NONABELIAN TENSOR PRODUCTS OF GROUPS: THE COMMUTATOR CONNECTION

LUISE-CHARLOTTE KAPPE

SUNY at Binghamton, New York 13902-6000, USA

Abstract

This is a progress report on some of the developments in nonabelian tensor products of groups since the appearance of the paper "Some Computations of Non-Abelian Tensor Products of Groups" by Brown, Johnson and Robertson, ten years ago.

In the spring of 1988 Ronnie Brown came to Binghamton and gave a talk about nonabelian tensor products, in particular about his paper with Johnson and Robertson [7] which had just appeared. I fell in love with tensor products on first sight and started my student Michael Bacon on this topic for his dissertation, and since then others have joined in these investigations.

This talk is an invitation for others to join in this research. There are many interesting and accessible problems and it appears likely that there are interesting applications to group theory, in the same way as regular tensors have been applied.

All this is provided you do not immediately get thrown off by the notation. In context with nonabelian tensor products left actions are used. Early on I contemplated switching to right action but decided against it. That would be like insisting on driving on the right in a country where everyone else drives on the left.

We use the following notation. For elements g, g', h, h' in a group G we set ${}^h g = hgh^{-1}$ for the conjugate of g by h, and $[h, g] = hgh^{-1}g^{-1}$ for the commutator of h and g. The familiar expansion formulas using left action appear as follows:

$$\begin{aligned} [gg', h] &= [{}^g g', {}^g h][g, h], \\ [g, hh'] &= [g, h][{}^h g, {}^h h']. \end{aligned}$$

Now we define the nonabelian tensor product of two groups, preceded by the definition of a compatible action which is intimately connected with nonabelian tensor products.

Definition 1. *Let G and H be a pair of groups acting upon each other in a compatible way, that is*

$$^{({}^g h)}g' = {}^g({}^h({}^{g^{-1}}g')) \ \textit{and} \ ^{({}^h g)}h' = {}^h({}^g({}^{h^{-1}}h'))$$

for $g, g' \in G$ and $h, h' \in H$, and acting upon themselves by conjugation.

Then the nonabelian tensor product $G \otimes H$ is the group generated by the symbols $g \otimes h$ with defining relations

$$\begin{aligned} gg' \otimes h &= ({}^g g' \otimes {}^g h)(g \otimes h), \\ g \otimes hh' &= (g \otimes h)({}^h g \otimes {}^h h') \end{aligned}$$

for $g, g' \in G, h, h' \in H$.

If $G = H$ and all actions are conjugation, then $G \otimes G$ is called the nonabelian tensor square of G.

As the reader might have already realized, a slight change in notation gets you from the commutator expansion formulas to the defining relations of the nonabelian tensor product. It was this visual similarity that lead me to look into this topic. Of course, there is a conceptual connection between commutators and nonabelian tensors, but this will remain in the background here. The idea is to explore tensor products (and squares) as group theoretical objects, and to explicitly determine nonabelian tensor squares and products for whole classes of groups, by using techniques similar to commutator calculus.

To illustrate my point, I will discuss a classical group theoretical problem originating with F. W. Levi [23] and a nonabelian tensor analogue. Levi considered groups in which the commutator operation is associative. Here is a characterization of this class of groups based on Levi's results.

Theorem 2. *For a group G the following conditions are equivalent:*

(i) $[[x, y], z] = [x, [y, z]] \ \ \forall\ x, y, z \in G$;

(ii) $[[x, y], z] = 1 \ \ \forall\ x, y, z \in G$;

(iii) $[xy, z] = [y, z][x, z]$ *and* $[x, yz] = [x, y][x, z] \ \ \forall\ x, y, z \in G$;

(iv) $G' \subseteq Z(G)$.

M. Bacon [3] looked at an analogue of Levi's result for nonabelian tensors. To formulate the analogue we need to consider the tensor center of a group which is a characteristic and central subgroup of every group and is defined as $Z^{\otimes}(G) = \{g \in G; g \otimes x = 1_{\otimes}\ \forall\ x \in G\}$. Replacing the center by the tensor center we may ask now for a characterization of groups in which the commutator subgroup is contained in the tensor center. We arrive at the following five equivalent conditions.

Theorem 3. [3] *The following are equivalent for a group G and its tensor square:*

(i) $[x, y] \otimes z = x \otimes [y, z] \ \ \forall\ x, y, z \in G$;

(ii) $[x, y] \otimes z = 1_{\otimes} \quad \forall\, x, y, z \in G$;

(iii) $xy \otimes z = (y \otimes z)(x \otimes z)$ *and* $x \otimes yz = (x \otimes y)(x \otimes z) \quad \forall\, x, y, z \in G$;

(iv) $G' \subseteq Z^{\otimes}(G)$;

(v) $G \otimes G \cong G/G' \otimes G/G'$.

The first four conditions bear a striking resemblance to Levi's characterization of groups in which the commutator operation is associative. In fact, the methods for their proofs are very similar. The fifth condition comes somewhat as a surprise and arises out of the question whether there are nonabelian groups whose nonabelian tensor square is isomorphic to the tensor square of its abelianization. The answer is yes, and such groups are exactly those groups whose commutator subgroup is contained in the tensor center. This follows immediately from a result due to Graham Ellis [13].

Proposition 4. *Let G be a group and $N \lhd G$. Then $G \otimes G \cong G/N \otimes G/N$ if and only if $N \leqslant Z^{\otimes}(G)$.*

The following example shows that the conditions of Theorem 3 can be satisfied in a nontrivial way.

Example 5. Let p be a prime and $G = \langle a \rangle \rtimes \langle b \rangle$, where $|a| = p^{\alpha}, |b| = p^{\beta}, |[a, b]| = p^{\gamma}, \alpha \geqslant 2\gamma, \beta \geqslant \gamma \geqslant 1, \alpha - \gamma \geqslant \beta$. Then G satisfies the conditions of Theorem 3.

More could be said in this context, but I want to move on and look briefly at the origins and the history. Nonabelian tensor products have their roots in algebraic K-theory as well as topology. Everyone agrees that some ideas can already be found in Whitehead's work [31]. The nonabelian tensor square appears in essence but not in name in the work of Keith Dennis [10] and is based on ideas of C. Miller [25]. Independently, Lue in [24] defines nonabelian tensor products in the setting of nilpotent groups. He extends earlier work by Ganea [16]. Brown and Loday in [8] and [9] can lay claim to be the inventors of the nonabelian tensor product of groups. It is a direct outgrowth of their involvement with generalized Van Kampen theorems. The nonabelian tensor squares of K-theory origin are a special case of nonabelian tensor products.

This brings us up to the 1987-paper of Brown, Johnson and Robertson [7], the starting point for looking at nonabelian tensor products as group theoretical objects. Almost simultaneously several other papers on this topic appear, e.g. by Aboughazi [1], Ellis [11] and Johnson [21]. I will come back to some of them later.

Here was the starting point for us in 1988. Initially, our research was guided by the 8 open problems at the end of [7]. I will report next on the progress that has been made during the past ten years in solving these problems.

1. Let G and H be finite groups acting compatibly on each other. Then is it true that $G \otimes H$ is finite?

Already Brown and Loday in [9] establish that the tensor square, $G \otimes G$, is finite for finite G. Ellis settled this question for tensor products affirmatively in [11], but no purely algebraic proof is known. In addition he shows that the tensor product $G \otimes H$ is of p-power order if G and H are of p-power order. N. Rocco in [27] gives a bound for the order of $G \otimes G$ if G has order p^n. In [15], Ellis and McDermott improve Rocco's bound and extend it to the case of nonabelian tensor products $G \otimes H$ of prime-power groups G and H. In particular it is shown that if G has order p^n and d is the minimal number of generators of G, then the order of $G \otimes G$ does not exceed p^{dn}.

These finiteness results open the door for determining tensor squares and products with the help of computers. Already in [7] the nonabelian tensor squares of all groups up to order 30 are computed in this way, using just the definition of a tensor square. In [14], Ellis and Leonard give a computer algorithm capable of determining the tensor squares of much larger groups, e.g. the Burnside group $B(2,4)$ which has order 4096.

2. Let $d(G)$ be the minimal number of generators for a group G. Can any general estimate of $d(G \otimes G)$ be found when G is finite?

This question was apparently prompted by the following result:

Proposition 6. [7] *Let F be free of rank 2, then $d(F \otimes F)$ is countably infinite.*

More generally, one can now ask under what conditions on G does $d(G)$ finite imply $d(G \otimes G)$ finite. As a sampler of some of our results related to this question I provide one due to M. Bacon [2]:

Theorem 7. *Let G be a group of nilpotency class two with $d(G) = n$, then*

$$d(G \otimes G) \leqslant \frac{n(n^2 + 3n - 1)}{3}.$$

The bound given in the above theorem is sharp. It is attained for the free group of nilpotency class 2 and rank n.

The third and fourth open problems in [7] address the solvability length and nilpotency class of $G \otimes G$, given such information about G.

3. If G is solvable of derived length $l(G)$, then $G \otimes G$ is solvable and $l(G \otimes G) = l(G)$ or $l(G) - 1$. Is there any intrinsic characterization of solvable groups of either type?

4. If G is nilpotent of class $cl(G)$, then $G \otimes G$ is nilpotent and $cl(G \otimes G) = cl(G')$ or $cl(G') + 1$. Can either of these types be characterized internally?

Here are some of our contributions to Problem 4.

Theorem 8. [4] *Let G be nilpotent of class 2. Then $G \otimes G$ is abelian.*

The nilpotency of $G \otimes G$ only depends on the nilpotency of the commutator subgroup G' of G as the following result shows.

Theorem 9. [5] *Let G be a group. If the derived subgroup G' is nilpotent of class $cl(G')$, then $G \otimes G$ is nilpotent with $cl(G \otimes G) = cl(G')$ or $cl(G') + 1$.*

As a byproduct of our investigations in [5] we have the following result.

Proposition 10. *Let T_n be the free nilpotent group of class 3 and rank n. Then $T_2 \otimes T_2$ is abelian and $T_n \otimes T_n$ is nilpotent of class 2 precisely for $n \geqslant 3$.*

The answer to Problem 4 seems not only related to the nilpotency class of G but also to the number of generators. Having now more examples gives hope that we will soon get an answer at least in the case of metabelian groups.

Before going into the specifics of the next three open problems in [7], all dealing with determinations of tensor squares, let me report on the second big topic addressed in our research, namely the explicit computation of tensor squares and products for whole classes of groups. The key to the method are crossed pairings (see [7]).

Definition 11. *Let G and L be groups. A function $\Phi : G \times G \to L$ is called a crossed pairing if*

$$\begin{aligned} \Phi(gg', h) &= \Phi({}^{g}g', {}^{g}h)\Phi(g, h), \\ \Phi(g, hh') &= \Phi(g, h)\Phi({}^{h}g, {}^{h}h') \end{aligned}$$

for $g, g', h, h' \in G$.

Proposition 12. *A crossed pairing Φ determines a unique homomorphism of groups $\Phi^* : G \otimes G \to L$ such that $\Phi^*(g \otimes h) = \Phi(g, h)$ for all $g, h \in G$.*

This method involves conjecturing L and Φ. It seems to look like an improbable task, but can be successfully managed if L can be shown to be abelian, as in the case of groups of nilpotency class 2. The key element in the analysis is the expansion of $g \otimes h$ which is obtained in a similar fashion as a commutator expansion. In [4] a complete classification of nonabelian tensor squares of 2-generator p-groups of class 2 was obtained in the case $p \neq 2$. We lacked at the time a classification of these groups for $p = 2$.

N. Sarmin recently obtained a classification of 2-generator 2-groups of class 2. We have now completed the computation of the tensor squares of these groups and are in the process of preparing a paper on these results [22].

If L is not commutative, conjecturing L is much more difficult and the calculations for the expansion of $g \otimes h$ and checking the crossed pairing are much more involved. Recently we were able to compute $\mathcal{E} \otimes \mathcal{E}$, a nonabelian group, where $\mathcal{E}$ is the free 3-generator 2-Engel group [5]. The group L was conjectured from our knowledge of the tensor square of $B(3,3)$, the 3-generator Burnside group of exponent 3, which is a finite homomorphic image of $\mathcal{E}$. The computations for $B(3,3) \otimes B(3,3)$ were done by Ellis using the algorithm in [14]. The extensive calculations for the expansion of $g \otimes h$ and the verification of the crossed pairing were done with the help of symbolic calculations in GAP [28].

In this context I want to mention a paper by M. Hartl [20]. Extending ideas from [1], he develops a method to determine nonabelian tensor squares for groups of nilpotency class two and computes some of them using this method.

I want to return to the remaining open problems from [7], the next three addressing the determination of nonabelian tensor squares.

5. *Examine the behavior of $G \otimes G$ under the formation of free products.*

As already mentioned in [7] this was done by Gilbert in [17].

6. Complete the evaluation of $G \otimes G$ for all metacyclic groups G.

Some results for split extensions can already be found in [7]. In [21], D.L. Johnson settled the question for finite metacyclic split extensions completely. This leaves us with the cases of infinite metacyclic groups and those which are not split extensions. James Beuerle, a Ph.D. student who just started working with me in the fall of 1996 has made good progress and we hope to have our results ready for publication soon [6]. All tensor squares of metacyclic groups are abelian. In addition to crossed pairings we also use the fact that the tensor square of a homomorphic image of a group G is a homomorphic image of $G \otimes G$ (see [7]).

7. Compute the tensor square of $GL(2,p)$ and other linear groups.

This problem was solved by Hannebauer [19] before we ever got a chance to look at it.

The first seven problems in [7] are of a group theoretical nature. The last problem concerns itself with the diagram (1) in [7] which we reproduce here for the convenience of the reader.

$$
(1)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & G \otimes G & \longrightarrow & G{\wedge}G & \longrightarrow & 1 \\
 & & & & {\scriptstyle \kappa}\downarrow & & {\scriptstyle \kappa'}\downarrow & & \\
 & & & & G' & \xrightarrow{=} & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

From results in [8] and [9] it follows that the above diagram is commutative with exact rows and central extensions as columns. Here Γ is Whitehead's quadratic functor [31], $G{\wedge}G$ is a generalized exterior product, and $H_n(G)$ is the n-th Eilenberg-MacLane homology of G with trivial integer coefficients. For further details see [7]. The last in the list of open problems in [7] is the following:

> 8. *Give an algebraic proof of the exactness of the top row of diagram (1)*:
>
> $$H_3(G) \to \Gamma(G^{ab}) \xrightarrow{\psi} J_2(G) \to H_2(G) \to 0$$
>
> *We suspect that this is an edge exact sequence of a spectral sequence of algebraic origin.*

A purely algebraic proof of the exactness of the sequence of Problem 8 has been given by Ellis in [12]. The material in this paper has since been developed further by T. Parashvili in [26].

Most of the open problems in [7] deal with nonabelian tensor squares, and as a consequence, so does this progress report. In the concluding paragraphs I want to take a look at general tensor products. Compared to the good progress made with tensor squares, the information on products is still very sparse. In [18], Gilbert and Higgins compute the first nonabelian tensor product. They establish that $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$ where the action is mutual inversion. In [14], Ellis and Leonard developed a method to compute $N \otimes M$, where N and M are embedded as normal subgroups in a group G and implemented it by computer for $|G| \leqslant 14$.

The largest class so far addressed are cyclic groups. M. Visscher is preparing these results for publication [29]. The methods employed are again crossed pairings $\Phi : G \times H \to L$ (see

[7]). Let $G = \mathbb{Z}_n$ and $H = \mathbb{Z}_m$, where $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are cyclic, and not necessarily finite, groups of order n and m. Then $d(\mathbb{Z}_n \otimes \mathbb{Z}_m) \leqslant 2$, and this bound is sharp even in the case that $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are both finite.

So far we do not have a single example for a nonabelian tensor product $G \otimes H$ where G and H are each nonabelian with nontrivial compatible actions. Compatability is still an enigma to us. We hope that we can say more about it in the near future.

A natural extension of Problems 3 and 4 in [7] is to consider questions about solvability length and nilpotency class in the setting of nonabelian tensor products. What can one say about the solvability length and nilpotency class of $G \otimes H$ if such information is given on G and H? M. Visscher addresses these questions in [30]. We conclude with a summary of these results.

Definition 13. *Let H and G be groups and let H act on G. We define*

$$D_H(G) = \langle g^h g^{-1} \mid g \in G, h \in H \rangle$$

as the derivative of H in G.

Proposition 14. *If H and G act compatibly on each other, then $D_H(G) \lhd G$ and $D_G(H) \lhd H$.*

Theorem 15. *Let G and H be groups acting compatibly on each other. If $D_H(G)$ is solvable then so are $G \otimes H$ and $D_G(H)$. Furthermore if $l(D_H(G))$ is the derived length of $D_H(G)$ then $l(D_H(G)) \leqslant l(G \otimes H) \leqslant l(D_H(G)) + 1$, and $l(D_H(G)) - 1 \leqslant l(D_G(H)) \leqslant l(D_H(G)) + 1$.*

Theorem 16. *Let G and H be groups acting compatibly on each other. If $D_H(G)$ is nilpotent then so are $G \otimes H$ and $D_G(H)$. Furthermore, if $cl(D_H(G))$ is the nilpotency class of $D_H(G)$, then $cl(D_H(G)) \leqslant cl(G \otimes H) \leqslant cl(D_H(G)) + 1$,*
and $cl(D_H(G)) - 1 \leqslant cl(D_G(H)) \leqslant cl(D_H(G)) + 1$.

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