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Homotopical Excision:
(work to be done!)

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Contents

1a. Background:

In homotopy theory identifications in low dimensions affect high dimensional invariants. How to model this?

Warning: sometimes you need information in dimension zero!

Example: $X = S^n \vee [0,2]$, $\pi_n(X,0) \cong \pi_n(S^n,0)$.

But $\pi_n(X,\{0,1,2\})$ is clearly a free $\pi_1([0,2],\{0,1,2\})$-module.

Then identify $0,1,2$ to a point:

$\pi_n(S^n \vee S^1 \vee S^1, 0)$ is a free $\pi_1(S^1 \vee S^1, 0)$-module on 1 generator.

However in this talk all spaces will be pointed!

Solution to general problem: try to find

algebraic objects with structure in dimensions $0, 1, \ldots, n$,

modelling spaces with analogous structure.

AIM: Colimit theorems for homotopically defined functors on Topological Data with values in strict higher groupoids, so enabling specific nonabelian calculations of some homotopy types and so of some classical invariants.

1b. Methodology

We consider functors

$$\begin{pmatrix}\text{Topological}\\ \text{Data}\end{pmatrix} \underset{\mathbb{B}}{\overset{\mathbb{H}}{\rightleftarrows}} \begin{pmatrix}\text{Algebraic}\\ \text{Data}\end{pmatrix}$$

such that

1) $\mathbb{H}$ is homotopically defined.
2) $\mathbb{H}\mathbb{B}$ is equivalent to 1.
3) The Topological Data has a notion of connected.
4) For all Algebraic Data A, $\mathbb{B}A$ is connected.
5) "Nice" colimits of connected Topological Data are :
   (a) connected, and
   (b) preserved by $\mathbb{H}$.

The last is a generalised Seifert-van Kampen Theorem.
We now recall an Excision Theorem of RB and Philip Higgins, JPAA, (1981)

2. Relative Homotopical Excision

Excision deals with
$Y = X \cup B, A = X \cap B$, and,
e.g., X, B are open in Y and
considers the pushout of pairs

$$\begin{array}{ccc} (A, A) & \xrightarrow{i} & (B, B) \\ \downarrow & & \downarrow \\ (X, A) & \xrightarrow[j]{} & (Y, B) \end{array}$$

Excision Theorem If (X, A) is $(n-1)$-connected, then so also is (Y, B) and we get a pushout of modules (crossed if $n = 2$)

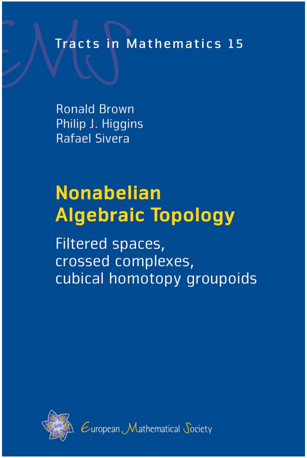
$$\begin{array}{ccc} (1, \pi_1(A)) & \xrightarrow{i_*} & (1, \pi_1(B)) \\ \downarrow & & \downarrow \\ (\pi_n(X, A), \pi_1(A)) & \xrightarrow[j_*]{} & (\pi_n(Y, B), \pi_1(B)) \end{array}$$

j_* is "induced" by i_*

This implies the Relative Hurewicz Theorem by taking
$B = CA$, a cone on A.

The case $n = 2$,
$X = CA$, $Y = B \cup CA$ and $A = \vee_i S^1_i$ is a wedge of circles gives a 1949 Theorem of Whitehead on free crossed modules.

This Excision Theorem is a special case, or application, of a Seifert-van Kampen Theorem for filtered spaces; see the book Nonabelian Algebraic Topology (EMS, 2011, 703 pp) which realises the methodology of the second slide to give an exposition of the border between homology and homotopy, using



and not requiring the usual singular homology.

3. Origin of my work with Loday

In November 1981 as part of a visit to France, and at the suggestion of Michel Zisman, I visited Jean-Louis Loday in Strasbourg to give a seminar on the work with Philip Higgins.
Jean-Louis got the point of the talk, and expected there could be a van Kampen Theorem for his cat^n-groups functor on n-cubes of pointed spaces.
He also told me of a conjecture he had.
I interpreted this conjecture as a
Triadic Hurewicz Theorem, which from the RB/PJH point of view should be deduced from a
Triadic Excision Theorem, which itself should be deduced from a
Triadic van Kampen Theorem.
Jean-Louis suggested a more general theorem could be easier to prove!

4. Squares of pointed spaces

J-L's methods involves in the first dimension a square of pointed spaces

$$Z := \begin{array}{ccc} A & \xrightarrow{b} & B \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Now we expand this to a diagram of fibrations,

$$\begin{array}{ccccc} F(Z) & \longrightarrow & F(x) & \longrightarrow & F(g) \\ \downarrow & & \downarrow & & \downarrow \\ F(b) & \longrightarrow & A & \xrightarrow{b} & B \\ \downarrow & & {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle g} \\ F(f) & \longrightarrow & X & \xrightarrow[f]{} & Y \end{array}$$

and say Z is connected if all these spaces are connected.
Then we use π_1 to form the square of groups

$$\Pi(Z) := \begin{array}{ccc} \pi_1(F(Z)) & \longrightarrow & \pi_1(F(x)) \\ \downarrow & & \downarrow \\ \pi_1(F(b)) & \longrightarrow & \pi_1(A) \end{array}$$

In the case $X, B \subseteq Y, A = X \cap B$ this is equivalent to the classical diagram

$$\begin{array}{ccc} \pi_3(Y; B, X) & \longrightarrow & \pi_2(X, A) \\ \downarrow & & \downarrow \\ \pi_2(B, A) & \longrightarrow & \pi_1(A) \end{array}$$

With the operations of $\pi_1(A)$ on the other groups
and the generalised Whitehead product

$$h : \pi_2(B, A) \times \pi_2(X, A) \to \pi_3(Y; B, X)$$

this gives a structure called a Crossed Square.
So we have a functor

$$\mathbb{H} : \text{(Squares of pointed spaces)} \to \text{(Crossed Squares)}.$$

We also say $(Y; B, X)$ is connected if the square Z of spaces on the previous slide is connected.

5. Crossed Squares

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow[\mu]{} & P \end{array} \qquad \text{and} \qquad \begin{array}{c} h : M \times N \to L \\ P \text{ acts on } L, M, N \\ \text{so } M, N \text{ act on } L, M, N \end{array}$$

A crossed square should be thought of as a crossed module of crossed modules. Two basic rules are:

$$\begin{aligned} h(mm', n) &= h({}^m m', {}^m n)h(m, n), \\ h(m, nn') &= h(m, n)h({}^n m, {}^n n'), \end{aligned}$$

h is a biderivation, cf a rule for commutators

6. Squares of squares

A standard trick is that a (pushout) square of pointed spaces

$$\mathsf{Z} := \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

can be turned into

$$\mathbb{Z} := \begin{array}{ccc} \begin{smallmatrix} A & A \\ A & A \end{smallmatrix} & \to & \begin{smallmatrix} A & B \\ A & B \end{smallmatrix} \\ \downarrow & & \downarrow \\ \begin{smallmatrix} A & A \\ X & X \end{smallmatrix} & \to & \begin{smallmatrix} A & B \\ X & Y \end{smallmatrix} \end{array}$$

a (pushout) square of squares. If Z is a connected square so also are all the other vertices of this square of squares.

Part of the van Kampen Theorem for squares gives the converse, concluding that the square Z is connected. Considering "squares of squares", or "cubes of cubes", is analogous to using skeleta of CW-complexes, but allows also *n*-cubes of *r*-cubes!

7. A nonabelian tensor product

By the algebraic part of the van Kampen Theorem for squares,
applying Π to the square of squares $\mathbb{Z}$ gives a pushout of crossed squares of the form

$$\begin{array}{ccc} \begin{pmatrix} 1 & 1 \\ 1 & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} 1 & N \\ 1 & P \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} 1 & 1 \\ M & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} L & N \\ M & P \end{pmatrix} \end{array}$$

What should be L?
It has to be home for a new $h : M \times N \to L$. This turns out to be the universal biderivation, so we write it
$h : M \times N \to M \otimes N$, a nonabelian tensor product.
This is an algebraic example
of identifications in dimensions < 3 producing structure in dimension 3.

Standard example: $M, N \trianglelefteq P$ are normal subgroups of P and

$$\kappa : M \otimes N \to P,$$
$$m \otimes n \mapsto [m, n].$$

is well defined. Special case:
$M = N = P$:
the crossed square

$$\begin{array}{ccc} P \otimes P & \longrightarrow & P \\ \big\downarrow & & \big\downarrow{\scriptstyle 1} \\ P & \xrightarrow[1]{} & P \end{array}$$

gives the homotopy 3-type of $SK(P, 1)$, allowing descriptions of π_2, π_3, and Whitehead product $\pi_2 \times \pi_2 \to \pi_3$ as $([x], [y]) \mapsto (x \otimes y)(y \otimes x)$.

So this brings the nonabelian tensor product into homotopy theory.
My web bibliography on the nonabelian tensor product
www.groupoids.org.uk/nonabtens.html
now has 160 entries, dating from 1952, with most interest from group theorists, because of the commutator connection, and the fun of calculating examples.

8. Categorical background

The forgetful functor

$$\Phi : \begin{pmatrix} L & N \\ M & P \end{pmatrix} \mapsto (M \to P, N \to P)$$

has a left adjoint

$$D : (M \to P, N \to P) \mapsto \begin{pmatrix} M \otimes N & N \\ M & P \end{pmatrix}$$

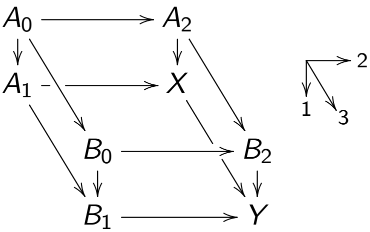
and Φ is a fibration and cofibration of categories. These aspects are relevant to homotopical excision
and to calculate colimits of crossed squares. Φ also has a right adjoint of the form

$$R : (M \to P, N \to P) \mapsto \begin{pmatrix} M \times_P N & N \\ M & P \end{pmatrix}$$

9. Excision for unions of 3 sets

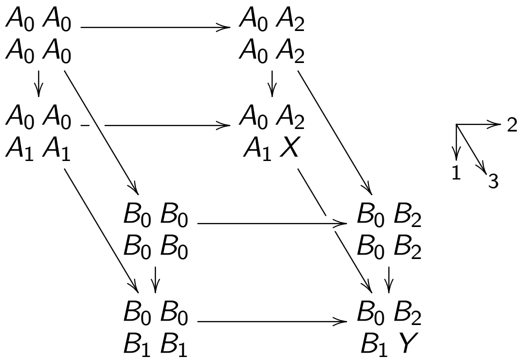
Now extend Homotopical Excision from $Y = X \cup B$
to the case $Y = X \cup B_1 \cup B_2$, all open in Y.
We set $B_0 = B_1 \cap B_2, A_i = X \cap B_i, i = 0, 1, 2,$
giving a 3-cube:

Z :=



This we regard as a map of squares $\partial_3^- \mathsf{Z} \to \partial_3^+ \mathsf{Z}$.
We assume that the square $\partial_3^- \mathsf{Z}$ is connected.
Now we turn Z into a map in direction 3 of squares of squares!

$\mathbb{Z} :=$



This is now a pushout 3-cube of squares of squares.
By the connectivity assumption, all squares of $\partial_3^- \mathbb{Z}$ are connected; then by the 2-d excision applied to the squares $\partial_i^- \mathbb{Z}$ for $i = 1, 2$ we get that the squares of $\partial_3^+ \mathbb{Z}$ except for the corner are connected.
It now follows from the van Kampen Theorem for 3-cubes of squares of spaces that the remaining corner square is connected and that Π applied to this diagram gives a 3-pushout of crossed squares.

This implies the

Triadic Hurewicz Theorem If the square diagram of $(X; A_1, A_2)$ is connected then $H_3(X; A_1, A_2)$ ($= \pi_3(X \cup CA_1 \cup CA_2; CA_1, CA_2)$) is obtained from $\pi_3(X; A_1, A_2)$ by factoring out the operations of $\pi_1(A_0)$ and the generalised Whitehead product.

Abelianisation!

More generally, we need to know how to

compute 2- and 3-pushouts of crossed squares!

There are some general principles, see Appendix B of Nonabelian Algebraic Topology, but currently a

lack of lots of simple computational examples for crossed squares!

We also expect further applications in homotopy theory,

and surely also in geometric topology!

10. Conclusion

The above argument extends to the case $Y = X \cup B_1 \cup \cdots \cup B_n$.
In this talk I have tried to indicate the flexibility and potential of the stated methodology for obtaining new precise results on homotopy types and so on homotopy invariants. It opens up lots of possibilities!
It was not till 1993 that I realised that I could not do some simple calculations with induced crossed modules, and asked for help from Chris Wensley. This resulted in three joint papers, and a Chapter in the NAT book.
Now I need some help, or independent work, on simple basic examples on pushouts and induced crossed squares, and work on higher dimensions!
I think this should be related to work on higher rewriting, on which there are a couple of conferences this coming September, in Oxford and in Marseilles-Luminy.

Some starter references

Brown, R. and Loday, J.-L. "Van Kampen theorems for diagrams of spaces". *Topology* **26** (3) (1986) 311–335. With an appendix by M. Zisman.

Brown, R. and Loday, J.-L. "Homotopical excision, and Hurewicz theorems, for n-cubes of spaces". *Proc. London Math. Soc. (3)* **54** (1) (1987) 176–192.

Brown, R. "Computing homotopy types using crossed n-cubes of groups". In 'Adams Memorial Symposium on Algebraic Topology, 1' (Manchester, 1990), *London Math. Soc. Lecture Note Ser.*, Volume 175. Cambridge Univ. Press, Cambridge (1992), 187–210.

(See `http://groupoids.org.uk/pdffiles/ADAMSVT.pdf` for a revised version with hyperref and some updated references.)