid structures $+_1, +_2$ over groupoids H and V, which are themselves groupoids on the common set

X. These are all subject to the compatibility condition that the structure maps of each structure on $\mathcal{D}$ are morphisms with respect to the other.

Elements of D are pictured as squares



in which $v_1, v_2 \in V$ are the source and target of w with respect to the horizontal structure $+_2$ on D, and $h_1, h_2 \in H$ are the source and target with respect to the vertical structure $+_1$. (Thus, our convention for directions is the same as that for matrices.) Note that we use addition for the groupoid compositions, but principally because it makes for the easier use of negatives rather than inverses. However, somewhat inconsistently, we shall tend use 1 rather than 0 for the unit for $+$. For further information on double groupoids, we refer to [3,9,10,13].

Double groupoids were introduced by Ehresmann in the early 1960s [15,16], but primarily as instances of double categories, and as a part of a general exploration of categories with structure. Since that time their main use has been in homotopy theory. Brown–Higgins [4] gave the earliest example of a 'higher homotopy groupoid' by associating to a pointed pair of spaces (X, A) a edge symmetric double groupoid with connection $\rho(X, A)$ in the sense of Brown and Spencer (see below). In such a double groupoid, the vertical and horizontal edge structures H and V coincide. In terms of this functor ρ, [4] proved a Generalised Van Kampen Theorem, and deduced from it a Van Kampen Theorem for the second relative homotopy crossed module $\pi_2(X, A) \to \pi_1(A)$.

We shall initially be interested in edge symmetric double groupoid, i.e. those in which the horizontal and vertical edge groupoids coincide. These were called special double groupoids in [13]. In this case we write G for $H = V$. The main result of Brown–Spencer [13] is that an edge symmetric double groupoid with connection (see later) whose double base is a singleton is entirely determined by a certain crossed module it contains; as explained above, crossed modules had arisen much earlier in the work of Whitehead [26] on 2-dimensional homotopy. This result of Brown and Spencer is easily extended to give an equivalence between arbitrary edge symmetric double groupoids with connection and crossed modules over groupoids; this is included in the results of [5].

We give this extended result as in [5,9]. The latter paper describes a more general class of double groupoids and the two crossed modules given below form part of the 'core diagram' of a double groupoid described in [9].

The method which is used here can be found in [13]. The sketch proof is as follows:

Let $\mathcal{D} = (D, H, V, X)$ be a double groupoid. We show that $\mathcal{D}$ determines two crossed modules over groupoids.

Let $x \in X$ and let

$$H(x) = \{a \in H : \alpha(a) = \beta(a) = x\}.$$

We define $V(x)$ similarly. We put

$$\Pi(D, H, x) = \{w \in D : \alpha_2 w = \beta_2(w) = \varepsilon(x),\ \beta_1(w) = \varepsilon(x)\}$$

and

$$\Pi(D, V, x) = \{v \in D : \alpha_1(v) = \beta_1(v) = \varepsilon(x),\ \alpha_2(v) = \varepsilon(x)\},$$

which have group structures induced from $+_2$, and $+_1$. Then $\Pi(D, H) = \{\Pi(D, H, x) : x \in X\}$ and $\Pi(D, V) = \{\Pi(D, V, x) : x \in X\}$ are totally intransitive groupoids over X.

Clearly, maps

$$\delta_H : \Pi(D, H) \to H \quad \text{and} \quad \delta_V : \Pi(D, V) \to V$$

defined by $\delta_H(w) = \alpha_1(w)$ and $\delta_V(v) = \alpha_0(v)$, respectively, are morphisms of groupoids.

For a double groupoid $\mathcal{D} = (D, H, V, X)$ it is shown in [13] that

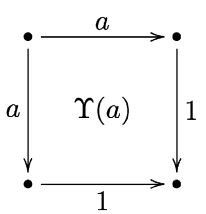
$$\gamma(\mathcal{D}) = (\Pi(D, H), H, \varepsilon) \quad \gamma'(\mathcal{D}) = (\Pi(D, V), V, \partial)$$

may be given the structure of crossed modules (see [3] for an exposition). So γ is a functor from the category of double groupoids to the category of crossed modules.

As we wrote, an *edge symmetric double groupoid* is a double groupoid $\mathcal{D}$ but with the extra condition that the horizontal and vertical edge groupoid structures coincide. These double groupoids will, from now on, be our sole concern, and for these it is convenient to denote the sets of points, edges and squares by X, G, D. The identities in G will be written $\varepsilon(x)$, 1_x or simply 1. The source and target maps $G \to X$ will be written α, β.

By a morphism $f : \mathcal{D} \to \mathcal{D}'$ of edge symmetric double groupoids is meant functions $f : D \to D'$, $f : G \to G'$, $f : X \to X'$ which commute with all three groupoid structures.

A *connection* for the double groupoid $\mathcal{D}$ is a function $\Upsilon : G \to D$ such that if $a \in G$ then $\Upsilon(a)$ has boundaries given by the following diagram:



and Υ satisfies the *transport law*: if $a, b \in G$ and $a+b$ is defined then

$$\Upsilon(a+b) = (\Upsilon(a) +_1 \varepsilon_2 b) +_2 \Upsilon(b) \qquad (*)$$

For further information on the transport law and its uses, see [10].

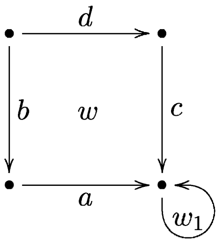
A morphism $f : \mathscr{D} \to \mathscr{D}'$ of edge symmetric double groupoid with special connections Υ, Υ' is said to preserve the connections if $f_2 \Upsilon' = \Upsilon f_1$.

The category *DGrpd*! has objects the pairs $(\mathscr{D}, \Upsilon)$ of an edge symmetric double groupoid $\mathscr{D}$ with connection Υ, and arrows the morphisms of edge symmetric double groupoids preserving the connection. If $(\mathscr{D}, \Upsilon)$ is an object of *DGrpd*!, then we have a crossed module $\gamma(\mathscr{D})$. Clearly γ extends to a functor also written γ from *DGrpd*! to *CrsMod*, the category of crossed modules.

We now show how edge symmetric double groupoids arise from crossed modules over groupoids.

Let $\mathscr{C} = (C, G, \delta)$ be a crossed module over groupoids with base set X. We define a edge symmetric double groupoid $\mathscr{D}(\mathscr{C})$ as follows. First, $H = V = G$ with its groupoid structure, base set X. The set $\mathscr{D}(\mathscr{C})$ of squares is to consist of quintuples

$$w = \begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix}$$



such that $w_1 \in C, a, b, c, d \in G$ and

$$\delta(w_1) = -a - b + d + c.$$

The vertical and horizontal structure on the set $\mathscr{D}(\mathscr{C})$ can be defined as follows.

The source and target maps on w yield d and a, respectively, and vertical composition is

$$\begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix} +_1 \begin{pmatrix} & a & \\ w_1' : b' & & c' \\ & a' & \end{pmatrix} = \begin{pmatrix} & d & \\ w_1' + w_1^{c'} : b + b' & & c + c' \\ & a' & \end{pmatrix}.$$

For the horizontal structure, the source and target maps on w yield b and c, and the compositions are

$$\begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix} +_2 \begin{pmatrix} & e & \\ v_1 : c & & j \\ & i & \end{pmatrix} = \begin{pmatrix} & d+e & \\ w_1^i + v_1 : b & & j \\ & a+i & \end{pmatrix}.$$

Then $\mathscr{D}(\mathscr{C})$ becomes a double groupoid with these structures. A connection on $\mathscr{D}(\mathscr{C})$ is given by

$$\Upsilon(a) = \begin{pmatrix} & a & \\ 1 : a & & 1 \\ & 1 & \end{pmatrix}.$$

The main result on double groupoids is:

Theorem 2.5. *The functor* $\gamma : DGrpd! \to CrsMod$ *is an equivalence of categories* [13].

We emphasise that [9] contains a considerable generalisation of this result.

We introduce a definition of linear coadmissible section for the special double groupoid $\mathscr{D}(\mathscr{C})$ as follows.

Definition 2.6. Let $\mathscr{C} = (C, G, \delta)$ be a crossed module and let $\mathscr{D}(\mathscr{C})$ be the corresponding double groupoid. A *linear coadmissible section* $\sigma = (\sigma_0, \sigma_1) : G \to \mathscr{D}(\mathscr{C})$ of $\mathscr{D}(\mathscr{C})$ also written

$$\sigma(a) = \begin{pmatrix} \sigma_1(a) : \sigma_0\alpha(a) & & \sigma_0\beta(a) \\ & a & \end{pmatrix}$$

is a pair of maps

$$\sigma_0 : X \to G, \quad \sigma_1 : G \to C$$

such that

(i) if $x \in X$, $\beta\sigma_0(x) = x$, and if $a \in G$, then $\beta\sigma_1(a) = \beta a$;
(ii) if $a, b, a + b \in G$, then

$$\sigma(a + b) = \sigma(a) +_2 \sigma(b);$$

(iii) $\alpha\sigma_0 : X \to X$ is a bijection, $\alpha_1\sigma : G \to G$ is an automorphism.

Let $\mathsf{LinSec}(\mathscr{D}(\mathscr{C}))$ denotes the set of all linear coadmissible sections. Then a group structure on $\mathsf{LinSec}(\mathscr{D}(\mathscr{C}))$ is defined by the multiplication

$$(\sigma * \tau)(z) = \begin{cases} (\sigma_0\alpha\tau_0(z)) + \tau_0(z) & \text{if } z \in X, \\ (\sigma\alpha_1(\tau(z))) +_1 \tau(z) & \text{if } z \in G \end{cases}$$

for $\sigma, \tau \in \mathsf{LinSec}(\mathscr{D}(\mathscr{C}))$.

It is easy to prove that the groups of linear coadmissible sections and of free invertible derivation maps (coadmissible homotopies) are isomorphic.

3. Local coadmissible homotopies and local linear sections

Our aim in this section is to 'localise' the concept of coadmissible homotopy given in the previous section, analogously to the way Ehresmann [14] in the 1-dimensional case localises a groupoid element to a local coadmissible section.

In order to cover both the topological and differentiable cases, we use the term C^r manifold for $r \geqslant -1$, where the case $r = -1$ deals with the case of topological spaces and continuous maps, with no local assumptions, while the case $r \geqslant 0$ deals as usual with C^r manifolds and C^r maps. Of course, a C^0 map is just a continuous map. We then abbreviate C^r to *smooth*. The terms *Lie group* or *Lie groupoid* will then involve smoothness in this extended sense. By a *partial diffeomorphism* $f : M \to N$ on C^r manifolds M, N we mean an injective partial function with open domain and range and such that f and f^{-1} are smooth.

One of the key differences between the cases $r = -1$ or 0 and $r \geqslant 1$ is that for $r \geqslant 1$, the pullback of C^r maps need not be a smooth submanifold of the product, and so differentiability of maps on the pullback cannot always be defined. We therefore adopt the following definition of Lie groupoid. Mackenzie [19] discusses the utility of various definitions of differential groupoid.

A *Lie groupoid* is a topological groupoid G such that

(i) the space of arrows is a smooth manifold, and the space of objects is a smooth submanifold of G,
(ii) the source and target maps α, β are smooth maps and are submersions.
(iii) the domain $G \sqcap_\beta G$ of the difference map is a smooth submanifold of $G \times G$, and
(iv) the difference map $\partial : (a, b) \mapsto a - b$ is a smooth map.

Recall that coadmissible homotopies were defined in the previous section. Here we define the local version.

Definition 3.1. Let $\mathscr{C} = (C, G, \delta)$ be a crossed module such that (G, X) is a Lie groupoid. A *local coadmissible homotopy* $s = (s_0, s_1)$ on U_0, U_1 consists of two partial maps

$$s_0 : X \to G, \quad s_1 : G \to C$$

with open domains $U_0 \subseteq X$, $U_1 \subseteq G$, say, such that $\alpha(U_1)$, $\beta(U_1) \subseteq U_0$ and

(i) If $x \in U_0$, then $\beta s_0(x) = x$.
(ii) If $a, b, a + b \in U_1$, then

$$s_1(a + b) = s_1(a)^b + s_1(b);$$

we say s_1 is a local derivation.
(iii) If $a \in U_1$ then $\beta s_1(a) = \beta(a)$.

(iv) If $\Delta(s) = (f_0, f_1, f_2)$ is defined by

$$f_0(x) = \alpha s_0(x), \quad x \in U_0,$$

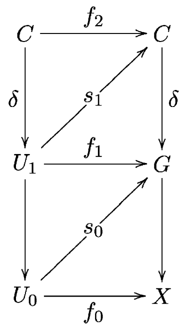
$$f_1(a) = s_0\alpha(a) + a + \delta s_1(a) - s_0\beta(a), \quad a \in U_1,$$

$$f_2(c) = (c + \delta s_1(c))^{-s_0\beta(c)}, \quad c \in C$$

then f_0, f_1 are partial diffeomorphisms and f_1^{-1}, f_1 are linear.

Note that f_1 injective implies f_2 injective by Theorem 2.4. We cannot put smooth conditions on f_2 as we do not have a topology on C.

A local coadmissible homotopy s defined as above can be illustrated by the following diagram:



Suppose given open subsets $V_0 \subseteq X$ and $V_1 \subseteq G$ such that $\alpha(V_1), \beta(V_1) \subseteq V_0$. Let $t : (V_0, V_1) \to (C, G)$ be a local coadmissible homotopy on V_0, V_1 with $\Delta(t) = g$. Let $\Delta(s) = f$ be as above. Now we can define a multiplication of s and t in the following way:

$$(s*t)(z) = \begin{cases} t_1(z) + s_1 g_1(z)^{t_0(\beta z)}, & z \in U_1, \\ s_0 g_0(z) + t_0(z), & z \in U_0. \end{cases}$$

We write $\mathsf{D}(s_\varepsilon)$ for the domain of a function s_ε.

Lemma 3.2. *The product $s*t$ is a local coadmissible homotopy.*

Proof. We will prove that the domain of $s*t$ is open. In fact, if $a \in U_1$, $g_1(a) \in V_1$, $\beta(a) \in U_0$, then $a \in U_1 \cap g_1^{-1}(V_1) \cap \beta^{-1}(U_0)$ is an open set in G and also if $x \in U_0$ and $g_0(x) \in V_0$ then $x \in U_0 \cap g_0^{-1}(V_0)$ is open in X, so the domain of $(s*t)$ is open. One can show that

$$\beta(s*t)_0(x) = \beta(x) \quad \text{for } x \in \mathsf{D}(s*t)_0,$$

$$\beta(s*t)_1(a) = \beta(a) \quad \text{for } a \in \mathsf{D}(s*t)_1$$

and $(s*t)_1$ is a derivation map as in [8, Proposition 2.4] i.e.,

$$(s*t)_1(a+b) = (s*t)_1(a)^b + (s*t)_1(b)$$

for $a, b, a+b \in \mathsf{D}(s*t)_1$. We define maps h_0, h_1 as follows:

$$h_0(x) = f_0 g_0(x) = \alpha(s*t)_0(x) \quad \text{for } x \in \mathsf{D}(s*t)_0,$$

$$h_1(a) = f_1 g_1(a) = (s*t)_0(\alpha a) + a + \delta(s*t)_1(a) - (s*t)_0(\beta a) \quad \text{for } a \in \mathsf{D}(s*t)_1.$$

Since h_0, h_1 are compositions of partial diffeomorphisms, they are partial diffeomorphisms. □

Proposition 3.3. *Let* $\mathsf{LFDer}^*(\mathscr{C})$ *denotes the set of all local coadmissible homotopies of a crossed module* $\mathscr{C} = (C, G, \delta)$ *such that* (G, X) *is a Lie groupoid. For each* $s, t \in \mathsf{LFDer}^*(\mathscr{C})$, $s*t \in \mathsf{LFDer}^*(\mathscr{C})$ *and for each* $s \in \mathsf{LFDer}^*(\mathscr{C})$, *let* $\Delta(s) = f$ *and let*

$$s^{-1}(z) = \begin{cases} -s_1(f_1^{-1}(z))^{s_0^{-1}(\beta z)} & \text{if } z \in U_1, \\ -s_0(f_0^{-1}(z)) & \text{if } z \in U_0. \end{cases}$$

Then $s^{-1} \in \mathsf{LFDer}^*(\mathscr{C})$, *and with this product and inverse element the set* $\mathsf{LFDer}^*(\mathscr{C})$ *of local coadmissible homotopies becomes an inverse semigroup.*

Proof. The proof is fairly straightforward. □

Recall that linear coadmissible sections were defined in the previous section. Here we define the local version.

Definition 3.4. Let $\mathscr{C} = (C, G, \delta)$ be a crossed module of groupoids with base space X and let $\mathscr{D}(\mathscr{C})$ be the corresponding edge symmetric double groupoid such that (G, X) is a Lie groupoid.

A *local linear section* $\sigma = (\sigma_0, \sigma_1) : G \to \mathscr{D}(\mathscr{C})$, written

$$\sigma(a) = \begin{pmatrix} \sigma_1(a) : \sigma_0\alpha(a) & & \sigma_0\beta(a) \\ & a & \end{pmatrix}$$

consists of two partial maps

$$\sigma_0 : X \to G, \quad \sigma_1 : G \to C$$

with open domains $U_0 \subseteq X$, $U_1 \subseteq G$, say, such that $\alpha(U_1)$, $\beta(U_1) \subseteq U_0$ and

(i) if $x \in U_0$, then $\beta\sigma_0(x) = x$, and if $a \in U_1$, then $\beta\sigma_1(a) = \beta a$;

(ii) if $a, b, a+b \in U_1$, then

$$\sigma(a+b) = \sigma(a) +_2 \sigma(b)$$

(this is the local linear condition);

(iii) if f_0, f_1 are defined by

$$f_0(x) = \alpha\sigma_0(x), \quad x \in U_0,$$

$$f_1(a) = \alpha_1\sigma(a), \quad a \in U_1,$$

then f_0, f_1 are partial diffeomorphisms and f_1, f_1^{-1} are linear.

Given open subsets $V_0 \subseteq X$ and $V_1 \subseteq G$ such that $\alpha(V_1), \beta(V_1) \subseteq V_0$, let τ be a local linear section with domain V_0 and V_1. Let σ be as above. Now we can define a multiplication of σ and τ in the following way

$$(\sigma * \tau)(z) = \begin{cases} \sigma(\alpha_1\tau)(z) +_1 \tau(z), & z \in U_1, \\ \sigma_0(\alpha\tau_0)(z) + \tau_0(z), & z \in U_0. \end{cases}$$

Lemma 3.5. *The product function $\sigma * \tau$ is a local linear coadmissible section.*

Proof. The key point is to prove that the domain of $\sigma * \tau$ is open. In fact, if $a \in U_1$, $\alpha_1\sigma(a) \in V_1$, $\beta(a) \in U_0$, then $a \in U_1 \cap (\alpha_1\sigma)^{-1}(V_1) \cap \beta^{-1}(U_0)$ is an open set in G and also if $x \in U_0$ and $\alpha\sigma_0(x) \in V_0$ then $x \in U_0 \cap (\alpha\sigma_0)^{-1}(V_0)$ is open in X, so domain of $(\sigma * \tau)$ is open.

The remaining part is easily done. □

Proposition 3.6. *Suppose $\mathscr{C} = (C, G, \delta)$ is a crossed module such that G is a Lie groupoid on X. Let $\Gamma(\mathscr{D}(\mathscr{C}))$ denote the set of all local linear coadmissible sections of $\mathscr{D}(\mathscr{C})$. For each $\sigma, \tau \in \Gamma(\mathscr{D}(\mathscr{C}))$, $\sigma * \tau \in \Gamma(\mathscr{D}(\mathscr{C}))$ and for each $\sigma \in \Gamma(\mathscr{D}(\mathscr{C}))$, let*

$$\sigma^{-1}(z) = \begin{cases} -_1\sigma(\alpha\sigma)^{-1}(z) & \text{if } z \in U_1, \\ -\sigma_0(\alpha\sigma_0)^{-1}(z)) & \text{if } z \in U_0. \end{cases} \tag{1}$$

Then with this product and inverse element, the set $\Gamma(\mathscr{D}(\mathscr{C}))$ of local linear coadmissible sections becomes an inverse semigroup.

Proof. The inverse of σ is a linear coadmissible section, because it is a composition of the linear maps σ and $(\alpha\sigma)^{-1}$. □

Proposition 3.7. *Let $\Delta(s_0, s_1) = f$ be a local coadmissible homotopy for a crossed module $\mathscr{C} = (C, G, \delta)$ and let $\mathscr{D}(\mathscr{C})$ be the corresponding double groupoid. A partial map s is defined by*

$$s = (s_0, s_1) : G \to \mathscr{D}(\mathscr{C}),$$

$$s(a) = (s_0, s_1)(a) = \begin{pmatrix} & f_1(a) & \\ s_1(a) : s_0(x) & & s_0(y) \\ & a & \end{pmatrix}.$$

Then s is a local linear coadmissible section.

Proof. This is easy to see from the definitions of local coadmissible sections and of local coadmissible homotopies [8]. □

Corollary 3.8. *The inverse semigroups of local coadmissible homotopies and local linear sections are isomorphic.*

Throughout the next sections, we will deal with linear coadmissible sections rather than coadmissible homotopies.

4. V-locally Lie double groupoids

Let $\mathscr{C} = (C, G, \delta)$ be a crossed module such that (G, X) is a Lie groupoid. Let $\mathscr{D}(\mathscr{C})$ be the corresponding double groupoid. Let $\Gamma(\mathscr{D}(\mathscr{C}))$ be the set of all local linear coadmissible sections and let W be a subset of C such that W has the structure of a manifold and $\beta : W \to X$ is a smooth surmersion.

Let W^G be the set

$$\left\{ \begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix} : w_1 \in W,\ a, b, c \in G,\ \beta(b) = \alpha(a),\ \beta(a) = \beta(c) = \beta(w_1), \right.$$

$$\left. d = b + a + \delta(w_1) - c \right\}. \qquad (*)$$

Here the set W^G can be considered as a repeated pullback, i.e., if

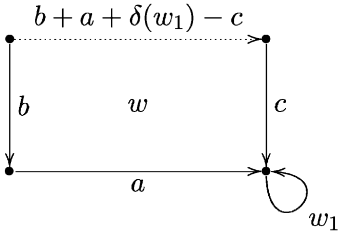
$$G^3 = \{(b, a, c) : \alpha(a) = \beta(b), \beta(a) = \beta(c)\}$$

is a pullback, then

$$\begin{array}{ccc} W^G & \longrightarrow & G^3 \\ \downarrow & & \downarrow{\scriptstyle \beta\pi_1} \\ W & \xrightarrow[\beta]{} & X \end{array}$$

is a pullback, so W^G has a manifold structure on it, because β and $\beta\pi_1$ are smooth and surmersions.

We can show each element $w \in W^G$ by the following diagram:



Clearly $W^G \subseteq W \times G \times G \times G$ and $W^G \subseteq \mathscr{D}(\mathscr{C})$.

A local linear coadmissible section $s = (s_0, s_1)$ as given in Proposition 3.7 is said to be smooth if $\mathrm{Im}(s_1) \subseteq W$ and s_0, s_1 are smooth. We emphasise that in our current context, such an s should be regarded as a 'local procedure'. Let $\Gamma^r(W^G)$ be the set of local linear smooth coadmissible sections. We say that the triple (α_1, β_1, W^G) has *enough smooth local linear coadmissible sections* if for each

$$w = \begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix} \in W^G,$$

there is a local linear smooth coadmissible section $\Delta(s) = f$ with domains (U_0, U_1) such that

(i) $s\beta_1(w) = w$, $\alpha_1(w) = f_1(a) = d$; $s_1\beta_1(w) = w_1 = s_1(a)$, $s_0\beta(a) = c, s_0\alpha(a) = b$;
(ii) the values of s lie in W^G;
(iii) s is smooth as a pair of function $U_0 = \mathsf{D}(s_0) \to G$ and $U_1 = \mathsf{D}(s_1) \to W^G$.

We call such an s a *local linear smooth coadmissible section through* w.

Definition 4.1. Let $\mathscr{C} = (C, G, \delta)$ be a crossed module over a groupoid with base space X and let $\mathscr{D}(\mathscr{C})$ be the corresponding double groupoid. A *V-locally Lie double groupoid* structure $(\mathscr{D}(\mathscr{C}), W^G)$ on $\mathscr{D}(\mathscr{C})$ consists of a smooth structure on G, X making (G, X) a Lie groupoid and a smooth manifold W contained as a set in C such that if W^G is as in (*), then

(S1) $W^G = -_1 W^G$;
(S2) $G \subseteq W^G \subseteq \mathscr{D}(\mathscr{C})$;
(S3) the set $(W^G \sqcap_{\beta_1} W^G) \cap \mathsf{d}^{-1}(W^G) = W^G_{\mathsf{d}}$ is open in $(W^G \sqcap_\beta W^G)$ and the restriction to W^G_{d} of the difference map

$$\mathsf{d} : \mathscr{D}(\mathscr{C}) \sqcap_{\beta_1} \mathscr{D}(\mathscr{C}) \to \mathscr{D}(\mathscr{C}),$$

$$(w, v) \mapsto w -_1 v$$

is smooth;

(S4) the restriction to W^G of the source and target maps α_1 and β_1 are smooth and the triple (α_1, β_1, W^G) has enough local linear smooth coadmissible sections,
(S5) W^G generates $\mathscr{D}(\mathscr{C})$ as a groupoid with respect to $+_1$.

Also one can define a locally Lie crossed module structure on a crossed module by considering the above Definition 4.1.

Let $\mathscr{C} = (C, G, \delta)$ be a crossed module over a groupoid with base space X. A *locally Lie crossed module structure* (C, W, δ) on $\mathscr{C}$ consists of a smooth structure on G, X making (G, X) a Lie groupoid and a subset W of C with a smooth structure on W such that W is G-equivariant and

(C1) (C, W) is a locally Lie groupoid,
(C2) $I(G) \subseteq W \subseteq C$,
(C3) the restriction to W of the map $\delta : C \to G$ is smooth,
(C4) the set $W_A = A^{-1}(W) \cap (W \sqcap_\beta G)$ is open in $W \sqcap_\beta G$ and the restriction to $A_W : W_A \to W$ of the action $A : C \sqcap_\beta G \to C$ is smooth.
(C5) Let

$$\underline{W} = \{(w; b, a, c) : w \in W, a, b, c \in G, \beta(a) = \beta(w), \alpha(a) = \beta(b), \beta(c) = \beta(a)\}.$$

We say that $\underline{W}$ has enough local smooth coadmissible homotopies if for all $(w; b, a, c) \in \underline{W}$ there exists a local smooth coadmissible homotopy (s_0, s_1) such that $s_1(a) = w$, $s_0\beta(a) = c$, $s_0\alpha(a) = b$.

Let us compare the above two definitions.

First of all, in the definition of locally Lie crossed module, conditions C3 and C4 gives rise to the difference map

$$\mathsf{d} : \mathscr{D}(\mathscr{C}) \sqcap_{\beta_1} \mathscr{D}(\mathscr{C}) \to \mathscr{D}(\mathscr{C}),$$

$$(w, v) \mapsto w -_1 v,$$

which is smooth. In fact,

$$\mathsf{d}\left(\left(\begin{matrix} & d & \\ w_1 : b & & c \\ & a & \end{matrix}\right), \left(\begin{matrix} & d' & \\ v_1 : b' & & c' \\ & a & \end{matrix}\right)\right)$$

$$= \begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix} -_1 \begin{pmatrix} & d' & \\ v_1 : b' & & c' \\ & a & \end{pmatrix}$$

$$= \begin{pmatrix} & d & \\ w_1 : b & & c \\ & a & \end{pmatrix} +_1 \begin{pmatrix} & a & \\ -v_1^{-c'} : -b' & & -c' \\ & d' & \end{pmatrix}$$

$$= \begin{pmatrix} & d & \\ (-v_1')^{-c'} + w_1^{-c'} : b - b' & & c - c' \\ & d' & \end{pmatrix}.$$

Since δ_W, $+$, A_W are smooth, d is smooth. This is equivalent to the two smooth conditions C3, C4 for a locally Lie crossed module, because the formulae for d involves $+$ and the action A_W.

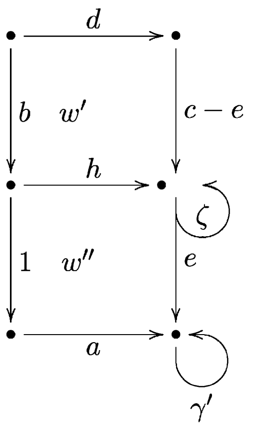
Condition C1 is that (C, W) is a locally Lie groupoid, which includes W generates C. The other equivalent condition can be stated as follows: We first prove that if W generates C and is G-equivariant, then W_G generates $\mathscr{D}(\mathscr{C})$ with respect to $+_1$.

Let $w = \begin{pmatrix} & d & \\ \gamma : b & & c \\ & a & \end{pmatrix} \in \mathscr{D}(\mathscr{C})$. We prove by induction that if γ can be expressed as a word of length n in conjugates of elements of W then w can be expressed as

$$w = w^1 +_1 \cdots +_1 w^n,$$

where $w^i = \begin{pmatrix} & d_i & \\ \gamma_i : b_i & & c_i \\ & a_i & \end{pmatrix} \in W^G$, for $i = 1, \ldots, n$, and $\gamma_i \in W$.

This is certainly true for $n = 1$, since $w \in W^G$ if and only if $\gamma \in W$.



Suppose $\gamma = \gamma' + \zeta^e$ where γ' can be expressed as a word of length n in conjugates of elements of W and $\zeta \in W$.

Let $h = a + \delta\gamma' - e$, and let $w'' = \begin{pmatrix} & h & \\ \gamma' : 1 & & e \\ & a & \end{pmatrix}$. Then $w'' \in \mathscr{D}(\mathscr{C})$ and so w'' can be expressed as a word of length n in elements of W^G, by the inductive assumption.

Let $w' = \begin{pmatrix} & d & \\ \zeta : b & & c - e \\ & h & \end{pmatrix}$. Then $w' \in \mathscr{D}(\mathscr{C})$, since

$$\begin{aligned} \delta\zeta &= \delta(-\gamma' + \gamma)^{-e} \\ &= -\delta(\gamma')^{-e} + \delta(\gamma)^{-e} \\ &= e - \delta\gamma' - e + e + \delta\gamma - e \\ &= e - (-a + h + e) - a - b + d + c - c \\ &= e - e - h + a - a - b + d + c - e \\ &= -h - b + d + c - e \end{aligned}$$

and $\zeta \in W$. Clearly $w = w' +_1 w''$, and so w can be expressed as a word of length $n+1$ in conjugates of elements of W^G.

Conversely, suppose W^G generates $\mathscr{D}(\mathscr{C})$ with respect to $+_1$.

Let $\gamma \in C$ and let $w = \begin{pmatrix} & 1 & \\ \gamma : 1 & & 1 \\ & 1 & \end{pmatrix}$. Then $w \in \mathscr{D}(\mathscr{C})$. Since W^G generates $\mathscr{D}(\mathscr{C})$, we can write

$$w = w^1 +_1 \cdots +_1 w^n,$$

where $w^i = \begin{pmatrix} \gamma_i : b_i & & c_i \\ & a_i & \end{pmatrix}$, $\gamma_i \in W$ for $i = 1, \ldots, n$ and $w^i \in W_G$. Then

$$\begin{aligned} w &= \begin{pmatrix} & d & \\ \gamma_1^{c_1} + \gamma_2^{c_2} + \cdots + \gamma_n^{c_n + \cdots + c_1} : b_1 + \cdots + b_n & & c_1 + \cdots + c_n \\ & a & \end{pmatrix} \\ &= \begin{pmatrix} & 1 & \\ \gamma : 1 & & 1 \\ & 1 & \end{pmatrix}. \end{aligned}$$

We get

$$\gamma = \gamma_1^{c_1} + \gamma_2^{c_1+c_2} + \cdots + \gamma_n^{c_n+\cdots+c_1}, \quad \gamma_i \in W.$$

So W generates C.

In the definition of V-locally Lie double groupoid, condition S4 transfers as follows: Let (α_1, β_1, W^G) have enough local linear smooth coadmissible sections. Then for each $w = \begin{pmatrix} \gamma : b & & c \\ & a & \end{pmatrix} \in W^G$ there exists a local linear smooth coadmissible section s such that $s\beta_1(w) = w$, i.e., there exists (s_0, s_1) that is a local coadmissible homotopy for the crossed module $\mathscr{C} = (C, G, \delta)$. So for $(w; b, a, c)$ defined as above, there exists a local smooth coadmissible homotopy $s = (s_0, s_1)$ such that $s_1(a) = w$, $s_0\alpha a = b$, $s_0\beta a = c$.

We state some deductions from axioms S1–S5.

(1) The inverse map $i : W^G \to W^G, w \mapsto -_1 w$ is smooth.
(2) The set $\mathsf{d}^{-1}(W^G) \cap (W^G_{\alpha_1} \sqcap_{\beta_1} W^G)$ is open $W^G_{\alpha_1} \sqcap_{\beta_1} W^G$ and the horizontal product map is smooth on this set.
(3) Let $k, s, t \in \Gamma(W^G)$, and suppose $a \in \mathsf{D}((k * s * t)_1)$ and $b = \beta_1 t(a)$ satisfy $(k * s)(b) \in W^G$ and $(k * s * t)(a) \in W^G$. Then there are restriction k', s', t' of k, s, t such that $a \in \mathsf{D}((k' * s' * t')_1)$ and $k' * s' * t' \in \Gamma(W^G)$ ([1, Compare 1.6]).

There is a lemma which we shall use later.

Lemma 4.2. *Suppose $s, t \in \Gamma^r(W^G)$, $a \in G$ and $s(a) = t(a)$. Then there is a pair of neighbourhoods (U_0, U_1), where U_0 is a neighbourhood both of $\alpha(a)$ and $\beta(a)$ and U_1 is a neighbourhood of a such that the restriction of $s * t^{-1}$ to (U_0, U_1) lies in $\Gamma^r(W^G)$.*

Proof. Since s and t are smooth and $s(a) = t(a)$, then $(s(a), t(a)) \in W^G \sqcap_{\beta_1} W^G$. This gives rise to maps

$$(s_0, t_0) : \mathsf{D}(s_0) \cap \mathsf{D}(t_0) \to G \sqcap_\beta G \quad \text{and} \quad (s, t) : \mathsf{D}(s_1) \cap \mathsf{D}(t_1) \to W^G \sqcap_{\beta_1} W^G,$$

which are smooth. But by condition S of Definition 4.1, $(W^G \sqcap_{\beta_1} W^G) \cap \mathsf{d}^{-1}(W^G)$ is open in $W^G \sqcap_{\beta_1} W^G$ and (G, X) is a (globally) Lie groupoid. Hence there exist open neighbourhoods U_1 of a in G , U_0 of $\alpha(a)$, $\beta(a)$ in X such that

$$(s, t)(U_1) \subseteq (W^G \sqcap_{\beta_1} W^G) \cap \mathsf{d}^{-1}(W^G) \quad \text{and} \quad (s_0, t_0)(U_0) \subseteq (G \sqcap_\beta G) \cap \partial^{-1}(G).$$

Hence $\mathsf{d}(s, t)(U_1)$ is contained in W^G and $\partial(s_0, t_0)(U_0)$ is contained in G. This gives $(s * t^{-1})(U_1) \subseteq W^G$ and $(s * t^{-1})_0(U_0) \subseteq G$. So $s * t^{-1} \in \Gamma^r(W^G)$. $\square$

4.1. Germs

Let s, t be two local linear coadmissible sections with domains (U_0, U_1) and (U'_0, U'_1), respectively, and let $a \in U_1 \cap U'_1$. We will define an equivalence relation as follows: set $s \sim_a t$ if and only if $U_1 \cap U'_1$ contains an open neighbourhood V_1 of a such that

$$s|_{V_1} = t|_{V_1}$$

and $\alpha(V_1), \beta(V_1) \subseteq V_0$.

Let $J_a(\mathscr{D}(\mathscr{C}))$ be the set of all equivalence classes of $\sim_a$ and let

$$J(\mathscr{D}(\mathscr{C})) = \bigcup \{J_a(\mathscr{D}(\mathscr{C})) : a \in G\}.$$

Each element of $J_a(\mathscr{D}(\mathscr{C}))$ is called a *germ* at a and is denoted by $[s]_a$ for $s \in \Gamma(\mathscr{D}(\mathscr{C}))$, and $J(\mathscr{D}(\mathscr{C}))$ is called the sheaf of germs of local linear coadmissible sections of the double groupoid $\mathscr{D}(\mathscr{C})$.

Proposition 4.3. *Let $J(\mathscr{D}(\mathscr{C}))$ denote the set of all germs of local linear coadmissible sections of the double groupoid $\mathscr{D}(\mathscr{C})$. Then $J(\mathscr{D}(\mathscr{C}))$ has a natural groupoid structure over G.*

Proof. Let $s, t \in \Gamma(\mathscr{D}(\mathscr{C}))$ and $\Delta(s) = f$, $\Delta(t) = g$. The source and target maps are

$$\alpha([s]_a) = f_1(a),$$

$$\beta([s]_a) = a$$

and the object map is $a \mapsto [1]_a$, the multiplication $*$ is

$$[s]_{g_1 a} * [t]_a = [(s * t)]_a$$

and the inversion map is

$$[s]_a^{-1} = [s^{-1}]_{f_1 a}. \qquad \square$$

Remark. One can give a sheaf topology on $J(\mathscr{D}(\mathscr{C}))$ defined by taking as basis the sets $\{[s]_a : a \in U_1\}$ for $s \in \Gamma(\mathscr{D}(\mathscr{C}))$, U_1 open in G. With this topology $J(\mathscr{D}(\mathscr{C}))$ is a topological groupoid. We do not use the sheaf topology since this will not give W^G embedded as an open set.

Suppose now that $(\mathscr{D}(\mathscr{C}), W^G)$ is a V-locally Lie double groupoid. Let $\Gamma^r(W^G)$ be the subinverse semigroup of $\Gamma(\mathscr{D}(\mathscr{C}))$ consisting of local linear coadmissible sections with values in W^G and which are smooth. Let $\Gamma^r(\mathscr{D}(\mathscr{C}), W^G)$ be the subinverse semigroup of $\Gamma(\mathscr{D}(\mathscr{C}))$ generated by $\Gamma^r(W^G)$.

Remark. In view of the discussion in the Introduction, we regard an element of this inverse semigroup as an 'iteration of local procedures'.

If $s \in \Gamma^r(\mathscr{D}(\mathscr{C}), W^G)$, then there are $s^i \in \Gamma^r(W^G)$, $i = 1, \ldots, n$ such that

$$s = s^n * \cdots * s^1.$$

So let $J^r(\mathscr{D}(\mathscr{C}))$ be the subsheaf of $J(\mathscr{D}(\mathscr{C}))$ of germs of elements of $\Gamma^r(\mathscr{D}(\mathscr{C}), W^G)$. Then $J^r(\mathscr{D}(\mathscr{C}))$ is generated as a subgroupoid of $J(\mathscr{D}(\mathscr{C}))$ by the sheaf $J^r(W^G)$ of germs of element of $\Gamma^r(W^G)$. Thus an element of $J^r(\mathscr{D}(\mathscr{C}))$ is of the form

$$[s]_a = [s^n]_{a_n} * \cdots * [s^1]_{a_1},$$

where $s = s^n * \cdots * s^1$ with $[s^i]_{a_i} \in J^r(W^G)$, $a_{i+1} = f_i(a_i)$, $i = 1, \ldots, n$ and $a_1 = a \in \mathsf{D}(s^1)$.

Let $\psi : J(\mathscr{D}(\mathscr{C})) \to \mathscr{D}(\mathscr{C})$ be the final map defined by

$$\psi([s]_a) = s(a) = \begin{pmatrix} & f_1(a) & \\ s_1(a) : s_0(x) & & s_0(y) \\ & a & \end{pmatrix},$$

where s is a local linear coadmissible section. Then ψ is a groupoid morphism. In fact, let $\Delta(s) = f$, $\Delta(t) = g$, then

$$\begin{aligned} \psi([s]_{g_1(a)} * [t]_a) &= \psi([s * t]_a) \\ &= (s * t)(a) \\ &= s\alpha_1 t(a) +_1 t(a) \\ &= s(g_1(a)) +_1 t(a) \\ &= \psi[s]_{g_1(a)} +_1 \psi[t]_a. \end{aligned}$$

Then

$$\psi(J^r(\mathscr{D}(\mathscr{C}))) = \mathscr{D}(\mathscr{C})$$

from the axiom S4 of a V-locally Lie double groupoid on $\mathscr{D}(\mathscr{C})$ in Definition 4.1.

Let $J_0 = J^r(W^G) \cap Ker\,\psi$, where as usual

$$Ker\,\psi = \{[s]_a : \psi[s]_a = 1_a\}.$$

We will prove that J_0 is a normal subgroupoid of $J^r(\mathscr{D}(\mathscr{C}))$. This allows us to define a quotient groupoid $J^r(\mathscr{D}(\mathscr{C}))/J_0$ in the next section.

Lemma 4.4. *The set $J_0 = J^r(W^G) \cap Ker\,\psi$ is a wide subgroupoid of the groupoid $J^r(\mathscr{D}(\mathscr{C}))$.*

Proof. Let $a \in G$. Recall that $\Delta(c) = I$ is the constant linear section. Then $[c]_a$ is the identity at a for $J^r(\mathscr{D}(\mathscr{C}))$ and $[c]_a \in J_0$. So J_0 is wide in $J^r(\mathscr{D}(\mathscr{C}))$.

Let $[s]_a, [t]_a \in J_0(a,a)$, where s and t are local linear smooth coadmissible sections with $a \in \mathsf{D}(s_1) \cap \mathsf{D}(t_1)$ and $\alpha(a), \beta(a) \in \mathsf{D}(s_0) \cap \mathsf{D}(t_0)$.

Since $J_0 = J^r(W^G) \cap Ker\,\psi$, then we have that

(i) $[s]_a, [t]_a \in J^r(W^G)$ and so we may assume that the images of s and t are both contained in W^G and s, t are smooth by definition of germs of local linear smooth coadmissible sections.
(ii) $[s]_a, [t]_a \in Ker\,\psi$ and this implies that $\psi([s]_a) = \psi([t]_a) = 1_a \in \mathscr{D}(\mathscr{C})$ which gives $s(a) = t(a) = 1_a$ by definition of the final map.

Therefore, $(s(a), t(a)) \in W^G \sqcap_{\beta_1} W^G$ and $\mathsf{d}(s(a), t(a)) = s(a) -_1 t(a) = 1_a \in W^G$ which implies that

$$(s(a), t(a)) \in (W^G \sqcap_{\beta_1} W^G) \cap \mathsf{d}^{-1}(W^G).$$

Since s and t are smooth, then the induced maps

$$(s_0, t_0) : \mathsf{D}(s_0) \cap \mathsf{D}(t_0) \to G \sqcap_{\beta} G \quad \text{and} \quad (s, t) : \mathsf{D}(s_1) \cap \mathsf{D}(t_1) \to W^G \sqcap_{\beta_1} W^G$$

are smooth. But, by condition S3 of Definition 4.1 , $(W^G \sqcap_{\beta_1} W^G) \cap \mathsf{d}^{-1}(W^G)$ is open in $W^G \sqcap_{\beta_1} W^G$. Since (G, X) is a (globally) Lie groupoid, there exist open neighbourhoods U_1 of a in G, U_0 of $\alpha(a)$, $\beta(a)$ in X and $\alpha(U_1), \beta(U_1) \subseteq U_0$ such that

$$(s,t)(U_1) \subseteq (W^G \sqcap_{\beta_1} W^G) \cap \mathsf{d}^{-1}(W^G) \quad \text{and} \quad (s_0, t_0)(U_0) \subseteq (G \sqcap_{\beta} G) \cap \partial^{-1}(G),$$

which implies that $(s,t)(U_1) \subseteq \mathsf{d}^{-1}(W^G)$ and $(s_0, t_0)(U_0) \subseteq \partial^{-1}(G)$. Thus $(s * t^{-1})(U_1) \subseteq W^G$ and $(s * t^{-1})_0(U_0) \subseteq G$, and hence $[s * t^{-1}]_a \in J^r(W^G)$. Since $s(a) = t(a)$, then $[s * t^{-1}]_a \in Ker\,\psi$. Therefore $[s * t^{-1}]_a \in J_0(a,a)$ and this completes the proof. □

Lemma 4.5. *The groupoid J_0 is a normal subgroupoid of the groupoid $J^r(\mathscr{D}(\mathscr{C}))$.*

Proof. Let $[k]_a \in J_0(a,a)$ and let $[s]_a \in J_0(b,a)$ for some $a, b \in G$ where k, s are local smooth coadmissible sections, $\Delta(s) = f$, with $b = f_1(a)$ and $\beta_1 k(a) = \alpha_1 k(a) = \beta_1 s(a) = a$. Moreover, $k(a) = 1_a$. Since $J^r(\mathscr{D}(\mathscr{C}))$ is generated by $J^r(W^G)$, then

$$[s]_a = [s^n]_{a_n} * \cdots * [s^1]_{a_1}, \quad s^i \in \Gamma^r(W^G),$$

where $a_1 = a, a_{i+1} = f_i(a_i), i = 1, \dots, n, [s^i]_{a_i} \in J^r(W^G)$, where we may assume that the images of the s^i, $i = 1, \dots, n$ are contained in W^G and are smooth. Then

$$\begin{aligned}
[s]_a[k]_a[s]_a^{-1} &= [s^n]_{a_n} * \cdots * [s^1]_{a_1} * [k]_a * ([s^n]_{a_n} * \cdots * [s^1]_{a_1})^{-1} \\
&= [s^n]_{a_n} * \cdots * [s^1]_{a_1} * [k]_a * [s^1]_{a_1}^{-1} * \cdots * [s^n]_{a_n}^{-1} \\
&= [s^n]_{a_n} * \cdots * [s^1]_{a_1} * [k]_a * [(s^1)^{-1}]_{f_1 a_1} * \cdots * [(s^n)^{-1}]_{f_1 a_n = b} \\
&= [s * k * s^{-1}]_b \in J_0(b, b).
\end{aligned}$$

In fact, now, since $k^{-1}(a) = -_1 k(I^{-1}(a)) = -_1 k(a)$, then $k^{-1}(a) = -_1 k(a)$. But $k(a) = 1_a$, by definition of J_0; hence $k^{-1}(a) = 1_a \in -_1 W^G$.

Since, by condition S1 of Definition 4.1, $W^G = -_1 W^G$, then $k(a) \in W^G$. Since $[s]_a \in J_0(b, a)$, then we may assume that the image of s is contained in W^G and s is a local linear smooth coadmissible section. So $s(a) \in W^G$, and therefore

$$(s(a), -_1 k(a)) \in W^G \sqcap_{\beta_1} W^G, \quad (s_0(x), -k_0(x)) \in G \sqcap_\beta G$$

and $\mathsf{d}(s(a), -k(a)) = s(a) +_1 k(a) = (s * k)(a) = s(a)$. Also $\partial(s_0(x), -k_0(x)) = s_0(x) + k_0(x) = (s_0 * k_0)(x) = s_0(x)$. Hence $(s(a), -_1 k(a)) \in W^G_{\mathsf{d}}$ and $(s_0(x), -k_0(x)) \in G_\partial$, for $x \in X$. By the smoothness of k^{-1} and s, the induced maps

$$(s, k^{-1}) : \mathsf{D}(s_1) \cap \mathsf{D}(k_1^{-1}) \to W^G \sqcap_{\beta_1} W^G \quad \text{and} \quad (s_0, k_0^{-1}) : \mathsf{D}(s_0) \cap \mathsf{D}(k_0^{-1}) \to G \sqcap_\beta G$$

are smooth. Hence, there exists a pair of open neighbourhood (U_0, U_1) where $\alpha(a), \beta(a) \in U_0$ in X, and $a \in U_1$ in G such that

$$(s, k^{-1})(U_1) \subseteq W^G_{\mathsf{d}}, \quad (s_0, k_0^{-1})(U_0) \subseteq G_\partial$$

$$(s(U_1) -_1 k(U_1)) \subseteq W^G, \quad (s_0(U_0) - k_0(U_0)) \subseteq G.$$

Therefore $[s * k]_a \in J_0(W^G)$.

Thus we may assume that the image of $s * k$ is contained in W^G and $s * k$ is a local linear smooth coadmissible section. Since $\beta_1(s * k)(a) = \beta_1 s(a) = \beta_1 k(a) = a$ and $(s * k)(a) = s(a)$. Then $((s * k)(a), s(a)) \in W^G \sqcap_{\beta_1} W^G$ and so $\mathsf{d}((s * k)(a), s(a)) = (k * s)(a) -_1 s(a) = 1_a \in W^G$, and $((s * k)(a), s(a)) \in W^G_{\mathsf{d}}$. Similarly $a \in G(x, y)$, for $x, y \in X$, $((s * k)_0(x), s_0(x)) \in G_\partial$. Since s and $s * k$ are smooth, then they induce smooth maps

$$((s * k), s) : \mathsf{D}(k_1) \cap \mathsf{D}(s_1) \to W^G \sqcap_{\beta_1} W^G, \quad ((s * k)_0, s_0) : \mathsf{D}(k_0) \cap \mathsf{D}(s_0) \to G \sqcap_\beta G.$$

But $W^G_{\rm d}$ and G_∂ are open in $W^G \sqcap_{\beta_1} W^G$ and $G \sqcap_\beta G$, respectively. Hence there exists a pair of neighbourhoods (U'_0, U'_1) of $\alpha(a), \beta(a) \in U'_0$ in X and $a \in U'_1$ in G such that

$$((s*k), s)(U'_1) \subseteq W^G \sqcap_{\beta_1} W^G, \quad ((s*k)_0, s_0)(U'_0) \subseteq G \sqcap_\beta G,$$

which implies that

$$(s*k)(U'_1) -_1 s(U'_1) \subseteq W^G \quad \text{and} \quad (s*k)_0(U'_0) - s_0(U'_0) \subseteq G.$$

Since $f_1(a) = b$, $[s*k]_a * [s]_a^{-1} = [s*k]_a * [s^{-1}]_b = [s*k*s^{-1}]_b \in J(b,b)$. But $[s*k*s^{-1}]_b \in (Ker\,\phi)(b,b)$, since $(s*k*s^{-1})(b) = 1_b$. Hence $[s*k*s^{-1}]_b \in J_0(b,b)$ and so J_0 is a normal subgroupoid of $J^r(\mathscr{D}(\mathscr{C}))$. $\square$

5. The holonomy groupoid of $(\mathscr{D}(\mathbf{C}), \mathbf{W}^{\mathbf{G}})$

In this section, we deal with some locally Lie structures on an edge symmetric double groupoid $\mathscr{D}(\mathscr{C})$ corresponding to a crossed module $\mathscr{C} = (C, G, \delta)$—namely such a locally Lie structure is a Lie groupoid structure on the groupoid (G, X) of $\mathscr{D}(\mathscr{C})$, and a manifold structure on a certain subset W^G of the set of squares, satisfying certain conditions. The Lie groupoid $Hol(\mathscr{D}(\mathscr{C}), W^G)$ we construct will be called the holonomy groupoid of the V-locally Lie double groupoid $(\mathscr{D}(\mathscr{C}), W^G)$. Further, we state a universal property of the Lie groupoid $Hol(\mathscr{D}(\mathscr{C}), W^G)$ in Theorem 5.4.

We state a part of a Lie version of the Brown–Spencer Theorem given in Brown–Mackenzie [9]. We give a definition of Lie crossed modules of groupoids and of double Lie groupoids.

Definition 5.1. A *Lie crossed module of groupoids* is a crossed module (C, G, δ) together with a Lie groupoid structure on C, G (so that G, X is also a Lie groupoid) such that $\delta : C \to G$ and the action of G on C are smooth.

Recall that a *double groupoid* consists of a quadruple of sets (D, H, V, X), together with groupoid structures on H and V, both with base X, and two groupoid structure on D, a horizontal with base V, and a vertical structure with base H, such that the structure maps (source, target, difference map, and identity maps) of each structure on D are morphisms with respect to the other.

Definition 5.2. A *double Lie groupoid* is a double groupoid $\mathscr{D} = (D; H, V, X)$ together with differentiable structures on D, H, V and X, such that all four groupoid structures are Lie groupoids and such that the double source map $D \to H \times_\alpha V = \{(h, v) : \alpha_1(h) = \alpha_2(v)\}$, $d \mapsto (\alpha_2(d), \alpha_1(d))$ is a surjective submersion, where α_2, α_1 are source maps on D vertically and horizontally, respectively.

In differential geometry, double Lie groupoids, but usually with one of the structure totally intransitive, have been considered in passing by Pradines [23,24]. In general, double Lie groupoids were investigated by Mackenzie [20] and Brown and Mackenzie [9].

Theorem 5.3 (Brown and Mackenzie [9]). *Let $\mathscr{C} = (C, G, \delta)$ be a Lie crossed module with base space X and let the anchor map $[,] : G \to X \times X$ be transversal as a smooth map. Then the corresponding edge symmetric double groupoid $\mathscr{D}(\mathscr{C})$ is a double Lie groupoid.*

We define the quotient groupoid

$$Hol(\mathscr{D}(\mathscr{C}), W^G) = J^r(\mathscr{D}(\mathscr{C}))/J_0$$

and call this the *holonomy groupoid* of the locally Lie groupoid $(\mathscr{D}(\mathscr{C}), W^G)$ on $\mathscr{D}(\mathscr{C})$.

We now state our main theorem.

Theorem 5.4. *Let $\mathscr{C} = (C, G, \delta)$ be a crossed module and let $\mathscr{D}(\mathscr{C})$ be the corresponding double groupoid. Let $(\mathscr{D}(\mathscr{C}), W^G)$ be a V-locally Lie double groupoid for the double groupoid $\mathscr{D}(\mathscr{C})$. Then there is a Lie groupoid structure on $Hol(\mathscr{D}(\mathscr{C}), W^G)$, a morphism*

$$\psi : Hol(\mathscr{D}(\mathscr{C}), W^G) \to \mathscr{D}(\mathscr{C})$$

of groupoids, and an embedding

$$i : W^G \to Hol(\mathscr{D}(\mathscr{C}), W^G)$$

of W^G to an open neighbourhood of $Ob(Hol(\mathscr{D}(\mathscr{C}), W^G)) = G$, such that

(i) *ψ is the identity on G, ψi is the identity on W^G, $\psi^{-1}(W^G)$ is open in $Hol(\mathscr{D}(\mathscr{C}), W^G)$, and the restriction $\psi_{W^G} : \psi^{-1}(W^G) \to W^G$ of ψ is smooth.*

(ii) *if $\mathscr{A} = (A, B, \delta')$ is a Lie crossed module and $\mu : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{C})$ is a morphism of groupoids such that*

(a) *μ is the identity on objects;*

(b) *the restriction $\mu_{W^G} : \mu^{-1}(W^G) \to W^G$ is smooth and $\mu^{-1}(W^G)$ is open in $\mathscr{D}(\mathscr{A})$ and generates $\mathsf{D}(\mathscr{A})$ with respect to $+_1$ as a groupoid.*

(c) *the triple $(\alpha_1, \beta_1, \mathscr{D}(\mathscr{A}))$ has enough local linear smooth coadmissible sections;*

then there is a unique morphism $\mu' : (\mathscr{D}(\mathscr{A}), B) \to Hol(\mathscr{D}(\mathscr{C}), W^G)$ of Lie groupoids such that $\psi\mu' = \mu$ and $\mu'(w) = (i\mu)(a)$ for $w \in \mu^{-1}(W^G)$.

The proof occupies the next two sections.

6. Lie groupoid structure on $Hol(\mathscr{D}(\mathscr{C}), W^G)$

The aim of this section is to construct a topology on the holonomy groupoid $Hol(\mathscr{D}(\mathscr{C}), W^G)$ such that $Hol(\mathscr{D}(\mathscr{C}), W^G)$ with this topology is a Lie groupoid. In the next section we verify that the universal property of Theorem 5.4 holds. The intuition is that first of all W^G embeds in $Hol(\mathscr{D}(\mathscr{C}), W^G)$, and second that $Hol(\mathscr{D}(\mathscr{C}), W^G)$ has enough local linear coadmissible sections for it to obtain a topology by translation of the topology of W^G.

Let $s \in \Gamma^r(\mathscr{D}(\mathscr{C}), W^G)$. We define a partial function $\chi_s : W^G \to Hol(\mathscr{D}(\mathscr{C}), W^G)$. The domain of χ_s is the set of $w \in W^G$ such that $\alpha_1(w) = a \in \mathsf{D}(s_1)$ and $\alpha(a), \beta(a) \in \mathsf{D}(s_0)$. The value $\chi_s(w)$ is obtained as follows. Choose a local linear smooth coadmissible section θ through w. Then we set

$$\chi_s(w) = \langle s \rangle_{\alpha_1(w)} \langle \theta \rangle_{\beta_1(w)} = \langle s * \theta \rangle_{\beta_1(w)}.$$

We have to show that $\chi_s(w)$ is independent of the choice of the local linear smooth coadmissible section θ. For this reason we state a lemma.

Lemma 6.1. *Let $w \in W^G$, and let s and t be local linear smooth coadmissible sections through w. Let $a = \beta_1 w$. Then $\langle s \rangle_a = \langle t \rangle_a$ in $Hol(\mathscr{D}(\mathscr{C}), W^G)$.*

Proof. By assumption $sa = ta = w$. Let $b = \alpha_1 w$. Without loss of generality, we may assume that s and t have the same domain (U_0, U_1) and have image contained in W^G and G, respectively. By Lemma 4.2, $s * t^{-1} \in \Gamma^r(W^G)$. So $[s * t^{-1}]_b \in J_0$. Hence

$$\langle t \rangle_a = \langle s * t^{-1} \rangle_b \langle t \rangle_a = \langle s * t^{-1} * t \rangle_a = \langle s \rangle_a. \qquad \square$$

Lemma 6.2. *χ_s is injective.*

Proof. Suppose $\chi_s v = \chi_s w$. Then $\beta_1 w = \beta_1 v = a$, say and $\alpha_1 s \alpha_1 v = \alpha_1 s \alpha_1 w$. By definition of s, $\alpha_1 v = \alpha_1 w = d$, say. Let θ be a local linear smooth coadmissible section through w. Then we now obtain from $\chi_s v = \chi_s w$ that

$$\langle s \rangle_d \langle \theta \rangle_a = \langle s \rangle_d \langle \theta' \rangle_a$$

and hence, since $Hol(\mathscr{D}(\mathscr{C}), W^G)$ is a groupoid, that $\langle \theta \rangle_a = \langle \theta' \rangle_a$. Hence, $v = \theta(a) = \theta'(a) = w \in W^G$. $\square$

Let $s \in \Gamma(\mathscr{D}(\mathscr{C}))$. Then s defines a left translation L_s on $\mathscr{D}(\mathscr{C})$ by

$$L_s(w) = s(\alpha_1(w)) +_1 w.$$

This is an injective partial function on $\mathscr{D}(\mathscr{C})$. The inverse L_s^{-1} of L_s is

$$v \mapsto -_1 s(\alpha_1 s)^{-1}(\alpha_1(v)) +_1 v$$

and $L_s^{-1} = L_{s^{-1}}$. We call L_s the *left translation corresponding to s.*

So we have an injective function χ_s from an open subset of W^G to $Hol(\mathscr{D}(\mathscr{C}), W^G)$. By definition of $Hol(\mathscr{D}(\mathscr{C}), W^G)$, every element of $Hol(\mathscr{D}(\mathscr{C}), W^G))$ is in the image of χ_s for some s. These χ_s will form a set of charts and so induce a topology on $Hol(\mathscr{D}(\mathscr{C}), W^G)$. The compatibility of these charts results from the following lemma, which is essential to ensure that W^G retains its topology in $Hol(\mathscr{D}(\mathscr{C}), W^G)$ and is open in $Hol(\mathscr{D}(\mathscr{C}), W^G)$. As in the groupoid case [1], this is a key lemma.

Lemma 6.3. *Let $s, t \in \Gamma^r(\mathscr{D}(\mathscr{C}), W^G)$. Then $(\chi_t)^{-1}\chi_s$ coincides with L_η, left translation by the local linear smooth coadmissible section $\eta = t^{-1} * s$, and L_η maps open sets of W^G diffeomorphically to open sets of W^G.*

Proof. Suppose $v, w \in W^G$ and $\chi_s v = \chi_t w$. Choose local linear smooth coadmissible sections θ and θ' through v and w, respectively, such that the images of θ and θ' are contained in W^G. Since $\chi_s v = \chi_t w$, then $\beta_1 v = \beta_1 w = a$ say. Let $\alpha_1 v = b$, $\alpha_1 w = c$.

Since $\chi_s v = \chi_t w$, we have

$$\langle s * \theta \rangle_a = \langle t * \theta' \rangle_a.$$

Hence there exists a local linear smooth coadmissible section ζ with $a \in \mathsf{D}(\zeta)$ such that $[\zeta]_a \in J_0$ and

$$[s * \theta]_a = [t * \theta']_a [\zeta]_a.$$

Let $\eta = t^{-1} * s$. Then in the semigroup $\Gamma^r(\mathscr{D}(\mathscr{C}), W)$ we have from the above that $\eta * \theta = \theta' * \zeta$ locally near a. So we get $w = (\theta' * \zeta)(a) = \theta'(a) +_1 \zeta(a) = \theta'(a) +_1 1_a = (\eta * \theta)a = \eta\alpha_1 v +_1 v$. This shows that $(\chi_t)^{-1}\chi_s = L_\eta$, left translation by $\eta \in \Gamma(\mathscr{D}(\mathscr{C}))$, i.e.,

$$\begin{aligned}(\chi_t)^{-1}(\chi_s)(v) &= (\chi_t)^{-1}(\langle s * \theta \rangle_{\beta_1 v = a}) \\ &= (t^{-1} * s * \theta)(a) \\ &= (\eta * \theta)(a), \text{ since } \eta = t^{-1} * s \\ &= \eta(\alpha_1(\theta(a)) +_1 \theta(a), \text{ by definition of } * \\ &= \eta(\alpha_1(v)) +_1 v, \text{ since } \theta(a) = v \\ &= L_\eta(v), \quad \text{by definition of } L_\eta.\end{aligned}$$

However, we also have $\eta = \theta' * \zeta * \theta^{-1}$ near $\alpha_1 v$. Hence $L_\eta = L_{\theta'} L_\zeta L_{\theta^{-1}}$ near v. Now $L_{\theta^{-1}}$ maps v to 1_a, L_ζ maps 1_a to 1_a, and $L_{\theta'}$ maps 1_a to w. We prove the first of these the others being similar

$$\begin{aligned} L_{\theta^{-1}}(v) &= \theta^{-1}(\alpha_1(v)) +_1 v \\ &= -_1 \theta(\alpha_1\theta)^{-1}(\alpha_1 v) +_1 v, \text{ by definition of } \theta^{-1} \\ &= -_1 \theta(\beta_1(v)) +_1 \theta(\beta_1 v), \text{ since } \theta(\beta_1 v) = v \\ &= 1_a. \end{aligned}$$

So these left translations are defined and smooth on open neighbourhoods of v, 1_a and 1_a, respectively. Hence L_η is defined and smooth on an open neighbourhood of v. □

We now impose on $Hol(\mathcal{D}(\mathcal{C}), W^G)$ the initial topology with respect to the charts χ_s for all $s \in \Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$. In this topology each element h has an open neighbourhood diffeomorphic to an open neighbourhood of $1_{\beta_1 h}$ in W^G.

Lemma 6.4. *With the above topology, $Hol(\mathcal{D}(\mathcal{C}), W^G)$ is a Lie groupoid.*

Proof. Source and target maps are smooth: In fact, for $w \in W^G$,

$$\beta_H(\chi_s(w)) = \beta_1(w), \quad \alpha_H(\chi_s(w)) = \alpha_1(s\alpha_1(w)).$$

It follows that α_H and β_H are smooth.

Now we have to prove that

$$\mathsf{d}_H : Hol(\mathcal{D}(\mathcal{C}), W^G) \sqcap_\beta Hol(\mathcal{D}(\mathcal{C}), W^G) \to Hol(\mathcal{D}(\mathcal{C}), W^G)$$

is smooth. Let $\langle s \rangle_a, \langle t \rangle_a \in Hol(\mathcal{D}(\mathcal{C}), W^G)$. Then $\chi_s(1_a) = \langle s \rangle_a, \chi_t(1_a) = \langle t \rangle_a$, and if $\eta = s * t^{-1}$, then $\chi_\eta(1_b) = \langle s * t^{-1} \rangle_b$, where $b = \beta_1 t(a)$. Let $v \in \mathsf{D}(\chi_s), w \in \mathsf{D}(\chi_t)$, with $\beta_1 v = \beta_1 w = c$, say and let θ and θ' be elements of $\Gamma^r(W^G)$ through v and w, respectively. Let $d = \beta_1(t * \theta')(c)$. Then

$$\begin{aligned} \chi_\eta^{-1} \mathsf{d}_H(\chi_s \times \chi_t)(v, w) &= \chi_\eta^{-1} \mathsf{d}_H(\chi_s(v), \chi_t(w)) \\ &= \chi_\eta^{-1} \mathsf{d}_H(\langle s * \theta \rangle_c, \langle t * \theta' \rangle_c), \quad \text{by definition of } \chi_s, \chi_t \end{aligned}$$

$$= \chi_\eta^{-1}(\langle (s*\theta)*(t*\theta')^{-1}\rangle_d), \quad \text{by definition of } \mathsf{d}_H$$

$$= (\eta^{-1})*(s*\theta)*(t*\theta')^{-1}(d), \quad \text{by definition of } \chi_\eta^{-1}$$

$$= ((s*t^{-1})^{-1})*(s*\theta)*(t*\theta')^{-1}(d), \text{ since } \eta = (s*t^{-1})$$

$$= (t*s^{-1}*s*\theta)*(t*\theta')^{-1})(d)$$

$$= ((t*\theta)*(t*\theta')^{-1})(d)$$

$$= ((t*\theta)(\alpha_1(t*\theta')^{-1}(d) +_1 (t*\theta')^{-1}(d)$$

$$= (t*\theta)_1(c) -_1 (t*\theta')(\alpha_1(t*\theta')^{-1})(d), \text{ since } \alpha_1(t*\theta')^{-1}(d) = c$$

$$= t(\alpha_1\theta_1(c) +_1 \theta(c) -_1 (t(\alpha_1\theta'(c) +_1 \theta'(c))$$

$$= (t(\alpha_1(v)) +_1 v -_1 (t(\alpha_1(w)) +_1 w)$$

$$= L_t(v) -_1 L_t(w)$$

$$= \Omega(v,w)$$

say. The smoothness of this map Ω at $(1_a, 1_a)$ is now easily shown by writing $t = t_n * \cdots * t_1$ where $t_i \in \Gamma^r(W^G)$ and using induction and a similar argument to that of Lemma 4.5. □

7. The universal property of $Hol(D(C), W^G)$

In this section we state and prove the main theorem of the universal property of the morphism $\psi : Hol(\mathscr{D}(\mathscr{C}), W^G) \to \mathscr{D}(\mathscr{C})$. Note that for the case of groupoids rather than crossed modules, Pradines stated a differential version involving germs of locally Lie groupoids in [22], and formulated the theorem in terms of adjoint functors. No information was given on the construction or proof. A version for locally topological groupoids was given in Aof–Brown [1], with complete details of the construction and proof, based on conversations of Brown with Pradines. The modifications for the differential case were given in Brown–Mucuk [11].

The main idea is when we are given a V-locally Lie double groupoid $(\mathscr{D}(\mathscr{C}), W^G)$ of a double groupoid $\mathscr{D}(\mathscr{C})$, coming from a Lie crossed module $\mathscr{C}$, a Lie crossed module $\mathscr{A}$ and a morphism

$$\mu : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{C})$$

with suitable conditions, we can construct a morphism

$$\mu' : \mathscr{D}(\mathscr{A}) \to Hol(\mathscr{D}(\mathscr{C}), W^G),$$

where $Hol(\mathscr{D}(\mathscr{C}), W^G))$ is a holonomy groupoid of a locally Lie crossed module, such that

$$\phi\mu' = \mu.$$

We prove that such a morphism μ' is well-defined, smooth and unique. Now let $(\mathscr{D}(\mathscr{C}), W^G)$ be a V-locally Lie double groupoid as above.

Theorem 7.1. *If $\mathscr{A} = (A, B, \delta')$ is a Lie crossed module and $\mu : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{C})$ is a morphism of groupoid such that*

(i) *μ is the identity on objects;*
(ii) *the restriction $\mu_{W^G} : \mu^{-1}(W^G) \to W^G$ of μ is smooth and $\mu^{-1}(W^G)$ is open in $\mathscr{D}(\mathscr{A})$ and generates $\mathscr{D}(\mathscr{A})$ as a groupoid with respect to $+_1$.*
(iii) *the triple $(\alpha_1, \beta_1, \mathscr{D}(\mathscr{A}))$ has enough local smooth coadmissible sections.*

Then there exists a unique morphism

$$\mu' : \mathscr{D}(\mathscr{A}) \to Hol(\mathscr{D}(\mathscr{C}), W^G)$$

of Lie groupoids such that $\psi\mu' = \mu$ and $\mu'(w) = i\mu(w)$ for $w \in \mu^{-1}(W^G)$.

Proof. Since, by condition (i), μ is the identity on G, then $G = B$ and $X = X'$ which implies that $\mu(G) = G$, $\mu(X) = X$. But $G \subseteq W^G \subseteq \mathscr{D}(\mathscr{C})$, by condition S2 of Definition 4.1. Hence $\mu(G) \subseteq W^G \subseteq \mathscr{D}(\mathscr{C})$. So it follows that $G \subseteq \mu^{-1}(W^G) \subseteq \mathscr{D}(\mathscr{A})$.

But, by condition (ii), $\mu^{-1}(W^G)$ is an open in $\mathscr{D}(\mathscr{A})$. Hence $\mu^{-1}(W^G)$ is open neighbourhood of G in $\mathscr{D}(\mathscr{A})$. Since $\mu^{-1}(W^G)$ generates $\mathsf{D}(\mathscr{A})$, we can write $w = w_n +_1 \cdots +_1 w_1$, where $\mu(w_i) \in W^G$, $i = 1, \ldots, n$.

Since $(\alpha_1, \beta_1, \mathscr{D}(\mathscr{A}))$ has enough local linear smooth coadmissible sections, by condition (iii), we can choose local linear smooth coadmissible sections θ_i through w_i, $i = 1, \ldots, n$, such that they are composable and their images are contained in $\mu^{-1}(W^G)$.

Because of condition (ii), the smoothness of μ on $\mu^{-1}(W^G)$ implies that $\mu\theta_i$ is a local linear smooth coadmissible section through $\mu(w_i) \in W^G$ whose image is contained in W^G. Therefore $\mu\theta \in \Gamma^r(\mathscr{D}(\mathscr{C}), W^G)$. Hence, we can set

$$\mu'(w) = \langle \mu\theta \rangle_{\beta_1(w)}. \qquad \square$$

Lemma 7.2. *$\mu'(w)$ is independent of the choices which have been made.*

Proof. Let $w = v_m +_1 \cdots +_1 v_1$, where $\mu v_j \in W^G$ and $j = 1, \ldots, m$. Choose a set of local linear smooth coadmissible sections θ'_j through v_j such that the θ'_j are composable and their images are contained in $\mu^{-1}(W^G)$.

Let $\theta' = \theta'_m * \cdots * \theta'_1$. Then $\mu\theta' \in \Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$, and so $\langle \mu\theta' \rangle_c \in Hol(\mathcal{D}(\mathcal{C}), W^G)$. Since by assumption, $\theta(c) = \theta'(c) = w \in \mathcal{D}(\mathcal{A})$, then $(\theta(c), \theta'(c)) \in \mathcal{D}(\mathcal{A}) \sqcap_{\beta_1} \mathcal{D}(\mathcal{A})$ and $\mathsf{d}_A(\theta(c), \theta'(c)) = \theta(c) -_1 \theta'(c) = 1_c$. Hence $(\theta(c), \theta'(c)) \in \mathsf{d}_A^{-1}\mu^{-1}(W^G)$ because $1_c \in \mu^{-1}(W^G)$.

Because $\mathcal{A}$ is a Lie crossed module and the corresponding double groupoid $\mathcal{D}(\mathcal{A})$ is a double Lie groupoid, the difference map $\mathsf{d}_A : \mathcal{D}(\mathcal{A}) \sqcap_{\beta_1} \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is smooth. Since $\mu^{-1}(W^G)$ is open in $\mathcal{D}(\mathcal{A})$, by condition (ii), then $\mathsf{d}_A^{-1}\mu^{-1}(W^G)$ is open in $\mathcal{D}(\mathcal{A}) \sqcap_{\beta_1} \mathcal{D}(\mathcal{A})$.

But, by the smoothness of θ and θ', the induced maps $(\theta, \theta') : \mathsf{D}(\theta_1) \cap \mathsf{D}(\theta'_1) \to \mathcal{D}(\mathcal{A}) \sqcap_{\beta_1} \mathcal{D}(\mathcal{A})$, $(\theta_0, \theta'_0) : \mathsf{D}(\theta_0) \cap \mathsf{D}(\theta'_0) \to G \sqcap_\beta G$ are smooth. Hence there exists open neighbourhoods N of c in G and N_0 of both $\alpha(c), \beta(c)$ such that $(\theta, \theta')(N) \subseteq (\mathsf{d}_A^{-1}\mu^{-1})(W^G)$ and $(\theta_0, \theta'_0)(N_0) \subseteq \partial_A^{-1}(G)$. This implies that $\theta * \theta'^{-1}(\alpha_1\theta' N) \subseteq \mu^{-1}(W^G)$ and $\theta_0 * \theta_0'^{-1}(N_0) \subseteq G$. So, after suitably restricting θ, θ', which we may suppose done without change of notation, we have that $\theta * \theta'^{-1}$ is a local linear smooth coadmissible section through $1_d \in \mathcal{D}(\mathcal{A})$ and its image is contained in $\mu^{-1}(W^G)$. So $\mu(\theta * \theta'^{-1})$ is a local linear smooth coadmissible section through $1_d \in W^G$, and its image is contained in W^G. Therefore $[\mu(\theta * \theta'^{-1})]_d \in J^r(W^G)$.

Since $\theta(c) = \theta'(c)$, then $\psi[\mu\theta]_c = \psi[\mu\theta']_c$. But ψ and μ are morphisms of groupoids; hence $\psi[\mu(\theta * \theta'^{-1})]_d = 1_d$, and so $[\mu(\theta * \theta'^{-1})]_d \in Ker\,\psi$. Therefore $[\mu(\theta * \theta'^{-1})]_d \in J^r(W) \cap Ker\,\psi = J_0$. Since μ is a morphism of groupoids, we have $[\mu(\theta * \theta'^{-1})]_d \in J^r$. Hence $\langle \mu(\theta * \theta'^{-1}) \rangle_d = 1_d \in Hol(\mathcal{D}(\mathcal{C}), W^G))$, and so

$$\langle \mu\theta \rangle_c = \langle \mu\theta \rangle_c \langle \mu(\theta * \theta'^{-1}) \rangle_d = \langle \mu\theta' \rangle_c,$$

which shows that $\mu' w$ is independent of the choices made. □

Lemma 7.3. *μ' is a morphism of groupoids.*

Proof. Let $u = w +_1 v$ be an element of $\mathcal{D}(\mathcal{A})$ such that $w = w_n +_1 \cdots +_1 w_1$ and $v = v_m +_1 \cdots +_1 v_1$, where $w_i, v_j \in \mu^{-1}(W^G)$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then $u = w_n +_1 \cdots +_1 w_1 +_1 v_m +_1 \cdots +_1 v_1$.

Let θ_i, θ'_j be local linear smooth coadmissible section through w_i and v_j, respectively, such that they are composable and their images are contained in $\mu^{-1}(W^G)$. Let $\theta = \theta_n * \cdots * \theta_1$ and $\theta' = \theta'_m * \cdots * \theta'_1$, $\kappa = \theta * \theta'$. Then κ is a local linear smooth coadmissible section through $u \in \mathcal{D}(\mathcal{A})$, and $\mu\theta$, $\mu\theta'$, $\mu\kappa \in \Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$, and $\mu\kappa = \mu\theta * \mu\theta'$, since μ is a morphism of groupoids.

Let $a = \beta_1 w$, $b = \beta_1 v$. Then $\langle \mu\kappa \rangle_a = \langle \mu\theta \rangle_a \langle \mu\theta \rangle_b$ and so μ' is a morphism. □

Lemma 7.4. *The morphism μ' is smooth, and is the only morphism of groupoids such that $\psi\mu' = \mu$ and $\mu' a = (i\mu)(a)$ for all $a \in \mu^{-1}(W^G)$.*

Proof. Since $(\alpha_1, \beta_1, \mathscr{D}(\mathscr{A}))$ has enough local linear smooth coadmissible section, it is enough to prove that μ' is smooth at 1_a for all $a \in G$. Let $\mathbf{c}$ denote the linear coadmissible section $\mathbf{c} : G \to \mathscr{D}(\mathscr{C})$, $a \mapsto 1_a$ and $c_0 : X \to G$, $x \mapsto 1_x$.

Let $a \in G$. If $w \in \mu^{-1}(W^G)$ and s is a local linear smooth coadmissible section through w, then $\mu' w = \langle \mu s \rangle_{\beta_1 w} = \chi_{\mathbf{c}}\mu(w)$. Since μ is smooth, it follows that μ' is smooth.

The uniqueness of μ' follows from the fact that μ' is determined on $\mu^{-1}(W^G)$ and that this set generates $\mathscr{D}(\mathscr{A})$. □

This completes the proof of our main result, Theorem 5.4.

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