“Not just an idle game” (examining some historical conceptual arguments in homotopy theory)

Ronald Brown

November 18, 2020

Abstract

Part of the title of this article is taken from writings of Einstein. which argues that we need to exercise our ability to analyse familiar concepts in science, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little . . .. The aim is to do this for the concepts of (i) the fundamental group of a pointed space, due to Poincaré; (ii) the fundamental groupoid of a space; (iii) the fundamental groupoid of a space with a set of base points, introduced by the author in 1967; and (iv) the search for higher dimensional versions of the fundamental group.

The history goes back at least to the ICM meeting in Zürich in 1932; the initial negative reactions to the seminar by E. Čech on higher homotopy groups; then the subsequent work of Hurewicz on these groups; the influence of this and subsequent work on the notion of space in topology; and also generalisations of the theorem of Van Kampen.

Introduction

Part of the title of this article is taken from writings of Einstein[1] (1879-1955) in the correspondence published in [24]:

> . . . the following questions must burningly interest me as a disciple of science: What goal will be reached by the science to which I am dedicating myself? What is essential and what is based only on the accidents of development? . . . Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. . . . It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little . . .

This quotation is about science, rather than mathematics, and it is well known for example in physics that there are still fundamental questions, such as the nature of dark matter, to answer. There should be awareness in mathematics that there are still some basic questions which have failed to be pursued for decades; thus we need to think also of educational methods of encouraging their pursuit.

[1]More information on most of the people mentioned in this article may be found in the web site `https://mathshistory.st-andrews.ac.uk/Biographies/`.

1 Homotopy groups at the ICM Zürich, 1932

Why am I considering this ancient meeting? Surely we have advanced since then? And the basic ideas have surely long been totally sorted?

Many mathematicians, especially Alexander Grothendieck (1928-2014), have shown us that basic ideas can be looked at again and in some cases renewed.

The main theme with which I am concerned in this paper is little discussed today, but is stated in [32, p.98]: it involves the introduction by the well respected topologist E. Čech (1893-1960) of homotopy groups $\pi_n(X, x)$ of a pointed space (X, x), and which were proved by him to be abelian for $n > 1$.

But it was argued that these groups were inappropriate for what was a key theme at the time, the development of higher dimensional versions of the fundamental group $\pi_1(X, x)$ of a pointed space as defined by H. Poincaré (1854-1912). In many of the known applications of the fundamental group in complex analysis and differential equations, the largely nonabelian nature of the fundamental group was a key factor.

Because of this abelian property of higher homotopy groups, Čech was persuaded by Heinz Hopf (1894-1971) to withdraw his paper, so that only a small paragraph appeared in the Proceeding [20]: However, the abelian homology groups $H_n(X)$ were known at the time to be well defined for any space X, and that if X was path connected, then $H_1(X)$ was isomorphic to the fundamental group $\pi_1(X, x)$ made abelian. Indeed P. Alexandrov (1896-1998) was reported to have exclaimed: "But my dear Čech, how can they be anything but the homology groups?" [2]

Remark 1.1 It is useful to give for $n = 2$ and in a modern form the argument for the abelian nature of the homotopy groups $\pi_2(X, x)$.

Proof We write I^2 for the unit square $[0.1]^2$, and define $G = \pi_2(X, x)$ as the set of homotopy classes rel the boundary of I^2 of maps $I^2 \to X$ which take all of the boundary ∂I^2 of I^2 to the base point x. It is then easy to define, analogously to the fundamental group, two compositions in "directions" 1 and 2 which we write as $a \circ_1 b, a \circ_2 b$. Both compositions have the structure of a group. We also have for these classes a property called the "interchange law", that each of these compositions is a morphism for the other: that is, for any $a, b, c, d \in G$

$$(a \circ_1 b) \circ_2 (c \circ_1 d) = (a \circ_2 c) \circ_1 (b \circ_2 d) \tag{1}$$

which, by writing

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \circ_1 b, \quad \begin{bmatrix} a & c \end{bmatrix} = a \circ_2 c$$

can be interpreted in matrix notation as that there is only one interpretation of the matrix of compositions

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}. \qquad \begin{array}{l} \ulcorner\!\!\longrightarrow 2 \\ \downarrow \\ 1 \end{array}$$

In this way a "2-dimensional formula" is easier to interpret than the "1-dimensional formula" (1).

Now let e_1, e_2 be the identities for $\circ_1, \circ_2$. Considering the matrix of compositions

$$\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}$$

[2] I heard of this comment in Tbilisi in 1987 from G. Chogoshvili (1914-1998) whose doctoral supervisor was Alexandrov. Compare also [1].It should also be said that Alexandrov and Hopf were two of the most respected topologists: their standing is shown by their invitation by S. Lefschetz (1884-1972) to spend the academic year 1926 in Princeton.

gives immediately that $e_1 = e_2$. So we write e for either. Now interpreting in turn the matrices

$$\begin{bmatrix} a & e \\ e & b \end{bmatrix} \qquad \begin{bmatrix} e & a \\ b & e \end{bmatrix}$$

gives first that $a \circ_1 b = a \circ_2 b$, so each are written ab, and second that $ab = ba$. This can be interpreted as "higher dimensional groups are abelian groups".[3]

Remark 1.2 Note however that this argument fails if the composition $e_1 \circ_2 e_2$ is not defined. This is relevant to Section 3.

W. Hurewicz (1906-1956) was at this ICM. With the publication of two notes [31] which shed light on the relation of the homotopy groups to homology groups, the interest in these homotopy groups started; with the growing study of the complications of the homotopy groups of spheres[4], which became seen as a major problem in algebraic topology, the idea of generalisations of the nonabelian fundamental group became disregarded, and it became easier to think of "space" and the "space with base point" necessary to define the homotopy groups, as in substance synonymous.

In 1968, Eldon Dyer (1934-1997), a topologist at CUNY, told me that Hopf told him in 1966 that the history of homotopy theory showed the danger of people being regarded as "the kings" of a subject and so key in deciding directions. There is a lot in this point, cf [1].

However it seems to be true that Aleksandrov and Hopf *were correct* in suggesting that the abelian homotopy groups are not what one would really like for a higher dimensional generalisation of the fundamental group! That does not mean that the higher homotopy groups are without interest; nor does it mean that the search for such higher dimensional generalisations should be completely abandoned.

2 Determining fundamental groups

One reason for this interest in fundamental groups was their known use in important questions relating complex analysis, covering spaces, integration and group theory. H.Seifert (1907-1996) proved useful relations between simplicial complexes and fundamental groups, [38], and a paper by E. H. Van Kampen (1908-1942) [33] gave a general result applied to the complement in a 3-manifold of an algebraic curve. However a modern proof was given by R.H. Crowell (1928-2006) in [22] following lectures of R.H. Fox (1913-1973). The result is often called the Van Kampen Theorem (VKT) and there are many excellent examples of applications of it in expositions of algebraic topology.

The usual statement of the VKT for the fundamental group is as follows.

Theorem 2.1 *Let the space X be the union of open sets U, V with intersection W ad assume U, V, W are path connected. Let $x \in W$. Then the following diagram of fundamental groups and morphisms induced by inclusions:*

$$\begin{array}{ccc} \pi_1(W,x) & \xrightarrow{j} & \pi_1(V,x) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle k} \\ \pi_1(U,x) & \xrightarrow[h]{} & \pi_1(X,x), \end{array} \tag{2}$$

is a pushout diagram of groups.

[3] Thus the title of the article [5] is an inrtended misnomer.

[4] This can be seen by a web search on this topic.

The term "pushout of groups" is defined entirely in terms of the notion of morphisms of groups; in terms of the diagram (2) it says that if G is a group and $f : \pi_1(U, x) \to G, g : \pi_1(V, x) \to G$ are morphisms of groups such that $fi = gj$, then there is a unique morphism of groups $\phi : \pi_1(X, x) \to G$ such that $\phi h = f, \phi g = k$. This property is called the "universal property" of a pushout, and proving it is called "verifying the universal property". In particular, such a verification need not involve a particular construction of the pushout, nor a proof that all pushouts of morphisms of groups exist.

The limitation to path connected spaces and intersections in Theorem 2.1 is also very restrictive.[5]. Because of the connectivity condition on W, this standard version of the Van Kampen Theorem for the fundamental group of a pointed space did not compute the fundamental group of the circle, which is after all **the** basic example in topology; the standard treatments instead make a detour into a small part of covering space theory by introducing the "winding number" of the map $p : \mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$ from the reals to the circle, which goes back to Poincaré in the 1890s.

3 From groups to groupoids

This is a theme with which I became involved in the years since 1965.

A *groupoid* is defined in modern terms as a small category in which every morphism is an isomorphism. It can be considered as a "group with many identities", or more formally as an algebraic structure with partial algebraic operations, [28]. I like to define "higher dimensional algebra" as the study of partial algebraic structures where the domains of the algebraic operations are defined by geometric conditions.

The simplest non trivial example of a groupoid is the groupoid say $\mathcal{I}$ which has two objects $0, 1$ and only one nontrivial arrow $\iota : 0 \to 1$. and hence also $\iota^{-1} : 1 \to 0$. This groupoid looks "trivial", but it is in fact the basic "transition operator". Groupoids had been defined by Brandt (1886-1954) in 1926, [2], in extending work of Gauss (1777-1815) on compositions of binary quadratic forms to the quaternionic case; their use in topology had been initiated by K. Reidemeister (1893-1971) in his 1932 book, [37].

The use of the *fundamental groupoid* $\pi_1(X)$ of a space X, defined in terms of homotopy classes rel end points of paths $x \to y$ in X was a commonplace by the 1960s. Students find it quite easy to see the idea of a path as a journey, not necessarily a return journey.

I was led to Philip Higgins' (1924-2015) paper on groupoids, [29]. I noticed that he utilised pushouts of groupoids, and so decided to insert in the book I was writing in 1965 an exercise on the Van Kampen Theorem for the fundamental groupoid $\pi_1(X)$. Then I thought I had better write out a proof; when I had done so it seemed so much better than my previous attempts that I decided to explore the relevance of groupoids.

It was still annoying that I could not deduce the fundamental group of the circle! I then realised we were in a "Goldilocks situation": one base point was *too small*; taking the whole space was *too large*; but for the circle taking two base points was *just right*! So, we needed a definition of the fundamental groupoid $\pi_1(X, S)$ for a **set** S of base points chosen according to the geometry of the situation: see the paper [3] and all editions of [4], as well as [30][6]

An inspiring conversation with G.W. Mackey (1916-2006) in 1967 at a Swansea BMC, where I gave an invited talk on the fundamental groupoid, informed me of the notion of "virtual groups", cf [35, 36] and their relation to groupoids; then led me to extensive work of C. Ehresmann (1905-1979) and his school, all showing that the idea of groupoid had much wider significance than I had suspected, cf [8]. See also more recent work

[5]Comments by Grothendieck on this restriction are quoted extensively in [6], see also [27, Section 2].

[6]For a discssion on this issue of many base poiits, see `https://mathoverflow.net/questions/40945/`.

on for example ‘Lie groupoids, Conway groupoids, groupoids and physics’.

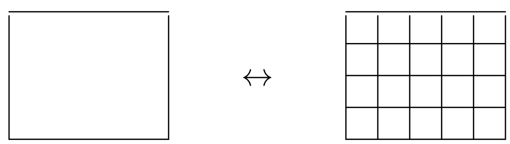
4 From groupoids to higher groupoids

As we have shown, “higher dimensional groups” are just abelian groups.

However this is no longer so for “higher dimensional groupoids”, [18].

It seemed to me in 1965 that some of the arguments for the VKT generalised to higher dimensions and this was prematurely claimed as a theorem in [3].[7]

One of these arguments comes under the theme or slogan of “algebraic inverses to subdivision”.

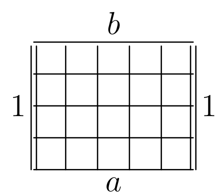


From left to right gives *subdivision*. From right to left should give *composition*. What we need for higher dimensional, nonabelian, local-to-global problems is:

Algebraic Inverses to Subdivision.

This aspect is clearly more easily treated by cubical methods than the standard simplicial or globular ones.

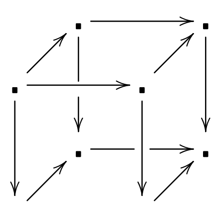
One part of the proof of the VKT for the fundamental group, namely the uniqueness of a universal morphism, is more easily expressed in terms of the double groupoid $\Box G$ of commutative squares in a group or groupoid G, which I first saw defined in [23]. The essence of its use is as follows: consider a diagram of morphisms in a groupoid:



(3)

Suppose each individual square is commutative, and the two vertical outside edges are identities. Then we easily deduce that $a = b$.

For the next dimension we therefore expect to need to know what is a “commutative cube”, and this is expected to be in a double groupoid[8]


(4)

We want the “composed faces” to commute! What can this mean?

[7]It could be more correctly called “ ideas for a proof in search of a theorem”.

[8]More explanation is given in [9] of the way this may be given in a “double groupoid” of squares G in which the horizontal edges G_h and vertical edges G_v come from the same groupoid: i.e. $G_h = G_v$.

We might say the top face is the "composite" of the other faces: so fold them flat to give the left hand diagram of Fig. 1, where the dotted lines show adjacent edges of a "cut". We indicate how to glue these edges back together in the right hand diagram of this Figure by means of extra squares which are a new kind of "degeneracy".

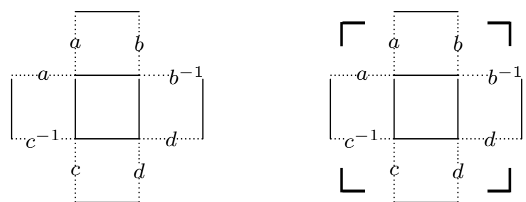
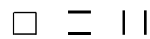


Figure 1: "Composing" five faces of a cube

Thus if we write the standard double groupoid identities in dimension 2 as



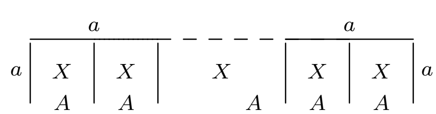
where a solid line indicates a constant edge, then the new types of square with commutative boundaries are written

.

These new kinds of "degeneracies" were called **connections** in [18], because of a relation to path connections in differential geometry. In a formal sense, and in all dimensions, they are constructed from the two functions $\max, \min : \{0,1\} \to \{0,1\}$.

It is explained in Section 8 of [9] how the introduction of these "connections" to the traditional theory of cubical sets remedied some main perceived deficiences of the cubical as against the standard simplicial theory, deficiences which had been known since 1955; yet the change allowed better control of homotopies and higher homotopies (because of the rule $I^m \times I^n \cong I^{m+n}$). It crucially allowed "algebraic inverses to subdivision", and so possibilities for Higher Van Kampen Theorems, starting in dimensions 1, 2 in [12, Part 1], and continuing there in all dimensions, so giving what amounts to a rewrite of much traditional singular and cellular algebraic topology.

We explain briefly the basic definition of $\pi_2(X, A, a)$. the second relative homotopy group of a based pair; it is defined as the homotopy classes relative to a base point say **0** of I^2, of maps $I^2 \to X$ which take $\{0\} \times I \cup \{0,1\} \times I$ to a. The composition of such classes is in the direction 2:



(5)

Whitehead proved that the boundary $\partial : \pi_2(X, A, a) \to \pi_1(A, a)$ and an operation of the group $\pi_1(A, a)$ on $\pi_2(X, A, a)$ have the structure of what he called a *crossed module*; these structures have proved important in studying higher groupoids. He proved in [39, Section 16] what we call Whitehead's free crossed module theorem that in the case X is formed from A by attaching 2-cells, then this crossed module is free on the characteristic maps of the 2-cells.

The definition of this crossed module involves choosing which vertex should be the base point of the square and which edges of the square should map to the base point a, so that the remaining edge maps into A. However it is a good principle to reduce, preferably completely, the number of choices used in basic definitions (though such choices are likely in developing consequences of the definitions). The paper [11], submitted in 1975, defined for a triple of spaces $X_* = (X_2, X_1, X_0)$ of spaces a structure $\varrho(X_*)$: this consisted in dimension 0 of X_0 (as a set); in dimension 1 of $\pi_1(X_1, X_0)$; and in dimension 2 of homotopy classes relative to the vertices of maps $(I^2, E, V) \to (X_2, X_1, X_0)$, where E, V are the spaces of edges and vertices of the standard square.[9]

This definition makes no choice of preferred direction. It is fairly easy and direct to prove that $\varrho(X_*)$ may be given the structure of double groupoid with connection containing the crossed module over a groupoid $\pi_2(X_*)$. That is, the proofs of the required properties of ϱ to make it a 2-dimensional version of the fundamental group as sought in the 1930s are fairly easy but not entirely trivial. The longer task, 1965-1974, was formulating the "correct" concepts, in the face of prejudice from some referees and editors. The proof of the corresponding Van Kampen Theorem allows a nonabelian result in dimension 2 which substantially generalises the work of Whitehead on free crossed modules: for example, it gives a result when X is formed from A by attaching a cone on B, Whitehead's case being B is a wedge of circles.

The paper [9] argues that one difficulty of obtaining such strict higher structures, and so colimits rather than homotopy colimits, is the difficulty of working with "bare" topological spaces. In order to calculate an invariant of a space one needs some information on that space; that information will have a certain structure which needs to be used. Because of the variety of convex sets in dimensions higher than 1, there is a variety also of potentially relevant higher algebraic structures. It turns out that some of these are non trivially equivalent, and can be described as "broad" and "narrow". The broad ones are elegant and symmetric, and useful for conjecturing and proving theorems; the narrow ones are useful for calculating and relating to classical methods; their non trivial equivalence allows one to get the best use of both. An example in [12] is the treatment of cellular methods, using filtered spaces and the related algebraic structures of crossed complexes and cubical ω-groupoids.

This is one explanation why the account in [12] can use and calculate with, strict rather than lax, i.e. up to homotopy, algebraic structures. In comparison, the paper [14] gives a result for all Hausdorff spaces, but so far has given no useful consequences.

To get nearer to a fully nonabelian theory we so far have only the use of n-cubes of pointed spaces as in [15, 16, 7]. It is this restriction to pointed spaces that is a kind of anomaly, and has been strongly criticised by Grothendieck as not suitable for modelling in algebraic geometry. However, the paper [25] gives an application to a well known problem in homotopy theory, *the first non-vanishing homotopy group of an n-ad*; also the *nonabelian tensor poduct of groups* from [15] has become a flourishing topic in group theory (and analogously for Lie algebras); a bibliography[10] 1952-2009 has 175 items.

See also [5].

It is now a commonplace that the further development of related higher structures are important for mathematics and particularly for applications in physics[11]. Note that the mathematical notion of group is deemed fundamental to the idea of symmetry, whose implications range far and wide. The bijections of a set S form a group $Aut(S)$. The automorphisms $Aut(G)$ of a group G form part of a crossed module $\chi : G \to Aut(G)$. The automorphisms of a crossed module form part of a "crossed square" [10]. These structures of set, group, crossed module, crossed square, are related to homotopy n-types for $n = 0, 1, 2, 3$.

The use in texts in English on algebraic topology of sets of base points for fundamental groupoids seems

[9] In that paper it was assumed that each loop in X_0 is contractible in X_1 but this later proved inconvenient and so the use of homotopies fixed on the vertices of I^2 and in general I^n was used in [12].

[10] See http://www.groupoids.org.uk/nonabtens.html

[11] This assertion is supported by a web search on "Institute of higher structures in maths".

currently restricted to [4, 12].

The argument over Čech's seminar to the 1932 ICM seems now able to be resolved through this development of groupoid and higher groupoid work, and he surely deserves credit for the first presentation on higher homotopy groups, as reported in [1, 20, 32][12].

Another way of putting Einstein's quote is that one should be wary of "received wisdom".

Acknowledgements

The projects described here have been supported by grants from SRC, EPSRC, British Council, University College of North Wales, Strasbourg University, Intas.

The perceptive comments of two referees have been helpful in revising this paper.

References

[1] Aleksandrov, P. S. and Finikov, S. P., 'Eduard Čech: Obituary.' *Uspehi Mat. Nauk* **16** (1 (97)) (1961) 119–126. (1 plate). 2, 3, 8

[2] Brandt, H., 'Über eine Verallgemeinerung des Gruppenbegriffes'. *Math. Ann.* **96** (4) (1926) 360–366. 4

[3] Brown, R., "Groupoids and Van Kampen's theorem", *Proc. London Math. Soc.* (3) **17** (1967) 385-401. 4, 5

[4] Brown, R., *Elements of Modern Topology*. McGraw-Hill Book Co., New York (1968). Revised version *"Topology and Groupoids"* (2006) available from amazon. 4, 8

[5] Brown, R., "Higher dimensional group theory", in — *Low dimensional topology, London Math Soc. Lecture Note Series* 48 (ed. R. Brown and T.L. Thickstun, Cambridge University Press, 1982), pp. 215-238. 3, 7

[6] Brown, R., "From groups to groupoids: a brief survey", Bull. London. Math. Soc. 19 (1987) 113-134. 4

[7] Brown, R., "Computing homotopy types using crossed n-cubes of groups", *Adams Memorial Symposium on Algebraic Topology*, Vol 1, edited N. Ray and G Walker, Cambridge University Press, 1992, 187-210. 7

[8] Brown, R., 'Three themes in the work of Charles Ehresmann: Local-to-global; Groupoids; Higher dimensions', *Proceedings of the 7th Conference on the Geometry and Topology of Manifolds: The Mathematical Legacy of Charles Ehresmann, Bedlewo (Poland)* 8.05.2005-15.05.2005, Banach Centre Publications 76, Institute of Mathematics Polish Academy of Sciences, Warsaw, (2007) 51-63. (math.DG/0602499). 4

[9] Brown, R., "Modelling and Computing Homotopy Types: I", Indagationes Math. (Special issue in honor of L.E.J. Brouwer) 29 (2018) 459-482. 5, 6, 7

[12] cf. Ecclesiastes 9.11.

[10] Brown, R. and Gilbert, N.D., "Algebraic models of 3-types and automorphism structures for crossed modules", *Proc. London Math. Soc.* (3) 59 (1989) 51-73. 7

[11] Brown, R. and Higgins, P.J., "On the connection between the second relative homotopy groups of some related spaces", *Proc. London Math. Soc.* (3) 36 (1978) 193-212. 7

[12] Brown, R., Higgins, P. J. and Sivera, R., *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids*. EMS Tracts in Mathematics Vol 15. European Mathematical Society (2011). 6, 7, 8

[13] Brown, R,. Janelidze, G., and Paechter, G., "Open covers, locally sectionable maps, sets of base points, and Van Kampen's theorem", *Theory App. Cat.* (to appear)

[14] Brown, R., Kamps, K.H. and Porter, T., "A homotopy double groupoid of a Hausdorff space II: a van Kampen theorem". Theory and Appl. Cat.14, (2005) 200-220.. 7

[15] Brown, R. and Loday, J.-L.. 'Van Kampen theorems for diagrams of spaces'. *Topology* **26** (3) (1986) 311–335. With an appendix by M. Zisman. Cited on p. 7

[16] Brown, R. and Loday, J.-L., 'Homotopical excision, and Hurewicz theorems for n-cubes of spaces'. *Proc. London Math. Soc. (3)* **54** (1) (1987) 176–192. 7

[17] Brown, R. and Spencer, C.B. "$\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group", *Proc. Kon. Ned. Akad. v. Wet.* 7 (1976) 296-302.

[18] Brown, R. and Spencer, C.B., "Double groupoids and crossed modules", Cah. Top. Géom. Diff. 17 (1976) 343-362. 5, 6

[19] Burke, R., *Preface to Brissot's Address to His Constituents*, (1794).

[20] Čech, E., *Höherdimensionale homotopiegruppen*. Verhandlungen des Internationalen Mathematiker-Kongresses Zurich, Band 2 (1932). 2, 8

[21] Chazan, D., 'Quasi-empirical views of mathematics and mathematics teaching', *Interchange* · 21(1):14-23 DOI: 10.1007/BF01809606

[22] Crowell, R. H., 'On the van Kampen theorem', *Pacific J. Math* 6 (1971) 257-276. 3

[23] Ehresmann, C., *Catégories et structures*. Dunod, Paris (1965). 5

[24] Einstein, A., 'The Einstein-Wertheimer Correspondence on Geometric Proofs and Mathematical Puzzles', *Math. Intell.* I, **12** (2)(1990) 35–43. 1

[25] Ellis, G. J. and Steiner, R., 'Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads'. *J. Pure Appl. Algebra* **46** (2-3) (1987) 117–136. 7

[26] Faria-Martin, J. and Picken, R., "Surface holonomy for non-abelian 2-bundles via double groupoids", *Adv. Math.* 226 (2011) 3309-3366.

[27] Grothendieck, A., *Esquisse d'un Programme* 54p. (1984). 4

[28] Higgins, P. J., 'Algebras with a scheme of operators'. *Math. Nachr.* **27** (1963) 115–132. 4

[29] Higgins, P. J., 'Presentations of groupoids, with applications to groups'. *Proc. Camb. Phil. Soc.* **60** (1964) 7–20. 4

[30] Higgins, P. J., *Notes on categories and groupoids*, *Mathematical Studies*, Volume 32. Van Nostrand Reinhold Co. London (1971); Reprints in Theory and Applications of Categories, No. 7 (2005) pp 1–195. 4

[31] Hurewicz, W., 'Beiträge zur Topologie der Deformationen'. *Nederl. Akad. Wetensch. Proc. Ser. A* **38** (1935) 112–119, 521–528. 3

[32] James, I.M., (editor) *History of Topology*, North-Holland (1994). 2, 8

[33] Kampen, H. Van, "On the connection between the fundamental groups of some related spaces", *Amer. J. Math.* 55 (1933) 261-267. 3

[34] Kline, M., *Mathematics: The loss of certainty?*, Oxford: (1980) Oxford University Press.

[35] Mackey, G.W., 'Ergodic theory and virtual groups', Math. Ann. 166 (1966), 187–207. 4

[36] Ramsay, A., 'Virtual groups and group actions', *Advances in Maths.* 6, 253-322 (1971). 4

[37] Reidemeister, K., *Introduction to Combinatorial Topology*, (2014), Translation by J. Stillwell of the 1932 German book. arXiv:1402.3906, 2014 4

[38] Seifert, H., 'Konstruction dreidimensionaler geschlossen Raume', *Berichte Sächs Akad. Leipzeig, Maths. Phys. Kl* 83 (1931) 26-66. 3

[39] Whitehead, J. H. C., 'Combinatorial homotopy. II'. *Bull. Amer. Math. Soc.* **55** (1949) 453–496. 6

[40] Wigner, E.P., 'The unreasonable effectiveness of mathematics in the natural sciences', *Comm. in Pure Appl. Math.* (1960), reprinted in Symmetries and reflections: scientific essays of Eugene P. Wigner, Bloomington Indiana University Press (1967).