"k-spaces"

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The notion of k-space has turned out to be important in various applications, and the analysis of this use brings out some interesting questions on the role of general topology. On the one hand, k-spaces are a particular kind of space, and so the analysis of their properties such as subspaces, products, and so on is of interest. On the other hand, the notion of topological space arose partly to give a useful setting to the notion of continuity, when it was found that metric spaces were inadequate for the applications. Thus from this point of view, we may not care what is the definition of topological space as long as it plays an adequate role for the purposes in mind. So then we have to ask how we define these purposes.

Since we need a notion of continuity, which defines a relation of a topological space to all other spaces, we are led to a categorical viewpoint, that is, the study of the category Top of all topological spaces and all continuous maps. This global viewpoint suggests asking if this category has all the properties we would desire, and if not, is there a 'better' candidate? It is not clear if the 'final answer' has yet been obtained, but the notion of k-space has played a key role in this investigation. An influential step was the remark in the R. Brown's 1963 paper [3]: 'It may be that the category of Hausdorff k-spaces is adequate and convenient for all purposes of topology.' Of course 'adequate' means that the category contains the basic spaces with which one wishes to deal, for example metric spaces. A key list of desirable properties for a **convenient category** was given in [4], which amounted to the property of being what is now called *cartesian closed* [17]. This term 'convenient category' was adopted in Steenrod's widely cited 1967 paper [18], and the scope of the idea was extended by other writers. For example, the 1998 book by Kriegl and Michor [15] provides a 'convenient setting for global analysis'. It is interesting to see from their account that workers in that field started by taking up the advantages of k-spaces, but eventually found a different setting was needed, and that is the main topic of [15]. But in certain applications, for example algebraic topology, it is often sufficient to have and to use a convenient category without knowing the specific details of its construction.

There are also a number of purely topological questions of interest in k-spaces and these we will come to later.

We first give the definition in the Hausdorff case. A Hausdorff space is said to be a **k-space** if it has the *final topology* with respect to all inclusions $C \to X$ of compact subspaces C of X, so that a set A in X is closed in X if and only if $A \cap C$ is closed in C for all compact subspaces C of X. Examples of k-spaces are *Hausdorff* spaces which are *locally compact*, or satisfy the *first axiom of countability*. Hence all metric spaces are k-spaces. Also all CW*-complexes* are k-spaces. A closed subspace of a k-space is again a k-space, but this is not true for arbitrary subspaces [E]. A space is **Frechet-Uryson** if, whenever a point x is in the closure of a subset A, there is a sequence from A converging to x; it is proved in [1] that a k-space X is hereditarily k, i.e. every subspace is a k-space, if and only if X is Frechet-Uryson.

The product of k-spaces need not be a k-space. Let W_A be the wedge of copies of the unit interval $[0,1]$ indexed by a set A, where $[0,1]$ is taken to have base point 0, say. (The **wedge** $\bigvee_{a \in A} X_a$ of a family $\{X_a\}_{a \in A}$ of pointed spaces is the space obtained from the disjoint union of all the X_a by shrinking the disjoint union of the set of base points to a point.) Consider the product $X = W_A \times W_B$ where A is an uncountable set and B is countably infinite. It is proved in [9] that X is not a CW-complex, although W_A, W_B are CW-complexes. Kelley in [14] states as an Exercise that the product of uncountably many copies of the real line $\mathbb{R}$ is not a k-space. A solution is in effect given in [2], since the example is used in showing that various topologies on $X \times Y$ are in general distinct.

One place k-spaces arise is with the Ascoli Theorem (see [E]): *Let X be a k-space, let $\mathcal{B}$ be the family of compact subsets of X, and let $(Y, \mathcal{U})$ denote a uniform space. Then a closed subspace F of the space $\mathsf{Top}(X,Y)$ of continuous functions $X \to Y$ with the compact-open topology is compact if and only if the following conditions hold:*
(a) $F|Z$ is equicontinuous for all $Z \in \mathcal{B}$,
(b) for all $x \in X$ the set $F(x) = \{f(x) : f \in F\}$ is a compact subset of Y.

However the most widespread applications of k-spaces are in algebraic topology, for dealing with identification spaces and function spaces.

A problem with identification maps is that the product of identification maps need not be an identification map. One example derives from Dowker's example mentioned above – another is $f \times 1 : \mathbb{Q} \times \mathbb{Q} \to (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$, where $\mathbb{Q}/\mathbb{Z}$ denotes here the space $\mathbb{Q}$ of rational numbers with the subspace $\mathbb{Z}$ of integers shrunk to a point. (A proof is given on p.105 of [3].)

Identification spaces present another problem in that an identification of a Hausdorff space need not be Hausdorff, and so there grew pressure, in view of the important applications of identification spaces, to extend the definition of k-space to non Hausdorff spaces. A useful definition was found by a number of writers as follows.

We now say that a space X is a k-space (also called **compactly generated**) if X has the final topology with respect to all maps $C \to X$ for all compact Hausdorff spaces C. We view maps t of this form as **test** maps. A consequence of this definition is that X is a necessary and sufficient condition for X to be a k-space is that for all spaces Y a function $f : X \to Y$ is continuous if and only if $ft : C \to Y$ is continuous for all test maps $t : C \to X$.

It may seem against common sense to have to test properties of a space X by considering all compact Hausdorff spaces, but in fact since X has only a set of closed subspaces, it is easy to show we can choose a *set* of test maps to determine if X is a k-space. This allows one to show that X is a k-space if and only if X is an identification space of a space which is a *sum* (disjoint union) of compact Hausdorff spaces.

Let kTop denote the subcategory of Top of k-spaces and continuous maps. The inclusion $i : \mathsf{kTop} \to \mathsf{Top}$ has a left adjoint $\mathsf{k} : \mathsf{Top} \to \mathsf{kTop}$ which assigns to any space X the space with the same underlying set but with the final topology with respect to all test maps $t : C \to X$. The adjointness condition means that there is a natural bijection

$$\mathsf{Top}(iX, Y) \to \mathsf{kTop}(X, \mathsf{k}Y)$$

for all k-spaces X and topological spaces Y. Consequently k preserves limits, and in particular the product $X \times_{\mathsf{k}} Y$ in the category kTop is the functor k applied to the usual product $X \times Y$. Further, i preserves colimits.

The importance for function spaces of the maps $\mathsf{k}X \to Y$ was first emphasised by Kelley [14] in the context of Hausdorff spaces. If we now work in the category kTop, it is natural to define the function space for k-spaces Y, Z to be the set of continuous maps $Y \to Z$ with a modification of the compact open topology to the *test open* topology. This has a subbase all sets

$$W(t, U) = \{f \in \mathsf{Top}(Y, Z) \mid ft(C) \subseteq U\}$$

for all U open in Z and all test maps $t : C \to Y$. Finally we apply k to this topology to get the function space $\mathsf{kTOP}(Y, Z)$. The major result is that the exponential correspondence gives a bijection

$$\mathsf{kTop}(X \times_{\mathsf{k}} Y, Z) \cong \mathsf{kTop}(X, \mathsf{kTOP}(Y, Z))$$

for all k-spaces X, Y, Z. For a detailed proof, see for example [8] or the references given there. Thus this result says that the category kTop is cartesian closed. A consequence is that in this category the product of identification maps is an identification map.

As an example of what can be proved formally from the cartesian closed property, we note that the composition map

$$\mathsf{kTOP}(X, Y) \times_{\mathsf{k}} \mathsf{kTOP}(Y, Z) \to \mathsf{kTOP}(X, Z)$$

is continuous. Consequently, $END(X) = \mathsf{kTOP}(X, X)$ is a monoid in the category kTop (using of course the product $- \times_{\mathsf{k}} -$).

Another important use of this law in algebraic topology is to be able to regard a homotopy as either a map $I \times X \to Y$ or as a path $I \to \mathsf{kTOP}(X, Y)$ in a space of maps. It is awkward for applications to have to restrict X to be for example locally compact, or Hausdorff.

As an example of the use of the category kTop we consider the construction of free products $G * H$ of **k-groups** G, H (by which is meant groups in the category kTop so that we require the difference map $(g', g) \mapsto g'g^{-1}$ on G to be continuous as a function $G \times_{\mathsf{k}} G \to G$). The group $G * H$ is constructed as a quotient set $p : W \to G * H$ where W is a monoid of words in $G \sqcup H$, and it is easy to show the multiplication $W \times_{\mathsf{k}} W \to W$ is continuous. Because $p \times_{\mathsf{k}} p$ is an identification map in kTop, it follows that the difference map on $G * H$ (which is given the identification topology) is also continuous. This type of application is pursued in [10].

There is another way of making this type of colimit construction, which also applies much more generally, for example to topological groupoids and categories, see [7]. This method, based on what is known as the adjoint functor theorem, see [17], supplies an object with the appropriate universal property but the method gives no easy information on the open sets of the constructed space. It can be argued that the universal property, since it defines the object uniquely, is all that is required and any more detailed information should be deduced from this property.

A more direct way has been found as follows. A topological space X is a **k_ω-space** if it has the final topology with respect to some countable increasing family $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$ of compact subspaces whose union is X. The following are some main results on these spaces, proved in [19].

Let X be a Hausdorff k_ω-space, and let $p : X \to Y$ be an identification map. Then the following are equivalent: (i) The graph of the equivalence relation associated with p is closed in $X \times X$. (ii) Y is a Hausdorff k_ω-space.

If $p : X \to Y$, $q : Z \to W$ are identification maps of Hausdorff k_ω-spaces X, Y, Z, W then $p \times q : X \times Z \to Y \times W$ is also an identification map of Hausdorff k_ω-spaces.

These results are used in [6] for the construction of free products of Hausdorff k_ω-groups (and more generally for constructions on topological groupoids).

There are other solutions to the problem of the inconvenient nature of the category Top. An early solution involved what are called *quasi-topologies* [16]. This was gradually thought to be unacceptable because the class of quasi-topologies on even a 2-point set did not form a set. However, this objection can be questioned.

Another solution involves the space $\mathbb{N}^\wedge$ which is the *one point compactification* of the discrete space of positive integers, that is it involves a sequential approach. For any topological space X one defines the s-test maps to be the continuous maps $\mathbb{N}^\wedge \to X$. The space X is said to be a **sequential space** (which we abbreviate here to s-space) if X has the final topology with respect to all s-test maps to X. The study of such spaces was initiated in [11]. By working in a manner analogous to that for k-spaces one finds the category sTop of s-spaces has a product $X \times_{\mathsf{s}} Y$ and a function space sTOP satisfying an exponential law

$$\mathsf{sTop}(X \times_{\mathsf{s}} Y, Z) \cong \mathsf{sTop}(X, \mathsf{sTOP}(Y, Z))$$

for all s-spaces X, Y, Z [20]. In fact the k-space and s-space exponential laws are special cases of a general exponential law defined by a chosen class of compact Hausdorff spaces satisfying a number of properties [8].

For sequential spaces, the property corresponding to Hausdorff is having unique sequential limits, which we abbreviate to 'has unique limits'. A space X has unique limits if and only if the diagonal is sequentially closed in $X \times X$.

Another advantage of sequential spaces is with regard to *proper maps*. The **one-point sequential compactification** $X^\wedge$ of a sequential space X is defined to be the space X with an additional point ω, say, and with a topology such that X is open in $X^\wedge$ and any sequence in X which has no convergent subsequence converges to the additional point ω. Any function $f : X \to Y$ has an extension $f^\wedge : X^\wedge \to Y^\wedge$ in which the additional point of X is mapped to the additional point of Y. Consider the following conditions for an s-map $f : X \to Y$: (a) $f^\wedge$ is an s-map; (b) $f \times 1 : X \times \mathbb{N}^\wedge \to Y \times \mathbb{N}^\wedge$ is sequentially closed; (c) $f \times 1 : X \times Z \to Y \times Z$ is sequentially closed for any s-space Z; (d) if s is a sequence in X with no subsequence convergent in X, then fs has no subsequence convergent in Y; (e) if B is a sequentially compact subset of Y, then $f^{-1}(B)$ is a sequentially compact subset of X; (f) if s is a convergent sequence in Y then $f^{-1}(\bar{s})$ is sequentially compact.

It is proved in [5] that (d)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(c)$\Rightarrow$(b), and that if X is T_1 then (b)$\Rightarrow$(d). Further, if X, Y have unique limits, then (a)$\Leftrightarrow$(c). It is reasonable therefore to call a map satisfying (a) **sequentially proper**.

A further advantage of the sequential theory as shown in [13] is that the category sTop can be embedded in a **topos**, that is a category which is not only cartesian closed but also has finite limits and a 'sub-object classifier'. This has a number of implications, including fibred exponential laws and spaces of partial maps, which are developed in books on topos theory and pursued further in this case in the thesis of Harasani [12]. We cannot go into these implications here, but they do suggest that extensions of the notion of topological space may be very important for future applications.

We now come to some other topological properties related to convergence of sequences.

A topological space X is called **Fréchet** if whenever $x \in X$ and $A \subseteq X$, then $x \in \bar{A}$ if and only if there is a sequence of points (x_n) of A such that $(x_n) \to x$. Then every Fréchet space is sequential but not conversely, and every first countable space is Fréchet.

Let $\mathbb{L}$ be the subspace of $[0, 1]$ consisting of 0 and the set of points $1/n, n = 1, 2, 3, \ldots$. Let $\mathbb{L}^* = \mathbb{L} \setminus \{0\}$. Let X be the set $[0, 1]$ retopologized as follows. The neighbourhoods of t in $(0, 1]$ are the usual neighbourhoods. The neighbourhoods of 0 are the usual neighbourhoods and also any set containing $\{0\} \cup U$ where U is a usual open neighbourhood of $\mathbb{L}^*$. Then X is sequential but not Fréchet, while $X \setminus \mathbb{L}^*$ is not sequential.

Let X be the space of of the previous paragraph but defined using $\mathbb{Q} \cap [0, 1]$ instead of $[0, 1]$. Then X is not sequential, but satisfies U is open in X if and only if $U \cap A$ is open in A for every countable subset A of X.

References

[1] A. Arkhangelskii, A characterization of very k-spaces, *Czech. Math. Journ.* 18 (1968) 392-395.

[2] R. Brown, *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid*, Ellis Horwood, Chichester (1988) 460 pp.

[3] R. Brown, Ten topologies for $X \times Y$, *Quart. J. Math.* (2) 14 (1963), 303-319.

[4] R. Brown, Function spaces and product topologies, *Quart. J. Math.* (2) 15 (1964), 238-250.

[5] R. Brown, Sequentially proper maps and a sequential compactification, *J. London Math Soc.* (2) 7 (1973) 515-522.

[6] R. Brown and J.P.L. Hardy, Subgroups of free topological groups and free products of topological groups, *J. London Math. Soc.* (2) 10 (1975) 431-440.

[7] R. Brown and J.P.L. Hardy, Topological groupoids I: universal constructions, *Math. Nachr.* 71 (1976) 273-286.

[8] P. Booth and J. Tillotson, Monoidal closed, cartesian closed and convenient categories of topological spaces, *Pacific J. Math.* 88 (1980) 35-53.

[9] C.H. Dowker, The topology of metric complexes, *Amer. J. Math.* 74 (1952) 555-577.

[10] E.J. Dubuc and H. Porta, Convenient categories of topological algebras and their duality theories, *J. Pure Appl. Algebra* 1 (1971) 281-316.

[11] S.P. Franklin, Spaces in which sequence suffice I, *Fund. Math.* 57 (1965) 107-116, II: *ibid* 61 (1967) 51-56.

[12] H.A. Harasani, *Topos theoretic methods in general topology*, University of Wales, Bangor, Ph.D. Thesis (1986).

[13] P. Johnstone, On a topological topos, *Proc. London Math. Soc.* (3) 38 (1979) 237-271.

[14] J.L. Kelley, *General Topology*, Van Nostrand, Princeton (1955).

[15] A. Kriegl and P.W. Michor, *The convenient setting for global analysis*, Mathematical Surveys and Monographs, 53. American Mathematical Society, Providence, RI, 1997.

[16] E. Spanier, Quasi topologies, *Duke Math. J.*, 30 (1963) 1-14.

[17] S. Mac Lane, *Categories for the working mathematician*, Springer Verlag, Berlin 1971.

[18] N.E. Steenrod, A convenient category of topological spaces, *Mich. Math. J.* 14 (1967) 133-152.

[19] B.L. Madison, Congruences in topological semigroups, *Second Florida Symposium in Automata and Semigroups, University of Florida (1971) Part II.*

[20] J.G. Tillotson, *The covenient category of sequential spaces*, Masters Thesis, Memorial University of Newfoundland, 1978.

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