"k-spaces"
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The notion of k-space has turned out to be important in various applications, and the analysis of this use brings out some interesting questions on the role of general topology. On the one hand, k-spaces are a particular kind of space, and so the analysis of their properties such as subspaces, products, and so on is of interest. On the other hand, the notion of topological space arose partly to give a useful setting to the notion of continuity, when it was found that metric spaces were inadequate for the applications. Thus from this point of view, we may not care what is the definition of topological space as long as it plays an adequate role for the purposes in mind. So then we have to ask how we define these purposes.

Since we need a notion of continuity, which defines a relation of a topological space to all other spaces, we are led to a categorical viewpoint, that is, the study of the category Top of all topological spaces and all continuous maps. This global viewpoint suggests asking if this category has all the properties we would desire, and if not, is there a ‘better’ candidate? It is not clear if the ‘final answer’ has yet been obtained, but the notion of k-space has played a key role in this investigation. An influential step was the remark in the R. Brown’s 1963 paper [3]: ‘It may be that the category of Hausdorff k-spaces is adequate and convenient for all purposes of topology.’ Of course ‘adequate’ means that the category contains the basic spaces with which one wishes to deal, for example metric spaces. A key list of desirable properties for a convenient category was given in [4], which amounted to the property of being what is now called cartesian closed [17]. This term ‘convenient category’ was adopted in Steenrod’s widely cited 1967 paper [18], and the scope of the idea was extended by other writers. For example, the 1998 book by Kriegl and Michor [15] provides a ‘convenient setting for global analysis’. It is interesting to see from their account that workers in that field started by taking up the advantages of k-spaces, but eventually found a different setting was needed, and that is the main topic of [15]. But in certain applications, for example algebraic topology, it is often sufficient to have and to use a convenient category without knowing the specific details of its construction.

There are also a number of purely topological questions of interest in k-spaces and these we will come to later.

We first give the definition in the Hausdorff case. A Hausdorff space is said to be a k-space if it has the final topology with respect to all inclusions \( C \to X \) of compact subspaces \( C \) of \( X \), so that a set \( A \) in \( X \) is closed in \( X \) if and only if \( A \cap C \) is closed in \( C \) for all compact subspaces \( C \) of \( X \). Examples of k-spaces are Hausdorff spaces which are locally compact, or satisfy the first axiom of countability. Hence all metric spaces are k-spaces. Also all CW-complexes are k-spaces. A closed subspace of a k-space is again a k-space, but this is not true for arbitrary subspaces [E]. A space is Frechet-Uryson if, whenever a point \( x \) is in the closure of a subset \( A \), there is a sequence from \( A \) converging to \( x \); it is proved in [1] that a k-space \( X \) is hereditarily k, i.e. every subspace is a k-space,
if and only if $X$ is Frechet-Uryson.

The product of $k$-spaces need not be a $k$-space. Let $W_A$ be the wedge of copies of the unit interval $[0,1]$ indexed by a set $A$, where $[0,1]$ is taken to have base point 0, say. (The wedge $\bigvee_{a \in A} X_a$ of a family $\{X_a\}_{a \in A}$ of pointed spaces is the space obtained from the disjoint union of all the $X_a$ by shrinking the disjoint union of the set of base points to a point.) Consider the product $X = W_A \times W_B$ where $A$ is an uncountable set and $B$ is countably infinite. It is proved in [9] that $X$ is not a CW-complex, although $W_A, W_B$ are CW-complexes. Kelley in [14] states as an Exercise that the product of uncountably many copies of the real line $\mathbb{R}$ is not a $k$-space. A solution is in effect given in [2], since the example is used in showing that various topologies on $X \times Y$ are in general distinct.

One place $k$-spaces arise is with the Ascoli Theorem (see [E]): Let $X$ be a $k$-space, let $B$ be the family of compact subsets of $X$, and let $(Y, \mathcal{U})$ denote a uniform space. Then a closed subspace $F$ of the space $\text{Top}(X,Y)$ of continuous functions $X \to Y$ with the compact-open topology is compact if and only if the following conditions hold:

(a) $F|Z$ is equicontinuous for all $Z \in B$,
(b) for all $x \in X$ the set $F(x) = \{f(x) : f \in F\}$ is a compact subset of $Y$.

However the most widespread applications of $k$-spaces are in algebraic topology, for dealing with identification spaces and function spaces.

A problem with identification maps is that the product of identification maps need not be an identification map. One example derives from Dowker’s example mentioned above – another is $f \times 1 : \mathbb{Q} \times \mathbb{Q} \to (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$, where $\mathbb{Q}/\mathbb{Z}$ denotes here the space of $\mathbb{Q}$ of rational numbers with the subspace $\mathbb{Z}$ of integers shrunk to a point. (A proof is given on p.105 of [3].)

Identification spaces present another problem in that an identification of a Hausdorff space need not be Hausdorff, and so there grew pressure, in view of the important applications of identification spaces, to extend the definition of $k$-space to non Hausdorff spaces. A useful definition was found by a number of writers as follows.

We now say that a space $X$ is a $k$-space (also called compactly generated) if $X$ has the final topology with respect to all maps $C \to X$ for all compact Hausdorff spaces $C$. We view maps $t$ of this form as test maps. A consequence of this definition is that $X$ is a necessary and sufficient condition for $X$ to be a $k$-space is that for all spaces $Y$ a function $f : X \to Y$ is continuous if and only if $ft : C \to Y$ is continuous for all test maps $t : C \to X$.

It may seem against common sense to have to test properties of a space $X$ by considering all compact Hausdorff spaces, but in fact since $X$ has only a set of closed subspaces, it is easy to show we can choose a set of test maps to determine if $X$ is a $k$-space. This allows one to show that $X$ is a $k$-space if and only if $X$ is an identification space of a space which is a sum (disjoint union) of compact Hausdorff spaces.

Let $k\text{Top}$ denote the subcategory of $\text{Top}$ of $k$-spaces and continuous maps. The inclusion $i : k\text{Top} \to \text{Top}$ has a left adjoint $k : \text{Top} \to k\text{Top}$ which assigns to any space $X$ the space with the same underlying set but with the final topology.
with respect to all test maps $t : C \to X$. The adjointness condition means that there is a natural bijection

$$\text{Top}(i_X, Y) \to k\text{Top}(X, kY)$$

for all $k$-spaces $X$ and topological spaces $Y$. Consequently $k$ preserves limits, and in particular the product $X \times_k Y$ in the category $k\text{Top}$ is the functor $k$ applied to the usual product $X \times Y$. Further, $i$ preserves colimits.

The importance for function spaces of the maps $kX \to Y$ was first emphasised by Kelley [14] in the context of Hausdorff spaces. If we now work in the category $k\text{Top}$, it is natural to define the function space for $k$-spaces $Y, Z$ to be the set of continuous maps $Y \to Z$ with a modification of the compact open topology to the test open topology. This has a subbase all sets

$$W(t, U) = \{ f \in \text{Top}(Y, Z) \mid f t(C) \subseteq U \}$$

for all $U$ open in $Z$ and all test maps $t : C \to Y$. Finally we apply $k$ to this topology to get the function space $k\text{TOP}(Y, Z)$. The major result is that the exponential correspondence gives a bijection

$$k\text{Top}(X \times_k Y, Z) \cong k\text{Top}(X, k\text{TOP}(Y, Z))$$

for all $k$-spaces $X, Y, Z$. For a detailed proof, see for example [8] or the references given there. Thus this result says that the category $k\text{Top}$ is cartesian closed. A consequence is that in this category the product of identification maps is an identification map.

As an example of what can be proved formally from the cartesian closed property, we note that the composition map

$$k\text{TOP}(X, Y) \times_k k\text{TOP}(Y, Z) \to k\text{TOP}(X, Z)$$

is continuous. Consequently, $END(X) = k\text{TOP}(X, X)$ is a monoid in the category $k\text{Top}$ (using of course the product $- \times_k -$).

Another important use of this law in algebraic topology is to be able to regard a homotopy as either a map $I \times X \to Y$ or as a path $I \to k\text{TOP}(X, Y)$ in a space of maps. It is awkward for applications to have to restrict $X$ to be for example locally compact, or Hausdorff.

As an example of the use of the category $k\text{Top}$ we consider the construction of free products $G \ast H$ of $k$-groups $G, H$ (by which is meant groups in the category $k\text{Top}$ so that we require the difference map $(g', g) \mapsto g'g^{-1}$ on $G$ to be continuous as a function $G \times_k G \to G$). The group $G \ast H$ is constructed as a quotient set $p : W \to G \ast H$ where $W$ is a monoid of words in $G \sqcup H$, and it is easy to show the multiplication $W \times_k W \to W$ is continuous. Because $p \times_k p$ is an identification map in $k\text{Top}$, it follows that the difference map on $G \ast H$ (which is given the identification topology) is also continuous. This type of application is pursued in [10].

There is another way of making this type of colimit construction, which also applies much more generally, for example to topological groupoids and
categories, see [7]. This method, based on what is known as the adjoint
functor theorem, see [17], supplies an object with the appropriate universal
property but the method gives no easy information on the open sets of the
constructed space. It can be argued that the universal property, since it defines
the object uniquely, is all that is required and any more detailed information
should be deduced from this property.

A more direct way has been found as follows. A topological space $X$ is a
$k_\infty$-space if it has the final topology with respect to some countable increasing
family $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$ of compact subspaces whose union
is $X$. The following are some main results on these spaces, proved in [19].

Let $X$ be a Hausdorff $k_\infty$-space, and let $p : X \to Y$ be an identification map.
Then the following are equivalent: (i) The graph of the equivalence relation
associated with $p$ is closed in $X \times X$. (ii) $Y$ is a Hausdorff
$k_\infty$-space.

If $p : X \to Y$, $q : Z \to W$ are identification maps of Hausdorff
$k_\infty$-spaces $X, Y, Z, W$ then $p \times q : X \times Z \to Y \times W$ is also an identification map of Hausdorff
$k_\infty$-spaces.

These results are used in [6] for the construction of free products of Hausdorff
$k_\infty$-groups (and more generally for constructions on topological groupoids).

There are other solutions to the problem of the inconvenient nature of the
category $\text{Top}$. An early solution involved what are called quasi-topologies [16].
This was gradually thought to be unacceptable because the class of quasi-
topologies on even a 2-point set did not form a set. However, this objection
can be questioned.

Another solution involves the space $\mathbb{N}^\alpha$ which is the one point compactifi-
cation of the discrete space of positive integers, that is it involves a sequential
approach. For any topological space $X$ one defines the $s$-test maps to be the con-
tinuous maps $\mathbb{N}^\alpha \to X$. The space $X$ is said to be a sequential space (which
we abbreviate here to $s$-space) if $X$ has the final topology with respect to all
$s$-test maps to $X$. The study of such spaces was initiated in [11]. By working in
a manner analogous to that for $k$-spaces one finds the category $\text{sTop}$ of $s$-spaces
has a product $X \times_s Y$ and a function space $\text{sSTOP}$ satisfying an exponential law

$$\text{sTop}(X \times_s Y, Z) \cong \text{sTop}(X, \text{sSTOP}(Y, Z))$$

for all $s$-spaces $X, Y, Z$ [20]. In fact the $k$-space and $s$-space exponential laws are
special cases of a general exponential law defined by a chosen class of compact
Hausdorff spaces satisfying a number of properties [8].

For sequential spaces, the property corresponding to Hausdorff is having
unique sequential limits, which we abbreviate to ‘has unique limits’. A space $X$
has unique limits if and only if the diagonal is sequentially closed in $X \times X$.

Another advantage of sequential spaces is with regard to proper maps. The
one-point sequential compactification $X^\omega$ of a sequential space $X$ is defined
to be the space $X$ with an additional point $\omega$, say, and with a topology such that
$X$ is open in $X^\omega$ and any sequence in $X$ which has no convergent subsequence
converges to the additional point $\omega$. Any function $f : X \to Y$ has an extension
$f^\omega : X^\omega \to Y^\omega$ in which the additional point of $X$ is mapped to the additional

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References

Let $X$ be a topological space and $A$ be a subset of $X$. Then $A$ is a retract of $X$ if there exists a continuous function $r: X \to A$ such that $r(a) = a$ for all $a \in A$. A topological space $X$ is called Fréchet if every subset of $X$ is a retract of $X$. A topological property is said to be hereditary if every subspace of a space with that property also has the property. A topological property is said to be a hereditary property if every subspace of a space with that property also has the property.

A topological property is a hereditary property if and only if it is a topological property.

Let $f: X \to Y$ be a continuous function. Then $f$ is a sectional map if $f$ is a sectional map on $X$ and $f$ is a sectional map on $Y$. A sectional map is a map $f: X \to Y$ such that for every subset $A$ of $X$, $f(A)$ is a sectional map of $A$ into $Y$. A sectional map is a map $f: X \to Y$ such that for every subset $A$ of $X$, $f(A)$ is a sectional map of $A$ into $Y$. A sectional map is a map $f: X \to Y$ such that for every subset $A$ of $X$, $f(A)$ is a sectional map of $A$ into $Y$.

For any topological space $X$, let $C(X)$ denote the set of all continuous functions on $X$. Then $C(X)$ is a vector space over the field of real numbers. A subset $S$ of $C(X)$ is said to be a subalgebra of $C(X)$ if $S$ is a subalgebra of $C(X)$. A subset $S$ of $C(X)$ is said to be a subalgebra of $C(X)$ if $S$ is a subalgebra of $C(X)$. A subset $S$ of $C(X)$ is said to be a subalgebra of $C(X)$ if $S$ is a subalgebra of $C(X).

Consider the following conditions for a map $f: X \to Y$.

(a) $A \subseteq X \Rightarrow f(A) \subseteq f(X)$
(b) $f(A) \subseteq f(B)$ for all $A \subseteq B$
(c) $f(A) = f(B)$ implies $A = B$
(d) $f$ is one-to-one
(e) $f$ is onto


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