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# On Künneth suspensions

BY R. BROWN
*University of Liverpool*

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0. *Introduction.* In (2) we defined the Künneth suspension of a cohomology operation —the Künneth suspension involves an arbitrary css-complex $Y$ rather than the 1-sphere $S^1$, as with the usual suspension of a cohomology operation. Now the suspension homomorphism is well known to be related to the operation of forming loop spaces (cf. (4)). The main object of this paper is to prove a similar result for the Künneth suspension.

Our results fall under the following general scheme. There is a natural function

$$\beta: [X, Z] \to [X^Y, Z^Y],$$

where square brackets denote homotopy classes of maps, and $X^Y$, $Y^Y$ are function complexes. Although this function is perfectly explicit, it is not obvious how to compute $\beta$ in general, part of the difficulty being that the spaces $X^Y$, $Z^Y$ have to be computed before $\beta$ can be. However, in the case when $Z = A$, a css-Abelian group, the homotopy type of $A^Y$ is very simply related to the cohomology of $Y$ and homotopy of $A$. Hence in this case, and when $X$ also is a css-Abelian group, we can hope for more convenient expressions for $\beta$; for example, when $X$ and $Z$ are both css-Abelian groups, we show how to express $\beta$ in terms of the Künneth suspension.

In section 3 we show how the methods given here may be used to determine the homotopy type of $X^Y$ by induction on the Postnikov system of $X$.

The problem from which the present work arose was pointed out to me by Dr M. G. Barratt; similar problems are considered by Thom in (5). The results of this paper, and of (2), formed part of an Oxford doctoral thesis written under the supervision of Dr Barratt, to whom I am deeply indebted for advice and criticism.

1. *Preliminaries.* We refer the reader to (2) for any notations and definitions not discussed here.

The category of css-complexes with base point is written $\mathscr{X}$. The correct product in $\mathscr{X}$ is the *collapsed* or *smash product*

$$X \divideontimes Y = Y \times Y/(X \times * \cup * \times Y).$$

The standard $q$-simplex $\Delta^q$ has no base point, and so we define the complex with base point

$$\Delta^q \divideontimes X = \Delta^q \times X/\Delta^q \times *.$$

The function complex $X^Y$ in $\mathscr{X}$ is defined by

$$(X^Y)_q = \operatorname{Map}(\Delta^q \divideontimes Y, X) \tag{1·1}$$

with the obvious css-operators and base point. (Maps and homotopies will always respect base point.) This complex satisfies the *exponential law*: for all $X$, $Y$, $Z$ in $\mathscr{X}$ there is a natural isomorphism (the *exponential map*)

$$\mu: X^{Z \divideontimes Y} \to (X^Y)^Z. \tag{1·2}$$

The proof of this fact is a simple modification of Cartan's proof (3) of the exponential law in the usual css-category.

In dimension 0, the exponential map reduces to a bijection

$$\mu: \operatorname{Map}(Z \divideontimes Y, X) \to \operatorname{Map}(Z, X^Y). \tag{1·3}$$

So the *evaluation map* $\epsilon: X^Y \divideontimes Y \to X$ is uniquely defined by the condition

$$\mu(\epsilon) = 1: X^Y \to X^Y.$$

When $X$ is Kan, the function complexes in (1·2) admit path components, and $\mu$ induces a bijection of these. In particular $\mu$ induces a bijection of homotopy classes

$$\mu: [Z \divideontimes Y, X] \to [Z, X^Y].$$

Let $X$, $Y$, $Z \in \mathscr{X}$. The function

$$\beta: \operatorname{Map}(X, Z) \to \operatorname{Map}(X^Y, Z^Y) \tag{1·4}$$

is the composition

$$\operatorname{Map}(X, Z) \xrightarrow{\epsilon} \operatorname{Map}(X^Y \divideontimes Y, Z) \xrightarrow{\mu} \operatorname{Map}(X^Y, Z^Y).$$

Explicitly, if $f: X \to Z$, and $g \in (X^Y)_q$, then $\beta(f)(g) \in (Z^Y)_p$ is the composition

$$\Delta^q \divideontimes Y \xrightarrow{g} X \xrightarrow{f} Z.$$

When $Z$ is Kan, $\beta$ induces a function

$$\beta: [X, Z] \to [X^Y, Z^Y]. \tag{1·5}$$

The case of interest to us is when $Z$ is an $FD$-complex (i.e. a css-Abelian group). Then $Z^Y$ is also an $FD$-complex, the sets $[X, Z], [X^Y, Z^Y]$ are Abelian groups and $\beta$ is a homomorphism.

Finally, we recall two facts from (2). First, for any $X$ in $\mathscr{X}$ and $FD$-complex $A$ there is a natural isomorphism

$$\gamma: [X, A] \to H^0(X; NA), \tag{1·6}$$

where $H^0(X; NA)$ is the 0-dimensional cohomology of $X$ with coefficients in the normalized chain complex $NA$ of $A$.

Secondly, for any $Y$ in $\mathscr{X}$ and $FD$-complexes $A$, $A'$ such that

$$H^r(Y; NA) \approx H_r(NA') \quad (r \geqslant 0),$$

there is an isomorphism

$$\kappa: H^0(X \divideontimes Y;\, NA) \to H^0(X;\, NA').$$

This isomorphism, which is natural with respect to maps of $X$, is called a *Künneth isomorphism of type* $(Y, NA;\, NA')$.

2. *First results.* Our object is to describe the homomorphism

$$\beta: [X, A] \to [X^Y, A^Y]$$

in a way suitable for computations. In practice we start off with $A$ a minimal css-Abelian group, for example, an Eilenberg–Maclane complex. Now $A^Y$ will not usually be minimal, but we know that there is a homotopy equivalence

$$\lambda: A^Y \to A',$$

where $A'$ is a minimal css-Abelian group, and so simply a product of Eilenberg–Maclane complexes (see, for example, 5·12 of (2)). For purposes of computation we seek to describe not $\beta$ but the composition

$$[X, A] \xrightarrow{\beta} [X^Y, A^Y] \xrightarrow{\lambda_*} [X^Y, A'].$$

The trouble is that in replacing $A^Y$ by $A'$ we have also altered the explicit homomorphism $\beta$.

This difficulty is overcome by the following theorem, in which the homotopy equivalence $\lambda$ is chosen to be closely related to other convenient maps.

Let $Y$ in $\mathscr{X}$ and $A$, $A'$ in $\mathscr{FD}$ be such that $H^*(Y;\, NA) \approx H(NA')$ (this is equivalent to $\pi_*(A^Y) \approx \pi_*(A')$), and let $\kappa$ be a Künneth isomorphism of type $(Y, NA;\, NA')$.

THEOREM 1. *There is a css-homotopy equivalence*

$$\lambda: A^Y \to A'$$

*such that the following diagram is commutative for all $X$ in $\mathscr{X}$.*

$$\begin{tikzcd}
{[X \divideontimes Y, A]} \arrow[r, "\gamma"] \arrow[d, "\mu"'] & H^0(X \divideontimes Y;\, NA) \arrow[dd, "\kappa"] \\
{[X, A^Y]} \arrow[d, "\lambda_*"'] & \\
{[X, A']} \arrow[r, "\gamma"] & H^0(X;\, NA')
\end{tikzcd} \tag{2·1}$$

*Proof.* Each map in (2·1) (except $\lambda_*$, which has not yet been chosen) is an isomorphism. Put $X = A^Y$ in (2·1) and let $\iota \in [A^Y, A^Y]$ be the class of the identity map.

We define $\lambda: A^Y \to A'$ to be any map in the class

$$\gamma^{-1}\kappa\gamma\mu^{-1}(\iota) \in [A^Y, A'].$$

Standard arguments using the naturality of the maps concerned show that diagram (2·1) is commutative. (The abstract result is that any natural transformation of representable functors is representable.) Hence $\lambda_*$ is an isomorphism for all $X$, and so $\lambda$ is a css-homotopy equivalence.

For the rest of this section, we assume $\lambda$ is chosen as in Theorem 1.

THEOREM 2. *The following diagram is commutative.*

$$\begin{tikzcd}
{[X,A]} \arrow[r, "\gamma"] \arrow[d, "\beta"'] & H^0(X;NA) \arrow[d, "\epsilon^*"] \\
{[X^Y,A^Y]} \arrow[d, "\lambda_*"'] & H^0(X^Y \divideontimes Y;NA) \arrow[d, "\kappa"] \\
{[X^Y,A']} \arrow[r, "\gamma"] & H^0(X^Y;NA')
\end{tikzcd} \tag{2·2}$$

*Proof.* We consider the following diagram.

$$\begin{tikzcd}
{[X,A]} \arrow[rr, "\gamma"] \arrow[dr, "\epsilon^*"] \arrow[dd, "\beta"'] & & H^0(X;NA) \arrow[d, "\epsilon^*"] \\
& {[X^Y \divideontimes Y,A]} \arrow[r, "\gamma"] \arrow[dl, "\mu"] & H^0(X^Y \divideontimes Y;NA) \arrow[dd, "\kappa"] \\
{[X^Y,A^Y]} \arrow[d, "\lambda_*"'] & & \\
{[X^Y,A']} \arrow[rr, "\gamma"] & & H^0(X^Y;NA')
\end{tikzcd}$$

Each cell of this diagram is commutative: the top cell by naturality of $\gamma$; the left-hand cell by definition of $\beta$; and the bottom cell by Theorem 1. Hence diagram (2·2) is commutative.

Since the Künneth isomorphism $\kappa$ is computable Theorem 2 throws the computation of $\beta$ onto the computation of the cohomology map induced by $\epsilon: X^Y \divideontimes Y \to X$. Our next theorem gives a description of $\epsilon$ when $X = A$ an $FD$-complex.

Recall that the *fundamental class* of an $FD$-complex $A$ is the class $\omega(A)$ in $H^0(A;NA)$ such that

$$\omega(A) = \gamma(\iota_A),$$

where $\iota_A$ in $[A,A]$ is the homotopy class of the identity map. The *evaluation class* of $\kappa$ is the class $e$ in $H^0(A' \divideontimes Y;NA)$ such that $\kappa(e) = \omega(A')$, the fundamental class in $H^0(A';NA)$.

THEOREM 3. *In the diagram*

$$\begin{tikzcd}
H^0(A;NA) \arrow[r, "\epsilon^*"] & H^0(A^Y \divideontimes Y;NA) \arrow[d, "\kappa"] & H^0(A' \divideontimes Y;NA) \arrow[l, "(\lambda \divideontimes 1)^*"'] \arrow[d, "\kappa"] \\
& H^0(A^Y;NA') & H^0(A';NA') \arrow[l, "\lambda^*"]
\end{tikzcd} \tag{2·3}$$

*we have* $$\kappa\epsilon^*\omega(A) = \lambda^*\omega(A') = \kappa(\lambda \divideontimes 1)^*(e).$$

*Proof.* The square of (2·3) is commutative by naturality of $\kappa$.

We consider the following diagram which is commutative by naturality of $\gamma$ and Theorem 1.

$$\begin{tikzcd}
{[A,A]} \arrow[r, "\gamma"] \arrow[d, "\epsilon^*"'] & H^0(A;NA) \arrow[d, "\epsilon^*"] \\
{[A^Y \divideontimes Y, A]} \arrow[r, "\gamma"] \arrow[d, "\mu"'] & H^0(A^Y \divideontimes Y; NA) \arrow[dd, "\kappa"] \\
{[A^Y, A^Y]} \arrow[d, "\lambda_*"'] & \\
{[A^Y, A']} \arrow[r, "\gamma"] & H^0(A^Y; NA') \\
{[A', A']} \arrow[r, "\gamma"] \arrow[u, "\lambda^*"] & H^0(A'; NA') \arrow[u, "\lambda^*"']
\end{tikzcd}$$

By definition of $\epsilon$,
$$\mu\epsilon^*(\iota_A) = \iota_{A^Y}.$$
Hence $\iota_A$, and $\iota_{A'}$ both map to the same element in $[A^Y, A']$ (namely the homotopy class of $\lambda$). Hence $\iota_A$ and $\iota_{A'}$ both map to the same element in $H^0(A^Y; NA')$. The required relation
$$\kappa\epsilon^*\omega(A) = \lambda^*\omega(A')$$
follows immediately. The relation $\lambda^*\omega(A') = \kappa(\lambda \divideontimes 1)^*(e)$ follows from the definition of $e$ and naturality of $\kappa$.

Computations of the evaluation class $e$ for simple $A$ and $Y$ were given in the Appendix of (2).

*Remark.* In the last paragraph of page 36 of (5) a formula is given for $g^*(\iota)$ where $g$ is an evaluation map and $\iota$ a fundamental class with reals as coefficient group. There is a gap in the argument (pointed out to me by M. G. Barratt) since, although it is clear that there are unique elements $u$, $v$ such that $g^*(\iota) = 1 \times \iota + d \times u + d^2 \times v$ (as stated), it is not obvious that $u$ and $v$ are non-zero and so generators of the cohomology groups they lie in. However, our Theorem 3 relates such an evaluation map to an evaluation class, and such classes were calculated in the Appendix of (2). In particular Theorem A. 8 (i) of (2) implies that the stated formula is true even in integral cohomology.

3. *Künneth suspensions.* In this section we prove the result intimated in the Introduction by characterizing
$$\beta: [A,B] \to [A^Y, B^Y]$$
(where $A, B$ are both $FD$-complexes) in terms of the Künneth suspension $\boldsymbol{\kappa}$ of section 8 of (2).

Let $\kappa_1$, $\kappa_2$ be Künneth isomorphisms of types $(Y, NA; NA')$, $(Y, NB; NB')$, respectively, determining homotopy equivalences
$$\lambda_1: A^Y \to A', \quad \lambda_2: B^Y \to B'$$
as in Theorem 1. It was shown in (2) that $\kappa_1$, $\kappa_2$ determine a Künneth suspension
$$\boldsymbol{\kappa}: \mathrm{Op}(NA, NB) \to \mathrm{Op}(NA', NB').$$

Now it follows from Theorems 7·2, 7·8 of (2) that there is a natural isomorphism

$$\Phi: [A, B] \to \mathrm{Op}\,(NA, NB).$$

THEOREM 4. *The following diagram, in which* $\Phi, \lambda_1^*, \lambda_{2*}$ *are isomorphisms, is commutative.*

$$\begin{array}{ccc}
[A,B] & \xrightarrow{\Phi} & \mathrm{Op}\,(NA,NB) \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\kappa} \\
[A^Y, B^Y] & & \mathrm{Op}\,(NA', NB') \\
\downarrow{\scriptstyle\lambda_{2*}} & & \uparrow{\scriptstyle\Phi} \\
[A^Y, B'] & \xleftarrow[\lambda_1^*]{} & [A', B']
\end{array}$$

*Proof.* Let $k \in [A, B]$ and let $l = \Phi(k) \in \mathrm{Op}\,(NA, NB)$. We consider the following diagram.

$$\begin{array}{ccccc}
[A,B] & \xrightarrow{\gamma} & H^0(A;\, NB) & \xleftarrow{l} & H^0(A;\, NA) \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\epsilon^*} & & \downarrow{\scriptstyle\epsilon^*} \\
[A^Y, B^Y] & & H^0(A^Y \divideontimes Y;\, NB) & \xleftarrow{l} & H^0(A^Y \divideontimes Y;\, NA) \\
\downarrow{\scriptstyle\lambda_{2*}} & & \downarrow{\scriptstyle\kappa_2} & & \downarrow{\scriptstyle\kappa_1} \\
[A^Y, B'] & \xrightarrow{\gamma} & H^0(A^Y;\, NB') & \xleftarrow{\kappa(l)} & H^0(A^Y;\, NA') \\
{\scriptstyle\lambda_1^*}\uparrow{\scriptstyle\approx} & & {\scriptstyle\lambda_1^*}\uparrow{\scriptstyle\approx} & & {\scriptstyle\approx}\uparrow{\scriptstyle\lambda_1^*} \\
[A', B'] & \xrightarrow{\gamma} & H^0(A';\, NB') & \xleftarrow{\kappa(l)} & H^0(A';\, NA')
\end{array} \tag{3·1}$$

This diagram is commutative: the left-hand cells by Theorem 1 and naturality of $\gamma$; the right-hand cells by naturality of cohomology operations and definition of $\kappa$.

Suppose that $k$ in $[A, B]$ maps by the left-hand vertical arrows down to $k'$ in $[A', B']$. By construction of $\Phi$

$$\gamma(k) = \Phi(k)\,(\omega(A)) = l(\omega(A)), \tag{3·2}$$

$$\gamma(k') = \Phi(k')\,(\omega(A')). \tag{3·3}$$

By Theorem 3, $\omega(A) \in H^0(A; NA)$ maps by the right-hand vertical arrows down to $\omega(A') \in H^0(A';\, NA')$. Now (3·2) implies that $k$ and $\omega(A)$ map by any path in diagram (3·1) to the same element in $H^0(A';\, NB')$. Hence (using (3·3))

$$\Phi(k')\,(\omega(A')) = \kappa(l)\,(\omega(A')).$$

This implies that $\Phi(k') = \kappa(l) = \kappa(\Phi(k))$, which is what we were required to prove.

*Remark.* It is well known that the suspension of a cohomology operation is additive. This result generalizes to Künneth suspensions as follows (using the notation of this section): if $Y$ is an $H'$-space in the sense of Hilton–Eckmann, then the image of the Künneth suspension $\kappa$ consists of additive operations. It would be interesting to know if there are additive operations which are not components of a Künneth suspension.

4. *Homotopy type of function spaces.* When $Y$ is finite dimensional, the previous results may be used to compute the homotopy type of $X^Y$ by induction on the Postnikov system of $X$.

The Postnikov system of a connected complex $X$ consists of (i) a sequence

$$p(n): X(n) \to X(n-1) \quad (n \geqslant 1)$$

of fibre bundles with fibre the Eilenberg–MacLane complex $K(\pi_n, n)$, $\pi_n = \pi_n(X)$; (ii) a sequence of maps $h(n): X \to X(n)$ such that $p(n)\,h(n) = h(n-1)$, $(n \geqslant 1)$. Further the $X(n)$ satisfy $\pi_r(X(n)) = 0\,(r > n)$, and the $h(n)$ are such that $h(n)_*: \pi_r(X) \to \pi_r(X(n))$ is an isomorphism for $r \leqslant n$. If $X$ is simply-connected, as we now assume, then the bundles $p(n): X(n) \to X(n-1)$ may be taken to be principal, so that they are entirely described by a classifying map $k(n+1): X(n-1) \to K(\pi_n, n+1)$.

We now consider this Postnikov system 'raised to the power $Y$'. For simplicity we do not consider all components of the function spaces, and $X^Y$ will denote only the component of the trivial map in the function complex of maps $Y \to X$.

First of all, $\pi_q(X^Y)$ is naturally isomorphic to $[S^q Y, X]$. This implies that the map $h(n)^Y: X^Y \to X(n)^Y$ induces isomorphisms $\pi_q(X^Y) \to \pi_q(X(n)^Y)$ for $q + \dim Y \leqslant n$.

Secondly, the principal bundle

$$K(\pi_n, n) \to X(n) \to X(n-1)$$

determines the principal bundle

$$K(\pi_n, n)^Y \to X(n)^Y \to X(n-1)^Y$$

with classifying map

$$k(n+1)^Y: X(n-1)^Y \to K(\pi_n, n+1)^Y.$$

If we assume inductively that $X(n-1)^Y$ and the evaluation map

$$\epsilon(n-1): X(n-1)^Y \circledast Y \to X(n-1)$$

are known, then Theorem 2 enables us to calculate $k(n+1)^Y$ and so $X(n)^Y$. The evaluation map $\epsilon(n)$ must then be computed from the diagram

$$\begin{tikzcd}
K(\pi_n, n)^Y \circledast Y \arrow[r] \arrow[d, "\epsilon"] & X(n)^Y \circledast Y \arrow[r] \arrow[d, "\epsilon(n)"] & X(n-1)^Y \circledast Y \arrow[d, "\epsilon(n-1)"] \\
K(\pi_n, n) \arrow[r] & X(n) \arrow[r] & X(n-1)
\end{tikzcd}$$

in which $\epsilon(n-1)$ is known by the inductive assumption and $\epsilon$ may be computed using Theorem 3.

By these algebraic methods it is possible to recover Barratt's results in (1) on the track group $[S^r Y, X]$ where $Y$ is an $A_2^n$-complex (i.e. $Y$ is $(n-1)$-connected and of dimension not greater than $n+1$): the computations for the case $r+n > 2$ were carried out in the author's thesis. Rather than repeat these computations here, we give a simple example which uses only the Künneth suspension and which illustrates a feature of Barratt's work, namely the way in which a certain extension is determined by the action of $Sq^1$ in $Y$. (The appearance of $Sq^1$ is due to the Cartan formula and the fact that if $X$ is $p$-connected $(p > 2)$ then the first $k$-invariant of $X$ is essentially $Sq^2$.) Our method also gives for free information on induced homomorphisms.

*Example.* Let $X = S^p \cup_2 e^{p+1}$ $(p > 2)$ and let $Y = S^{p-r} \cup_2 e^{p-r+1}$, so that $S^r Y = X$. Now $\pi_q(X^Y) = 0$ for $q + p - r + 1 < p$, that is, for $q < r - 1$. Barratt showed in (1) that $\pi_{r-1}(X^Y) = Z_2$ and that $\pi_r(X^Y) = Z_4$—the latter result was obtained by very geometric methods. We re-prove this result and also prove non-triviality of the map $\eta^*:\pi_{r-1}(X^Y) \to \pi_r(X^Y)$ induced by composition with the generator $\eta \in \pi_r(S^{r-1})$ ($r \geqslant 3$).

Let $W = X^Y$, $W_n = X(n)^Y$. Then $\pi_r(X^Y)$ is isomorphic to $\pi_r(W_{p+1})$. Now $X(p+1)$ is the total space of a fibration with fibre $K(Z_2, p+1)$, base $K(Z_2, p)$ and classifying map $Sq^2: K(Z_2, p) \to K(Z_2, p+2)$. Therefore $W_{p+1}$ is the total space of a fibration with fibre $K(Z_2, p+1)^Y$, base $K(Z_2, p)^Y$ and classifying map

$$(Sq^2)^Y: K(Z_2, p)^Y \to K(Z_2, p+2)^Y.$$

For general $Y$,

$$\pi_q(K(\pi, n)^Y) \approx [S^q Y, K(\pi, n)] \approx H^{q-n}(Y; \pi),$$

the last isomorphism following from (1·6) (our conventions on grading give $Y$ non-trivial cohomology only in non-positive dimensions). So in our particular case there are homotopy equivalences

$$\lambda_1: K(Z_2, p)^Y \to K(Z_2, r-1) \times K(Z_2, r) = A', \quad \text{say},$$

$$\lambda_2: K(Z_2, p+2)^Y \to K(Z_2, r+1) \times K(Z_2, r+2) = B', \quad \text{say}.$$

By Theorem 4, $\lambda_1$ and $\lambda_2$ may be chosen to transform $(Sq^2)^Y$ into $\kappa(Sq^2): A' \to B'$ where $\kappa$ is a Künneth suspension.

In the Appendix of (2) we calculated a Künneth suspension of this type and found that $\kappa(Sq^2)$ is given by the diagram

$$\begin{tikzcd} K(Z_2, r-1) \arrow[d, "Sq^2"'] \arrow[r, phantom, "\times"] & K(Z_2, r) \arrow[d, "Sq^2"] \arrow[dl, "Sq^1"'] \\ K(Z_2, r+1) \arrow[r, phantom, "\times"] & K(Z_2, r+2). \end{tikzcd} \tag{4·1}$$

We have immediately that $\pi_{r-1}(W) = Z_2$ and that $\pi_r(W)$ is an extension of $Z_2$ by $Z_2$. This extension is determined by the map $K(Z_2, r) \to K(Z_2, r+1)$; since $Sq^1$ is non-trivial the extension is non-trivial and $\pi_r(W) = Z_4$.

Finally, the map $K(Z_2, r-1) \to K(Z_2, r+1)$ determines the function $\eta^*$ in the induced fibration. Hence $\eta^*: \pi_{r-1}(W) \to \pi_r(W)$ is non-trivial if $r \geqslant 3$. (It also follows from (4·1) that $\eta^*: \pi_r(W) \to \pi_{r+1}(W)$ is non-trivial.)

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