# Graphs of Groups: Word Computations and Free Crossed Resolutions

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## Summary

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# Contents

# Introduction

The motivation for this work came from attempting to use groupoids to obtain a normal form for elements of the trefoil group. In fact a normal form theorem using groupoids had already been achieved by P. Higgins [17] in a little known paper *The fundamental Groupoid of a Graph of Groups.* Higgins paper includes well known results on the fundamental group of a graph of groups as given by Serre [23] as consequences of the fundamental groupoid of a graph of groups.

The groupoid approach seems more natural than a group only approach and suits computation better by splitting the reduction processes of words into smaller manageable pieces.

The aim of this thesis is to obtain a free crossed resolution of a fundamental groupoid of a graph of groups by adapting results on graphs of CW-complexes. We highlight results on computations used to obtain normal forms for words in graphs of groups and the construction of free crossed resolutions. The categories of groupoids and of crossed complexes are the natural setting for this work.

We use a combination of the graph theory and algebra of groupoids. The graph theory shows clearly how elements of a groupoid can be composed using paths in the underlying graph of the groupoid, and the idea of "passing elements along edges" of a graph of groups is developed in determining a normal form for the fundamental groupoid of a graph of groups. A crossed complex is a sequence of groupoids and morphisms satisfying certain conditions and can hold information about the presentation of a groupoid. An important example of a crossed complex is induced by a filtered space. The connection between crossed complexes and filtered spaces is the key to obtaining a free crossed resolution of a graph of groups.

We begin this thesis by defining a groupoid and using its underlying graph to define groupoid words. We give groupoid constructions that are analogous to standard group theory structures and aim to show that groupoids are a better setting to work in than groups in this context. We exploit groupoid techniques to convince group theorists that groupoids is the setting to work in. For a comprehensive overview of the history of groupoids and examples in different applications refer to R. Brown's survey, *From Groups to Groupoids* [4].

The groupoids that will play an important role in this thesis are called unit, free, universal and fundamental. The unit groupoid models the unit interval and will be used to define total groupoids. Free groupoids are defined on a graph and a word in a free groupoid is given by a path in the graph the groupoid is defined on. A normal form theorem exists for words in an universal groupoid, and this is further developed to obtain a normal form for an element of a fundamental groupoid of a graph of groups.

We develop P. J. Higgins paper, *The fundamental groupoid of a graph of groups* in which using a graph of groups a normal form for for an arrow of the fundamental groupoid of a graph of groups is obtained and a reduction process of a graph of groups word to a unique normal form is given. Higgins' paper is expanded upon and a new proof of his normal form theorem using techniques in common with rewriting algorithms. From Higgins' normal form theorem a normal form for elements of the trefoil group can be obtained and the general normal form coincides with the general normal form for free products with amalgamation as given in the group theory literature.

Using graphs of groups to achieve a normal form the process of reduction is split into smaller reduction processes. The group elements at the groups associated to the vertices are reduced using Knuth-Bendix methods and then linked by the graph structure and the group isomorphisms associated to the edges of the graph. Implementing the representation of a graph of groups and the reduction process has enabled a reduction of a graph of groups word to a unique normal form.

We link the fundamental groupoid of a graph of groups to a total groupoid for a graph of groups. The key to this work is cylinders and we give full details of groupoids obtained using mapping and double mapping cylinders. The main example is the double mapping cylinder which can be used to model free products with amalgamation and HNN-extensions. We give details of the homotopy equivalence of the fundamental groupoid of a graph of groups and the total groupoid of the same graph of groups.

We continue by defining graphs of spaces and the total space of a graph of spaces which is constructed using cylinders. The spaces that we restrict to are CW-complexes and by applying the homotopy crossed complex we have a connection between aspherical spaces and free crossed resolutions. The key result due to Scott and Wall [22] that given a graph of aspherical spaces, the total space is aspherical is used to prove the main result of this thesis that given a graph of free crossed resolutions the total crossed complex is a free crossed resolution.

Free crossed resolutions of a group contain information about the presentation of a group and we give calculations using free crossed resolutions to obtain free crossed resolutions for free products with amalgamation and HNN-extensions and give full details of the generators and boundary relations of the free crossed resolution of the trefoil group.

# Chapter 1

# Groupoids

The structure of a groupoid combines geometry and algebra. Geometrically a groupoid can be thought of in terms of objects and arrows: a directed arrow with two objects (source and target), and two arrows can be composed together if the target of the first is the source of the second. Algebraically we can impose conditions of associativity, left and right identities and inverses.

This chapter begins with a section on the construction of a category and a groupoid from a graph and continues with a section on groupoids with motivating examples. The construction of groupoids and analogous group structures in groupoid theory is developed to adapt group techniques on normal forms.

## 1.1 Graphs, Categories, and Groupoids

This section defines graphs, categories and groupoids using graphs. Graph theory has two roles in this exposition; firstly graphs are used to define graphs of groups, spaces and crossed complexes, and secondly, free structures for groupoids are defined on graphs.

Categories and groupoids have an underlying graph structure and many of the terms from graph theory are used to describe the related attributes and properties in category and groupoid theory.

Category theory provides a general framework which we can then apply to specific mathematical structures such as groups, groupoids and spaces.

### 1.1.1 Graphs

Graph theory is used in this chapter to define categories and groupoids. In later chapters we use graphs to construct graphs of groups, groupoids, $CW$-complexes and crossed complexes. We begin by defining a graph, its attributes and properties.

**Definition 1.1.1** A *directed graph* $\Gamma$ consists of a set of vertices $V(\Gamma)$, a set of edges $E(\Gamma)$, and two maps $s$ and $t$, (source and target) from $E(\Gamma)$ to $V(\Gamma)$.

We denote the edge $e$ from $s(e) = u$ to $t(e) = v$ as $e : u \longrightarrow v$ or $u \xrightarrow{e} v$. We write $\Gamma(u, v)$ for the set of edges from $u$ to $v$. For the purposes of this exposition we abbreviate a "directed graph" to "graph."

A *subgraph* of $\Gamma$ is a graph whose vertices and edges are subsets of the vertices and edges of $\Gamma$.

If we have two graphs $\Gamma_1$ and $\Gamma_2$ a *graph map* $\phi : \Gamma_1 \to \Gamma_2$ is a pair of maps $\phi_V : V(\Gamma_1) \to V(\Gamma_2)$ and $\phi_E : E(\Gamma_1) \to E(\Gamma_2)$ such that $s\phi_E(e) = \phi_V s(e)$ and $t\phi_E(e) = \phi_V t(e)$ for all $e \in E(\Gamma_1)$.

A *directed path* in a graph $\Gamma$ is either an ordered set $p = (e_1, \ldots, e_n)$ for some $n \geqslant 1$ of edges $e_i$ such that $t(e_i) = s(e_{i+1})$ for $i = 1, \ldots n-1$ or an empty path at $v \in V(\Gamma)$, denoted $()_v$.

If $p = (e_1, \ldots, e_n)$ and $q = (e'_1, \ldots, e'_m)$ are directed paths in $\Gamma$ then $pq = (e_1, \ldots, e_n, e'_1, \ldots, e'_m)$ is a directed path from $s(e_1)$ to $t(e'_m)$ if $t(e_n) = s(e'_1)$, and $()$ acts as an identity. This defines a multiplication of paths

$$\overrightarrow{\Gamma}(u, v) \times \overrightarrow{\Gamma}(v, w) \to \overrightarrow{\Gamma}(u, w)$$

where $\overrightarrow{\Gamma}(u, v)$ denotes the set of all directed paths from $u$ to $v$.

An *involution* of a graph $\Gamma$ is a map $E(\Gamma) \to E(\Gamma)$ which sends each edge $e$ to an edge $\overline{e}$ where $s(\overline{e}) = t(e)$, $t(\overline{e}) = s(e)$ and $\overline{\overline{e}} = e$ for all $e \in E(\Gamma)$.

To each graph $\Gamma$ we can associate a graph $\overline{\Gamma}$ with involution. Let $\Gamma'$ be a graph anti-isomorphic to $\Gamma$ where each edge $e$ of $\Gamma$ corresponds to an edge $\overline{e}$ of $\Gamma'$ which is involutary to $e$. We assume that $\Gamma$ and $\Gamma'$ have no edges in common. We define the graph with involution $\overline{\Gamma}$ to be the graph with vertices $V(\overline{\Gamma}) := V(\Gamma)$ and edges $\overline{\Gamma}(v, v') := \Gamma(v, v') \cup \Gamma'(v, v')$ for all $v,\ v' \in V(\Gamma)$.

We define a *path* in $\Gamma$ to be a directed path in the graph $\overline{\Gamma}$ with involution. A path $p = (e_1, \ldots, e_n)$ such that $s(e_1) = t(e_n)$ is called a *circuit* and a path of length 1 which is a circuit is called a *loop*.

Given a path $p = (e_1, \ldots, e_n)$ and $s(e_1) = v$ and $t(e_n) = v'$ then the path is said to *connect* vertices $v$ and $v'$. Connected vertices determines an equivalence relation on the vertices of $\Gamma$ defined by $v \equiv v'$ if and only if there is a path in $\overline{\Gamma}$ that connects $v$ and $v'$. The equivalence classes partition the graph into connected components. If the graph has one connected component it is called *connected*.

A graph without circuits is called *acyclic*. A graph is a *tree* if it is acyclic and connected. A *spanning tree* in a connected graph $\Gamma$ is a subgraph which is a tree and has the same vertex set as $\Gamma$.

In categories and groupoids we will use the terms *object* and *arrow* to describe elements of the vertex and edge sets respectively of their underlying graphs.

### 1.1.2 Categories and Groupoids

Category theory is the abstraction of the study of structures and structure preserving maps. We will use two approaches to category theory, categories of structures and a category as an algebraic object. The mathematical structures that we will study are groupoids, groups, topological spaces and crossed complexes. The structure preserving maps are called functors and are tools that allow comparison between different mathematical structures. We will use functors to translate results from one category to another.

We refer the reader to MacLane's *Categories for the Working Mathematician* [20] for a full account of category theory, and Higgins' *Categories and Groupoids* [16] which combines the theory and algebra.

Since groupoids are a special case of category theory we give the definition of a groupoid in this subsection. The theory of groupoids is developed in Section 1.2.

This subsection begins with the definition of a category and a functor and then gives examples of categories of structures. We then give details of the coproduct, coequaliser and colimit constructions. We end this subsection with a brief account of a category as an algebraic object in which we describe elements of a category of directed paths.

**Definition 1.1.2** A *category* $\mathcal{C}$ consists of a class of objects $\mathrm{Ob}(\mathcal{C})$ and a set of arrows $\mathrm{Arr}(\mathcal{C})$ with an underlying graph $\Gamma\mathcal{C}$ where $\mathrm{Ob}(\mathcal{C}) := V(\Gamma\mathcal{C})$ and $\mathrm{Arr}(\mathcal{C}) := E(\Gamma\mathcal{C})$ together with a family of multiplications:

$$\begin{aligned} \mathrm{Arr}(u,v) \times \mathrm{Arr}(v,w) &\rightarrow \mathrm{Arr}(u,w) \\ (a,b) &\mapsto ab \end{aligned}$$

satisfying the following axioms.

1. If $a : u \rightarrow v$, $b : v \rightarrow w$ and $c : w \rightarrow x$ then $(ab)c = a(bc)$.
2. For all $v \in \mathrm{Ob}(\mathcal{C})$ there is an element $1_v \in \mathrm{Arr}(v,v)$ such that $1_v b = b$ and $a1_v = a$ whenever these multiplications are defined.

We use the notation $\mathcal{C}(u,v)$ to denote the set of arrows $\mathrm{Arr}(u,v)$ in $\mathcal{C}$. A *small category* $\mathcal{C}$ is category where the objects of $\mathcal{C}$ form a set. (A set is a class which is a member of some other class.)

**Definition 1.1.3** A *groupoid* $\mathcal{G}$ is a small category in which every arrow has an inverse: for all $u, v \in \mathrm{Ob}(\mathcal{G})$ and $a : u \rightarrow v$ there is an element $a^{-1} : v \rightarrow u$ such that $aa^{-1} = 1_u$ and $a^{-1}a = 1_v$.

If $\mathcal{C}$ is a category we define its *opposite category* $\mathcal{C}^{op}$ as follows. The objects $u^{op}$ of $\mathcal{C}^{op}$ are in one-to-one correspondence with the objects $u$ of $\mathcal{C}$, the arrows $a^{op} \in \mathcal{C}^{op}(u^{op}, v^{op})$ are in one-to-one correspondence with arrows $a \in \mathcal{C}(v,u)$ and composition is defined by $a^{op}b^{op} = (ba)^{op}$. We note that $(\mathcal{C}^{op})^{op} = \mathcal{C}$ and $\Gamma'\mathcal{C} = \Gamma\mathcal{C}^{op}$ where $\Gamma'$ is the anti-isomorphic graph of $\Gamma$ as described in Subsection 1.1.

Let $X$ be any construction of a category $\mathcal{C}$, then the *dual* of $X$ is the construction defined for the category $\mathcal{C}$ by defining $X$ in $\mathcal{C}^{op}$ and reversing all the arrows.

Given two categories $\mathcal{C}$ and $\mathcal{D}$ we define their *product* $\mathcal{C} \times \mathcal{D}$ to be the category whose objects are ordered pairs $(c,d)$ of objects $c \in \mathrm{Ob}(\mathcal{C})$, $d \in \mathrm{Ob}(\mathcal{D})$ and arrows $(c,d) \rightarrow (c', d')$ are pairs $(f,g)$ where $f \in \mathcal{C}(c,c')$, $g \in D(d,d')$ with composition defined by $(f,g)(f',g') = (ff', gg')$.

**Definition 1.1.4** If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ assigns to each object $u$ of $\mathcal{C}$ an object $F(u)$ of $\mathcal{D}$, and to each arrow $g \in \mathcal{C}(u,v)$ an arrow

$F(g) \in \mathcal{D}(F(u), F(v))$, in such a way that $F(1_u) = 1_{F(u)}$ for each $u \in \mathrm{Ob}(\mathcal{C})$ and $F(gh) = F(g)F(h)$ whenever $gh$ is defined. The functor $F$ is called a *covariant functor*.

We note that a functor is a graph map which preserves products and identity elements. In contrast we also have a *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ which assigns to each object $u$ of $\mathcal{C}$ an object $F(u)$ of $\mathcal{D}$, and to each arrow $g \in \mathcal{C}(u, v)$ an arrow $F(g) \in \mathcal{D}(F(v), F(u))$, in such a way that $F(1_u) = 1_{F(u)}$ for each $u \in \mathrm{Ob}(\mathcal{C})$ and $F(gh) = F(h)F(g)$ whenever $gh$ is defined. We can define a contravariant functor as a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$.

The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined to be the identity map on objects and arrows.

An important class of functors are forgetful functors. We obtain forgetful functors from a groupoid to a category to a graph, $\mathcal{G} \to \mathcal{C} \to \Gamma\mathcal{C}$, by thinking of a groupoid as a category and a category as a graph.

We now give examples of categories in which the objects are structures and the arrows are mappings between them. Composition is given by composition of mappings and the identity arrows are given by the identity maps.

The category of groups, which we denote $\mathcal{G}p$ has as objects all groups and as arrows all homomorphisms of groups. We also note that a group $G$ can be thought of as a category with one object whose arrows are the elements of $G$.

The category of graphs, $\mathcal{G}ph$, has as objects all graphs and arrows all graph maps. Composition is given by composing the object and arrow maps in the obvious way. The identity arrow for each object $\Gamma$ is the identity graph map on $\mathrm{Ob}(\Gamma)$ and $\mathrm{Arr}(\Gamma)$.

Similarly we can define the categories $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{G}pd$, and $\mathcal{C}rs$ of sets, topological spaces, groupoids and crossed complexes respectively.

We also have functors between categories of structures which translate one structure to another. The fundamental groupoid is a functor from the category of topological spaces to the category of groupoids. We define the fundamental groupoid of a graph in subsection 1.2.2 and the fundamental groupoid of a space in Subsection 3.2.1.

We now define coproducts and coequalisers which are used to construct pushouts and colimits.

**Definition 1.1.5** The *coproduct* of a pair of objects $A$ and $B$ in a category $\mathcal{C}$ is an object of $\mathcal{C}$ denoted $A \sqcup B$ together with morphisms $i_A : A \to A \sqcup B$ and $i_B : B \to A \sqcup B$ with the following universal property: given any object $C$ of $\mathcal{C}$ and morphisms $i'_A : A \to C$ and $i'_B : B \to C$ there is a unique morphism $\phi : A \sqcup B \to C$ such that $\phi i_A = i'_A$ and $\phi i_B = i'_B$.

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "i'_A"'] & A \sqcup B \arrow[d, "\phi"] & B \arrow[l, "i_B"'] \arrow[dl, "i'_B"] \\ & C & \end{tikzcd}$$

The coproduct of objects in the categories of $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{G}p$ and $\mathcal{G}pd$ are disjoint union of sets, disjoint union of spaces, free product of groups and disjoint union of groupoids.

**Definition 1.1.6** A *coequaliser* of a pair of arrows $f, g : A \to B$ in a category $\mathcal{C}$ is an arrow $u : B \to C$ such that $uf = ug$ and whenever $h : B \to E$ satisfies $hf = hg$ then $h = h'u$ for a unique arrow $h' : C \to E$.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "u"] \arrow[dr, "h"'] & C \arrow[d, "h'"] \\ & & E \end{tikzcd}$$

In $\mathcal{S}et$ the coequaliser of two functions $f, g : X \to Y$ is the projection $p : Y \to Y/E$ on the quotient set of $Y$ by the least equivalence relation $E \subset Y \times Y$ which contains all pairs $(fx, gx)$ for $x \in X$. A similar construction for topological spaces gives the coequaliser in $\mathcal{T}op$. The coequaliser of two groupoid morphisms is given in Subsection 1.2.6.

**Definition 1.1.7** A *pushout* of a pair of arrows, $f : A \to B$ and $g : A \to C$ with common source, is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i"] \\ C \arrow[r, "j"'] & P \end{tikzcd}$$

such that to every other commutative square built on $f$ and $g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i'"] \\ C \arrow[r, "j'"'] & P' \end{tikzcd}$$

there is a unique morphism $u : P \to P'$ with $ui = i'$ and $uj = j'$.

In the category $\mathcal{S}et$ elements of pushouts are sets of equivalence classes. Similarly for topological spaces. An interesting class of pushouts in $\mathcal{T}op$ are adjunction spaces. In Chapter 3 we use a type of adjunction space called a "mapping cylinder" which is used to model a groupoid mapping cylinder.

The motivating example of this thesis, the free product with amalgamation of groups is given by a pushout of injective maps of groups. In Subsection 1.2.6 we will give details on free products of groups and HNN-extensions using pushouts of groupoids.

The dual construction of a pushout is called a *pullback*.

We now construct a structure that builds a category from a graph. Let $\mathcal{C}$ be a category and $\Gamma$ a non-empty graph. A $\Gamma$-*diagram* $\mathbb{D}$ *in* $\mathcal{C}$ is a graph map $\mathbb{D} : \Gamma \to \mathcal{C}$. If $\Gamma$ has vertex set $V$ and edge set $E$, then the diagram $\mathbb{D}$ consists of $\{D_i\}_{i\in V}$ of objects of $\mathcal{C}$ and a family $\{\alpha_e\}_{e\in E}$ of arrows in $\mathcal{C}$, where $\alpha_e \in \mathbb{D}(D_i, D_j)$ if $e$ is an edge from $i$ to $j$.

If $\mathbb{D}'$ is another $\Gamma$-diagram in $\mathcal{C}$ with objects $D'_i$ and arrows $\alpha'_e$ then a diagram map $f : \mathbb{D} \to \mathbb{D}'$ is a family $f = \{f_i\}_{i\in V}$ of morphisms $f_i : D_i \to D'_i$ in $\mathcal{C}$, such that for every edge $e$ of $\Gamma$ from $i$ to $j$, $f_i\alpha'_e = \alpha_e f_j$.

A *constant* $\Gamma$-*diagram in* $\mathcal{C}$ is one in which all the vertices of $V(\Gamma)$ are mapped to the same object $D$ and all edges are mapped to the identity arrow on $D$. There is up to isomorphism just one such diagram for each object $D$ of $\mathcal{C}$, and we denote it $k(D)$. We now define a colimit.

**Definition 1.1.8** Suppose we are given a $\Gamma$-diagram $\mathbb{D}$ in $\mathcal{C}$, an object $L$ of $\mathcal{C}$ and a diagram map $f : \mathbb{D} \to k(L)$. We say that $f$ is a *colimit* of the diagram $\mathbb{D}$ if it has the following property: for every object $C$ of $\mathcal{C}$ and every diagram map $g : \mathbb{D} \to k(C)$ there is a unique morphism $\gamma : L \to C$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[dr, "g"'] & k(L) \arrow[d, "k(\gamma)"] \\ & k(C) \end{tikzcd}$$

where $k(\gamma)$ is the diagram map $k(L) \to k(C)$.

If $\Gamma$ is a graph with no edges, the $\Gamma$-diagram in a category $\mathcal{C}$ is a family of objects $\{D_v\}_{v \in V(\Gamma)}$. If the colimit exists it is the coproduct of the objects $D_v$. The colimit (if it exists) of the $\Gamma$-diagram $D_1 \xleftarrow{\alpha_1} D_0 \xrightarrow{\alpha_2} D_3$ is the pushout of $\alpha_1$ and $\alpha_2$.

The following theorem is used to show that the category of groupoids admits colimits in Subsection 1.2.6.

**Theorem 1.1.9** *For categories $\mathcal{J}$ and $\mathcal{C}$, if $\mathcal{C}$ has coequalisers for all pairs of arrows and all coproducts indexed by the sets $\mathrm{Ob}(J)$ and $\mathrm{Arr}(J)$ then $\mathcal{C}$ has a colimit for every functor $F : \mathcal{J} \to \mathcal{C}$.*

**Proof** The result follows by dualising the proof of Theorem 1, MacLane [20] $\square$

MacLane restates the theorem in terms of a coequaliser of two morphisms of coproducts on arrows and objects.

**Theorem 1.1.10** *The colimit of $F : \mathcal{J} \to \mathcal{C}$ is the coequaliser of*

$$\bigsqcup_a F_{s(a)} \underset{g}{\overset{f}{\rightrightarrows}} \bigsqcup_u F_u$$

*for $u \in \mathrm{Ob}(J)$ and $a \in \mathrm{Arr}(J)$, where $fi_a = i_{s(a)}$ and $gi_a = i_{t(a)}F_a$.*

We now apply this to pushouts. Hence a pushout of the following diagram in a category $A \xleftarrow{\alpha} C \xrightarrow{\beta} C$ can be given as the following coequaliser.

$$C \sqcup C \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \sqcup C \sqcup B$$

We recall that a category can be viewed as an algebraic object. In Section 1.2 we will be studying groupoids as algebraic objects and considering the notions of generators, relations, free groupoids and word problems. For words in a groupoid we need the following category.

The set of directed paths in a graph $\Gamma$ defines a *category of directed paths* $P\Gamma$ where $\mathrm{Ob}(P\Gamma) := V(\Gamma)$, $\mathrm{Arr}(P\Gamma)(u, v) := \overrightarrow{\Gamma}(u, v)$ and identities are empty paths.

## 1.2 Groups to Groupoids

In this section we define groupoid structures that model group structures. We use groupoids because the algebra is closer to the geometry; a word in a groupoid corresponds to a path in the underlying graph of the groupoid.

The category of groupoids also provide an algebraic analogue of the unit interval which will be used to construct cylinders and mapping cylinders for groupoids and crossed complexes in Chapter 3.

We refer the reader to Higgin's [16] and Brown's *Topology* [4] for full accounts on groupoids and applications of groupoids.

This section gives examples of groupoids that will be used throughout this exposition, and details of how group concepts of free groups, normal subgroups, quotient groups, cosets and presentations are modelled in groupoid theory. We also give the construction of the universal groupoid which enables free products with amalgamation and HNN-extensions to be defined as pushouts of groupoids.

### 1.2.1 Examples and Properties of Groupoids

Groupoids are a generalisation of groups. A group is a groupoid since it is a category where every arrow has an inverse. It it is profitable to study groups in the context of groupoids. In Chapter 2 we construct the fundamental groupoid of a graph of groups which shows how more complicated groups can be constructed from less complicated groups in the context of groupoids.

The unit groupoid acts as the unit interval in the theory of groupoids and will be used to construct mapping cylinders and total crossed complexes in this thesis.

**Example 1.2.1** *Unit Groupoid*

The *unit groupoid* $\mathcal{I}$ has two objects; 0 and 1, two non-identity arrows; $\iota$ and $\iota^{-1}$ which are inverse to each other and the following underlying graph. $\Diamond$

$$\begin{tikzcd} 0 \arrow[loop left, "1_0"] \arrow[r, bend left, "\iota"] & 1 \arrow[l, bend left, "\iota^{-1}"] \arrow[loop right, "1_1"] \end{tikzcd}$$

Figure 1.1: Unit Groupoid

An important family of groupoids are tree groupoids $\mathcal{I}_n$. The unit groupoid is isomorphic to $\mathcal{I}_2$.

**Example 1.2.2** *Tree Groupoids*
The groupoid $\mathcal{I}_n$ has objects $\{0, ..., n-1\}$ and arrows $\{(i,j) : \ 0 \leqslant i, j \leqslant n-1\}$. The source of $(i,j)$ is $i$, and the target $j$. Composition is defined by $(i,j)(j,k) = (i,k)$. Identities $1_i := (i,i)$ and inverses $(i,j)^{-1} = (j,i)$ exist for all $i, j$. $\Diamond$

Any set of arrows forming a tree is a generating set $\mathcal{X}(\mathcal{I}_n)$ for $\mathcal{I}_n$, and we usually choose $\mathcal{X}(\mathcal{I}_n) = \{(0,i) : \ 1 \leqslant i \leqslant n-1\}$ called the star of 0 in $\mathcal{I}_n$. For an arbitrary groupoid $\mathcal{G}$ the *star* of $u$ in $\mathcal{G}$ is the union of sets $\mathcal{G}(u,v)$ for all objects $v$ of $\mathcal{G}$. We denote the star of $u$ in $\mathcal{G}$ by $\mathrm{St}_{\mathcal{G}}(u)$.

We now combine the groupoids; group and a tree groupoid to form a new groupoid.

**Example 1.2.3** *Direct Product of a Group and a Tree Groupoid*
If $G$ is a group and $\mathcal{I}_n$ a tree groupoid, the direct product $G \times \mathcal{I}_n$ is a groupoid. The objects of $G \times \mathcal{I}_n$ are $\{(\cdot, i) : \ \cdot \in \mathrm{Ob}(G), \ i \in \mathrm{Ob}(\mathcal{I}_n), \ 0 \leqslant i \leqslant n-1\}$ and the arrows of $G \times \mathcal{I}_n$ are $(g, (i,j)) : (\cdot, i) \to (\cdot, j)$ where $g$ is an arrow in $G$ and $(i,j)$ an arrow in $\mathcal{I}_n$. Composition is defined by $(g_1, (i,j))(g_2, (j,k)) = (g_1 g_2, (i,k))$ for all $g_1, g_2 \in G$ and $(i,j), \ (j,k) \in \mathcal{I}_n$. The identity arrows are $(e, (i,i))$ and the inverse of $(g, (i,j))$ is $(g^{-1}, (j,i))$. $\Diamond$

The groupoid $G \times \mathcal{I}_n$ has generating set $\mathcal{X}(G) \times \mathcal{X}(\mathcal{I}_n)$ where $\mathcal{X}(G)$ is a generating set of $G$.

A *subgroupoid* $\mathcal{H}$ of a groupoid $\mathcal{G}$ is a groupoid with $\mathrm{Ob}(\mathcal{H}) \subseteq \mathrm{Ob}(\mathcal{G})$, $\mathrm{Arr}(\mathcal{H}) \subseteq \mathrm{Arr}(\mathcal{G})$ and induced multiplication on $\mathcal{H}$. A subgroupoid is *full* if for any two objects

$u$, $v$ of $\mathcal{H}$ the two sets of arrows $\mathcal{H}(u,v)$ and $\mathcal{G}(u,v)$ are equal. The full subgroupoid $\mathcal{G}(u,u)$ is a group called the *vertex group* at $u$ which we denote $\mathcal{G}(u)$.

As groupoids have an underlying graph structure we use graph theory language to describe different properties and attributes of groupoids. A groupoid is *connected* if for any objects $u$, $v$ of $\mathcal{G}$ there is a path in $\mathcal{G}$ from $u$ to $v$ or, equivalently, $\mathcal{G}(u,v) \neq \emptyset$. The *components* of a groupoid $\mathcal{G}$ are the full connected subgroupoids of $\mathcal{G}$. If the components are all vertex groups, then $\mathcal{G}$ is *totally disconnected*. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is *wide* if $\mathrm{Ob}(\mathcal{G}) = \mathrm{Ob}(\mathcal{H})$. A groupoid $\mathcal{G}$ is *discrete* if $\mathcal{G}$ is totally disconnected and for each object $u$ of $\mathcal{G}$, the vertex group $\mathcal{G}(u)$ is the identity group.

The proof of the following proposition appears in 6.3.1 of [4].

**Proposition 1.2.4** *The vertex groups of a connected groupoid are all isomorphic.*

**Proof** If $u$ and $v$ are objects of a connected groupoid $\mathcal{G}$ then $\mathcal{G}(u,v) \neq \emptyset$. For any $g \in \mathcal{G}(u,v)$ the map $h \mapsto g^{-1}hg$ is an isomorphism $\mathcal{G}(u,u) \to \mathcal{G}(v,v)$. $\square$

Every connected groupoid with a finite number of vertices is determined up to isomorphism by a vertex group and object set. This result will be restated as Proposition 1.2.20 a result of free products of groupoids.

A *groupoid morphism* $f : \mathcal{G} \to \mathcal{H}$ is a functor of groupoids. As for groups we have the notions of injections, surjections and isomorphisms.

A morphism $f : \mathcal{G} \to \mathcal{H}$ is *injective (surjective)* if both $f(\mathrm{Ob}(\mathcal{G}))$ and $f(\mathrm{Arr}(\mathcal{G}))$ are injections (surjections). A morphism $f$ of groupoids is an *isomorphism* if and only if $f$ is both an injection and a surjection.

A groupoid morphism $f : \mathcal{G} \to \mathcal{H}$ is *faithful* (or an *embedding*) when to every pair $u$, $v$ of objects of $\mathcal{G}$ and to every pair of arrows $g_1$, $g_2 : u \to v$ of $\mathcal{G}$ the equality $fg_1 = fg_2 : fu \to fv$ implies $f_1 = f_2$.

The *kernel* of $f$, $\mathrm{Ker} f$, is the set of arrows of $g \in \mathcal{G}$ such that $f(g)$ is an identity of $\mathcal{H}$ and the *image* of $f$, $\mathrm{Im} f$, is the set of arrows $f(g) \in \mathcal{H}$, $g \in \mathcal{G}$.

If $f : \mathcal{G} \to \mathcal{H}$ is a groupoid morphism the image of $f$ need not be a subgroupoid of $\mathcal{H}$. For example, let $\phi : \mathcal{I} \to A$ where $A$ is the free group on one generator $a$. We define $\phi(1_0) = \phi_(1_1) = 1$, $\phi(\iota) = a$ and $\phi(\iota^{-1}) = a^{-1}$. The image of $\phi$ has three elements and hence is not a subgroupoid of $A$.

The proof of the next result is taken from 8.3.2 of [4].

**Proposition 1.2.5** *If $f : \mathcal{G} \to \mathcal{H}$ is a faithful morphism of groupoids, the image of $f$ is a subgroupoid of $\mathcal{H}$.*

**Proof** We need to show that if $c, d \in \mathrm{Im}(f)$ and $d^{-1}c \in \mathcal{H}$ then $d^{-1}c \in \mathrm{Im}(f)$.

Suppose $c = f(a)$, $d = f(b)$ where $a \in \mathcal{G}(u, v)$, $b \in \mathcal{G}(v, w)$. Since $d^{-1}c$ is defined $f(v) = f(w)$ which implies since $\mathrm{Ob}(f)$ is injective that $v = w$. Hence $b^{-1}a$ is defined and $d^{-1}c = f(b^{-1}a)$ which belongs to $\mathrm{Im}f$. $\square$

A morphism $f : \mathcal{G} \to \mathcal{H}$ of groupoids is said to *kill* a subgraph $\Gamma$ of $\Gamma\mathcal{G}$ if $f(\Gamma)$ is a discrete groupoid of $\mathcal{H}$.

A *homotopy* $h : f \simeq g$ of a pair of groupoid morphisms $f, g : \mathcal{G} \to \mathcal{H}$ is a functor $\mathcal{G} \times \mathcal{I} \to \mathcal{H}$ in $\mathcal{G}pd$ such that $hi_0(\mathcal{G}) \simeq f$ and $hi_1(\mathcal{G}) \simeq g$ where $i_0$ and $i_1$ are the inclusions of $\mathcal{G}$ into $\mathcal{G} \times \mathcal{I}$. A groupoid morphism $\rho : \mathcal{G} \to \mathcal{H}$ where $\mathcal{H}$ is a subgroupoid of $\mathcal{G}$ with inclusion map $\mu : \mathcal{H} \to \mathcal{G}$ is a *deformation retract* if and only if $\rho\mu = 1_{\mathcal{H}}$ and $\mu\rho \simeq 1_{\mathcal{G}}$ where $1_{\mathcal{H}}$ and $1_{\mathcal{G}}$ are identity functors.

### 1.2.2 Free Groupoid and Words

Free groupoid and words will be used extensively in Chapter 2 to obtain groupoid normal forms.

We define categories and groupoids by adding structure to a graph. We can also recover a graph from a category or a groupoid by using forgetful functors. We can think of a groupoid as a category and then think of a category as a graph. We move from graphs to groupoids and vice versa when constructing groupoid words.

Free groupoids are used to present groupoids in terms of generators and relations; and provide a groupoid setting for words and normal forms. Whereas free groups are defined on sets, free groupoids are defined on graphs.

**Definition 1.2.6** Let $X$ be a graph in the underlying graph $\Gamma\mathcal{G}$ of a groupoid $\mathcal{G}$. We say $\mathcal{G}$ is *free* on $X$ if $X$ is wide in $\mathcal{G}$ and for any groupoid $\mathcal{H}$, any graph morphism $\gamma : X \to \Gamma\mathcal{H}$ extends uniquely to a groupoid morphism $\mathcal{G} \to \mathcal{H}$. If such an $X$ exists we say $\mathcal{G}$ is a *free groupoid on $X$*.

The techniques used here for groupoid words are similar to the techniques used in group theory. We refer the reader to Cohen's *Combinatorial Group Theory* [11]. The following account for groupoids is adapted from Chapter 4 of [16].

Let $p = (e_1, \ldots, e_n)$ be a path in a graph $\Gamma$ i.e. a directed path in $\Gamma \cup \Gamma^{op}$. If for some $i$, $e_i = \overline{e}_{i+1}$ or $\overline{e}_i = e_{i+1}$ then the path $p$ *reduces* to $(e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_n)$ which is also a path in $\Gamma$. This is called a *simple reduction* of $p$. The simple reduction of $p = (e, \overline{e})$ is the empty word $()$ at $s(e)$.

We write $p \equiv p'$ if there is a finite sequence of paths $p = p_0, \ldots, p_k = p'$ $(k \geqslant 0)$ such that for $r = 0, 1, \ldots, k-1$; $p_r$ is a simple reduction of $p_{r+1}$ or vice versa. This defines an equivalence relation on the paths and we write $[p]$ for the equivalence class containing $p$. Equivalent paths have the same source and target so we can assign these as the source and target of the equivalence class.

The equivalence classes form a groupoid. The objects are the vertices of $\Gamma$ and arrows equivalence classes. Composition is defined by $[p][q] = [pq]$ whenever $pq$ is defined in $P\Gamma$ the category of paths in a graph. The identity elements are the classes $[()_u]$ where $()_u$ is the empty path at $u$. For any path $p = (e_1, \ldots, e_n)$ we also have a path $\overline{p} = (\overline{e}_n, \ldots \overline{e}_1)$ where $s(p) = t(\overline{p})$ and $t(p) = s(\overline{p})$. Hence $p\overline{p}$ and $\overline{p}p$ are equivalent to empty paths, so $[\overline{p}]$ is inverse to $[p]$. This groupoid is called the *fundamental groupoid of the graph* $\Gamma$ which we denote $\mathcal{F}(\Gamma)$. We note that this groupoid is also the free groupoid on $\Gamma$.

If $p = (e_1, \ldots, e_n)$ is a path in $\Gamma$ then the product $[e_1] \ldots [e_n]$ is defined in $\mathcal{F}(\Gamma)$ and we call it the *value* of $p$ in $\mathcal{F}(\Gamma)$; the value of the empty word at $u$ is the identity element $1_u$ of $\mathcal{F}(\Gamma)$.

To decide whether two paths have the same image in $\mathcal{F}(\Gamma)$ we consider words which are called reduced words.

**Definition 1.2.7** A path $p = (y_1, \ldots, y_n)$ where $y_i = e_i$ or $\overline{e}_{i+1}$ is a *reduced path* or *reduced word* in $\Gamma$ if $y_{i+1} \neq \overline{y}_i$ for $i = 1, 2, \ldots, n-1$; or if $p = ()_u$.

**Proposition 1.2.8** *Each equivalence class of paths in a graph* $\Gamma$ *contains exactly one reduced path.*

**Proof** We refer the reader to Corollary 1.2.18 on page 23. □

Since every path is equivalent to one reduced word two paths have the same value in $\mathcal{F}(\Gamma)$ if they reduce to the same reduced word. Hence every arrow of $\mathcal{F}(\Gamma)$ is either an identity arrow or is uniquely expressed as a product $[e_1] \dots [e_n]$ where $e_i$ is an edge of $\Gamma$ and $[e_{i+1}] \neq [\overline{e}_i]$ for $i = 1, \dots, n-1$.

### 1.2.3 Normal Subgroupoids and Quotient Groupoids

We use quotient groupoids to obtain the fundamental groupoid of a graph of groups in Chapter 2. To construct a quotient groupoid we factor a groupoid by a normal subgroupoid.

**Definition 1.2.9** A subgroupoid $\mathcal{N}$ of $\mathcal{G}$ is called *normal* if $\mathrm{Ob}(\mathcal{N}) = \mathrm{Ob}(\mathcal{G})$ and for $u, v \in \mathrm{Ob}(\mathcal{G})$ and $a \in \mathcal{G}(u, v)$, $a^{-1}\mathcal{N}(u)a \subseteq \mathcal{N}(v)$.

We define a quotient groupoid for two cases; where the normal subgroupoid is totally disconnected and for arbitrary normal subgroupoids.

If the normal subgroupoid is totally disconnected the quotient groupoid $\mathcal{G}/\mathcal{N}$ consists of objects $\mathrm{Ob}(\mathcal{G}/\mathcal{N}) = \mathrm{Ob}(\mathcal{G})$ and for arrows, if $u, v \in \mathrm{Ob}(\mathcal{G})$ and $a \in \mathcal{G}(u, v)$ we define $(\mathcal{G}/\mathcal{N})(u, v)$ to be the cosets $\mathcal{N}(u)a$. If $a \in \mathcal{G}(u, v)$ and $b \in \mathcal{G}(v, w)$ then normality gives $(\mathcal{N}(u)a)(\mathcal{N}(v)b) = \mathcal{N}(u)a(a^{-1}\mathcal{N}(u)a)b = \mathcal{N}(u)ab$ which defines multiplication of cosets. The identity elements of $(\mathcal{G}/\mathcal{N})$ are $\mathcal{N}(u)$ for $u \in \mathrm{Ob}(\mathcal{G})$ and the inverse of $\mathcal{N}(u)a$ is $\mathcal{N}(v)a^{-1}$.

Given an arbitrary normal subgroupoid the quotient groupoid $\mathcal{G}/\mathcal{N}$ consists of objects which are equivalence classes determined by the connected components of $\mathcal{N}$. The classes form a partition of $\mathrm{Ob}(\mathcal{G})$ and we write $u$ for the class containing $u$. Similarly the arrows are determined by equivalence classes of elements of $\mathrm{Arr}(\mathcal{G})$. If $a, b \in \mathrm{Arr}(\mathcal{G})$ the equivalence is given by $a \equiv b$ if $a = xby$ for some $x, y \in \mathrm{Arr}(\mathcal{N})$. For more details we refer the reader to Chapter 12 of [16].

The special case where $\mathcal{N}$ is totally disconnected follows from this more general construction. In the special case if $a = xby$, then $y \in \mathcal{N}(v)$ and $byb^{-1} = x' \in \mathcal{N}(u)$ so $a \equiv b$ implies $a = xx'b \in \mathcal{N}(u)b$.

### 1.2.4 Groupoid Cosets and Transversals

In the above subsection on normal subgroupoids we have introduced sets of arrows $\mathcal{N}(u)a$ called cosets for totally disconnected normal subgroupoids. In this subsection we define cosets for a wide subgroupoid of a groupoid.

Features of group cosets that we want to capture in groupoid cosets are that the cosets partition the groupoid and the elements of the groupoid can be written uniquely as a product of a coset representative and an element of the subgroupoid.

We use groupoid cosets and transversals in Subsection 2.1.6 to give a normal form theorem for graphs of groupoids words.

**Theorem 1.2.10** *Let $\mathcal{H}$ denote a wide subgroupoid of a groupoid $\mathcal{G}$, and define a relation $\equiv$ on $\mathcal{G}$ as follows $g \equiv g'$ if and only if $g = g'h$ for some $h \in \mathcal{H}$. Then $\equiv$ is an equivalence relation.*

**Proof**

*Reflexive* If $g \in \mathcal{G}$ then $g \equiv g$ because $g = g1_{t(g)}$ and $\mathcal{H}$ contains all the identity elements of $\mathcal{G}$.

*Symmetric* If $g \equiv g'$ then $g = g'h$ for some $h \in \mathcal{H}$. Since $\mathcal{H}$ is a groupoid $h^{-1} \in \mathcal{H}$ and $gh^{-1} = g'$. Hence $g' \equiv g$.

*Transitive* If $g_1 \equiv g_2$ and $g_2 \equiv g_3$ then $g_1 = g_2h$ and $g_2 = g_3h'$ for some $h$ and $h'$ in $\mathcal{H}$. Since $t(g_2) = s(h) = t(h')$ we have $g_1 = g_2h = g_3h'h$. As composition is closed in $\mathcal{H}$, $h'h \in \mathcal{H}$ and $g_1 \equiv g_3$. $\square$

We call the equivalence classes *cosets* which we can define as sets of arrows in $\mathcal{G}$

$$g\mathcal{H} = \{gh \ : \ h \in \mathcal{H}, \ t(g) = s(h)\}.$$

The left coset containing $g$ is denoted $g\mathcal{H}$. We note that the stars are all cosets, $g\mathcal{H} = \{g \ \mathrm{St}_{\mathcal{H}}(t(g))\}$ and $\mathcal{H}$ itself does not form a coset. As in group theory we can choose a set of coset representatives called a *transversal*. For each left coset choose a coset representative.

**Proposition 1.2.11** *If $\mathcal{H}$ is a wide subgroupoid of a groupoid $\mathcal{G}$ and $l, \ k \in \mathcal{G}$ then $l \in k\mathcal{H}$ if and only if $k \in l\mathcal{H}$ which implies $k\mathcal{H} = l\mathcal{H}$.*

**Proof** Consider the following diagram of arrows in $\mathcal{G}$.

$$\begin{tikzcd} \cdot \arrow[r, "k"] \arrow[dr, "l"'] & \cdot \arrow[r, "h''"] \arrow[d, shift right, "h"'] & {} \\ & \bullet \arrow[u, shift right, bend right, "h_1"'] \arrow[r, "h'"'] & {} \end{tikzcd}$$

If $l \in k\mathcal{H}$ then $l = kh$ for some $h \in \mathcal{H}$. Rearranging we get $k = lh^{-1}$. Hence $k \in l\mathcal{H}$ since $h^{-1} \in \mathcal{H}$. Similarly, if $k \in l\mathcal{H}$ then $k = lh_1$ for some $h_1 \in \mathcal{H}$. Therefore $k{h_1}^{-1} = l$ and $l \in k\mathcal{H}$.

Since $lh' = khh'$, $lh' \in l\mathcal{H}$ and $khh' \in k\mathcal{H}$ then $l\mathcal{H} = k\mathcal{H}$. Similarly, since $kh'' = lh_1h''$, $kh'' \in k\mathcal{H}$ and $lh_1h'' \in l\mathcal{H}$ then $k\mathcal{H} = l\mathcal{H}$. $\square$

Since the cosets partition the groupoid and any element of a coset can be written as $lh$ where $l$ is a coset representative and $h$ a subgroupoid element then any element of the groupoid $\mathcal{G}$ can be written uniquely as $lh$.

Similarly we can define right cosets by $\mathcal{H}g := \{hg : h \in \mathcal{H},\ t(h) = s(g)\}$ and choose suitable right transversals. For work on graphs of groupoids in Subsection 2.1.6 we will restrict to the use of left cosets and left transversals.

**Example 1.2.12** *Left Cosets and Transversal*
Suppose we are given the groupoid $\mathcal{G} = C_2 \times \mathcal{I}_3$ (where the group $C_2$ is given by the presentation $\langle x \mid x^2 = 1\rangle$, and the subgroupoid $\mathcal{H} = \{(1,(0,0),(1,(0,1)),(1,(1,0))$ $(1,(1,1)),(1,(2,2),(x,(2,2))\}$. We then have the following left cosets:

$$\begin{array}{ll} (1,(0,0))\mathcal{H} = \{(1,(0,0)),(1,(0,1))\} & (x,(0,0))\mathcal{H} = \{(x,(0,0)),(x,(0,1))\} \\ (1,(1,1))\mathcal{H} = \{(1,(1,0)),(1,(1,1))\} & (x,(1,1))\mathcal{H} = \{(x,(1,0)),(x,(1,1))\} \\ (1,(1,2))\mathcal{H} = \{(1,(1,2)),(1,(1,2))\} & (1,(2,1))\mathcal{H} = \{(1,(2,1)),(1,(2,0))\} \\ (x,(2,1))\mathcal{H} = \{(x,(2,1)),(x,(2,0))\} & (e,(0,2))\mathcal{H} = \{(1,(0,2)),(x,(0,2))\} \\ (1,(2,2))\mathcal{H} = \{(1,(2,2)),(x,(2,2))\} & \end{array}$$

We choose the following left transversal.

$$\begin{aligned} L &= \{(1,(0,0)),(1,(1,1)),(1,(2,2)),(x,(0,0)),(x,(1,1)), \\ &\quad (1,(1,2)),(1,(2,1)),(x,(2,1)),(1,(0,2))\} \end{aligned}$$

which allows elements of $C_2 \times \mathcal{I}_3$ to have a unique decomposition of a transversal and subgroup element. $\Diamond$

We now give an example that illustrates that groupoid cosets do not have necessarily have the same cardinality.

**Example 1.2.13** *Coset Sizes*
Let $\mathcal{G} := S_3 \times \mathcal{I}$ where $S_3$ be the group with presentation $\langle a, b \mid a^3 = b^2 = abab = 1\rangle$. If $\mathcal{H} = \{(1, 1_0), (a, 1_0), (a^2, 1_0), (1, 1_1), (b, 1_1)\}$ then we have the following left cosets:

$$
\begin{aligned}
(1, 1_0)\mathcal{H} &= \{(1, 1_0), (a, 1_0), (a^2, 1_0)\} & (1, \iota^{-1})\mathcal{H} &= \{(1, \iota^{-1}), (a, \iota^{-1}), (a^2, \iota^{-1})\} \\
(b, 1_0)\mathcal{H} &= \{(b, 1_0), (ab, 1_0), (a^2b, 1_0)\} & (b, \iota^{-1})\mathcal{H} &= \{(b, \iota^{-1}), (ab, \iota^{-1}), (a^2b, \iota^{-1})\} \\
(1, 1_1)\mathcal{H} &= \{(1, 1_1), (b, 1_1)\} & (1, \iota)\mathcal{H} &= \{(1, \iota), (b, \iota)\} \\
(a, 1_1)\mathcal{H} &= \{(a, 1_1), (a^2b, 1_1)\} & (a, \iota)\mathcal{H} &= \{(a, \iota), (a^2b, \iota)\} \\
(a^2, 1_1)\mathcal{H} &= \{(a^2, 1_1), (ab, 1_1)\} & (a^2, \iota)\mathcal{H} &= \{(a^2, \iota), (ab, \iota)\}.
\end{aligned}
$$

The cosets are of size 2 and 3. $\Diamond$

If $\mathcal{G}$ is a connected groupoid and $\mathcal{H}$ a totally disconnected subgroupoid of $\mathcal{G}$ with subgroups $H_v \subseteq \mathcal{G}(v)$ and $t(g) = v$ then $|g\mathcal{H}| = \{\mathcal{G} : H_v\}$.

### 1.2.5 Universal Groupoids

The universal groupoid is a special case of a colimit for groupoids and is a method used to obtain a new groupoid from an old groupoid by making identifications on objects and arrows. Free groupoids and free products of groups can be obtained using universal groupoid methods. A key result is that the word problem can be solved for universal groupoids and we use this result to solve the word problem for graphs of groups in Chapter 2.

Given a groupoid $\mathcal{G}$ and an object mapping from $\mathrm{Ob}(\mathcal{G})$ to a set $X$, considered as a groupoid, we can construct a groupoid on $X$ which is generated by $\mathcal{G}$ under the object mapping. We will use this construction in Chapter 2 to construct the universal group of the fundamental groupoid of a graph of groups.

**Definition 1.2.14** Given a groupoid $\mathcal{G}$ with object set $\mathrm{Ob}(\mathcal{G})$, a set $X$ and a mapping $\sigma : \mathrm{Ob}(\mathcal{G}) \to X$, there exists a groupoid $\mathcal{U} = \mathcal{U}_\sigma(\mathcal{G})$, which we call the *universal groupoid*, with object set $X$ and a morphism of groupoids $\sigma^* : \mathcal{G} \to \mathcal{U}$ with object mapping $\sigma$ such that:

1. $\sigma^*(\mathcal{G})$ generates $\mathcal{U}$ as a groupoid with object set $X$, and
2. given any groupoid $\mathcal{H}$ with object set $X$ and any morphism of groupoids $\theta : \mathcal{G} \to \mathcal{H}$ with object mapping $\sigma$, there is a morphism of groupoids $\theta^* : \mathcal{U} \to \mathcal{H}$ where $\mathrm{Ob}(\mathcal{U}) = \mathrm{Ob}(\mathcal{H})$ and $\theta$ is the identity object mapping, with $\sigma^*\theta^* = \theta$.

If $X$ consists of one element we obtain the *universal group* of the groupoid $\mathcal{G}$. Since free groupoids are a special case of universal groupoids (see proof of Corollary 1.2.18) it is clear that the following construction of groupoids words is more general than the groupoid words of subsection 1.2.2.

The following account of constructing an universal groupoid is based on Chapter 10 of [16]. Let $\mathcal{G}$ be a groupoid and $\sigma : \mathrm{Ob}(\mathcal{G}) \to X$ a map. We view $\mathcal{G}$ as a graph $\Gamma\mathcal{G}$ and form the graph $\mathcal{G}^\sigma$ as follows. Let $V(\mathcal{G}^\sigma) = X$, $E(\mathcal{G}^\sigma) = E(\Gamma\mathcal{G})$ and $s, t : E(\mathcal{G}^\sigma) \to V(\mathcal{G}^\sigma)$ are defined by $s(e^\sigma) = \phi s'(e)$ and $t(e^\sigma) = \phi t'(e)$ where $s'$, $t'$ are the source and target maps of $\Gamma\mathcal{G}$. The identity map on $E(\Gamma\mathcal{G})$ and $\sigma$ give a graph map $\mathcal{G} \to \mathcal{G}^\sigma$.

We also have the category of directed paths $P\mathcal{G}^\sigma$ in $\mathcal{G}^\sigma$. If $g$ is an arrow of $\mathcal{G}$ we let $g^\sigma$ denote the corresponding edge of $\mathcal{G}^\sigma$. We note that if $g_1g_2 = g_3$ in $\mathcal{G}$ then $g_1^\sigma g_2^\sigma$ is defined in $P\mathcal{G}^\sigma$ and is a path of length 2 so it cannot be equal to $g_3^\sigma$ which is a path of length 1. We therefore replace $P\mathcal{G}^\sigma$ by a groupoid where the elements $g_1^\sigma g_2^\sigma$ and $g_3^\sigma$ become equal. If $i^\sigma = j$ where $i \in \mathrm{Ob}(\mathcal{G})$ and $j \in X$ then $1_i^\sigma = 1_j$.

Let $p = (g_1^\sigma, g_2^\sigma, \ldots, g_n^\sigma)$ be an arrow of $P\mathcal{G}^\sigma$, that is a path of length $n$ in $\mathcal{G}^\sigma$ from $u = s(g_1^\sigma)$ to $v = t(g_n^\sigma)$, say. If for some $i$ in $1 \leqslant i < n$ the product $g_ig_{i+1}$ is defined in $\mathcal{G}$ and has the value $g$, then $(g_1^\sigma, \ldots, g_{i-1}^\sigma, g^\sigma, g_{i+2}^\sigma, \ldots, g_n^\sigma)$ is a path from $u$ to $v$. Also if some $g_i$ is an identity element of $\mathcal{G}$ then $(g_1^\sigma, \ldots, g_{i-1}^\sigma, g_{i+1}^\sigma, \ldots, g_n^\sigma)$ is a path from $u$ to $v$. If $n = 1$ this is the empty path at $u$. We call these two reduced processes *elementary reductions* of $p$, and we write $p \equiv p'$ if there exists a finite sequence of paths $p = p_0, \ldots, p_k = p'$ $(k \geqslant 0)$ such that for $r = 1, \ldots, k-1$; $p_r$ is an elementary reduction of $p_{r+1}$ or vice versa. We say that a path in $\mathcal{G}^\sigma$ is *$\sigma$-reduced* if it has no elementary reductions. This defines an equivalence relation on $P\mathcal{G}^\sigma$ and we write $[p]$ for the equivalence class containing $p$. Equivalent paths have the same source and target so we can assign these as the source and target of the equivalence class.

The equivalence classes form a graph with vertex set $X$. We denote this graph by $\mathcal{U}_\sigma(\mathcal{G})$. The objects are the elements of $X$ and arrows are equivalence classes. If $p \equiv p'$ and $q \equiv q'$ and $pq$ is defined in $P\mathcal{G}^\sigma$, then $p'q'$ is defined and $pq \equiv p'q'$, so $\mathcal{U}_\sigma(\mathcal{G})$ is a groupoid with multiplication $[p][q] = [pq]$, the identity elements are the classes $[()_u]$ where $()_u$ is the empty path at $u$ and $[\overline{p}]$ is inverse to $[p]$.

We now identify edges of $\mathcal{G}$ with their images in $\mathcal{G}^\sigma$ to simplify notation. A path $p$ in $\mathcal{G}^\sigma$ is either one of the empty paths $()_u$ or is of the form $(g_1, g_2, \ldots, g_n)$ where the $g_i$ are arrows of $\mathcal{G}$ satisfying source and target relations $\sigma(t(g_i)) = \sigma(s(g_{i+1}))$ for $i = 1, \ldots, n-1$.

**Definition 1.2.15** A *$\sigma$-reduced word* is either the empty path $()_u$ for $u \in X$ or a path $(g_1, g_2, \ldots, g_n)$ for $n \geqslant 1$ where the $g_i$ are non-identity edges of $\mathcal{G}$ satisfying $\sigma(t(g_i)) = \sigma(s(g_{i+1}))$ for $i = 1, \ldots, n-1$ but the products $g_i g_{i+1}$ are not defined in $\mathcal{G}$.

$$\begin{tikzcd}
 & t(g_i) \arrow[dr, mapsto, "\sigma"] & & s(g_{i+1}) \arrow[dl, mapsto, "\sigma"'] \arrow[dr, "g_{i+1}"] & \\
s(g_i) \arrow[ur, "g_i"] & & \bullet & & t(g_{i+1})
\end{tikzcd}$$

The following theorem, corollaries and proofs in this subsection appear in Chapter 10 of [16] and are used in Section 2.1.4 of this exposition to obtain normal forms for elements of the fundamental groupoid of a graph of groups.

**Theorem 1.2.16** *Each arrow of $\mathcal{U}_\sigma(\mathcal{G})$ is represented by exactly one $\sigma$-reduced path.*

**Proof** The proof is given on page 74 of [16]. The proof of theorem 2.1.7 on page 35 of this exposition could also be modified to prove the above theorem. $\square$

**Corollary 1.2.17** *If two distinct arrows of $\mathcal{G}$ have the same image in $\mathcal{U}_\sigma(\mathcal{G})$ then they are identity elements at vertices $v$, $v'$ such that $\sigma(v) = \sigma(v')$.* $\square$

**Corollary 1.2.18** *If $\mathcal{G}$ is the free groupoid on a graph $\Gamma$ then*

*(i)* $\mathrm{Ob}(\mathcal{G}) = V(\Gamma)$,

*(ii)* $\Gamma$ *is embedded in* $\mathcal{G}$,

*(iii) every non-identity of $\mathcal{G}$ is uniquely expressible in the form $g_1^{\varepsilon_1} g_2^{\varepsilon_2} \dots g_n^{\varepsilon_n}$ for $n \geqslant 1$ where $g_i$ are edges of $\Gamma$, the $\varepsilon_i$ are $\pm 1$, and no adjacent pairs $gg^{-1}$ or $g^{-1}g$ occur, and*

*(iv) the identities of $\mathcal{G}$ are not expressible in the above form.*

*Conversely, if $\mathcal{G}$ satisfies (i), (iii), and (iv) for a subgraph $\Gamma$, then $\Gamma$ generates $\mathcal{G}$ freely.* $\square$

**Proof** The free groupoid on $\Gamma$ can be constructed as $\mathcal{U}_\sigma(\mathcal{G}')$ where $\mathcal{G}'$ is the disjoint union of unit groupoids, $\mathcal{I}$, one for each edge of $\Gamma$ and $\sigma$ is determined by the source and target maps of $\Gamma$. We denote the non-identity arrows of $\mathcal{I}$ (associated to each edge $g$ of $\Gamma$) by $g$ and $g^{-1}$. Then the only products of non-identity edges which are defined in $\mathcal{G}'$ are the products $gg^{-1}$ and $g^{-1}g$, so the $\sigma$-reduced paths are the empty paths and the paths $g_1^{\varepsilon_1} g_2^{\varepsilon_2} \dots g_n^{\varepsilon_n}$ as described above. The corollary follows from the theorem. $\square$

We use universal groupoids to construct free products of groupoids.

**Example 1.2.19** *Free Product of Groupoids*
Let $i_1 : \mathcal{G}_1 \to \mathcal{H}$ and $i_2 : \mathcal{G}_2 \to \mathcal{H}$ be groupoid morphisms with object maps $\rho_1 : \mathrm{Ob}(\mathcal{G}_1) \to \mathrm{Ob}(\mathcal{H})$ and $\rho_2 : \mathrm{Ob}(\mathcal{G}_2) \to \mathrm{Ob}(\mathcal{H})$. We call $\mathcal{H} = \mathcal{G}_1 * \mathcal{G}_2$ the free product of groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ with respect to the morphisms $i_1$ and $i_2$ if the following universal property holds: for $j_1 : \mathcal{G}_1 \to \mathcal{K}$ and $j_2 : \mathcal{G}_2 \to \mathcal{K}$ whose object maps are of the form $\theta\rho_1$ and $\theta\rho_2$ where $\theta : \mathrm{Ob}(\mathcal{H}) \to \mathrm{Ob}(\mathcal{K})$ there is a unique morphism $\phi : \mathcal{H} \to \mathcal{K}$ such that $\phi i_1 = j_1$ and $\phi i_2 = j_2$.

To construct $\mathcal{H}$ explicitly we take $X = \mathrm{Ob}(\mathcal{G}_1) \cup \mathrm{Ob}(\mathcal{G}_2)$ and let $\mathrm{Ob}(\mathcal{G}_1) \sqcup \mathrm{Ob}(\mathcal{G}_2) \to$

$X$ be the natural map. Then

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "i_1"] \arrow[dr, "j_1"'] & \mathcal{G}_1 \sqcup \mathcal{G}_2 \arrow[d, "\sigma^*"] & \mathcal{G}_2 \arrow[l, "i_2"'] \arrow[dl, "j_2"] \\ & \mathcal{H} & \end{tikzcd}$$

where $i_1$ and $i_2$ are the injections of the coproduct, $\mathcal{H} = \mathcal{U}_\sigma(\mathcal{G}_1 \sqcup \mathcal{G}_2)$, $j_1 = \sigma^* i_1$ and $j_2 = \sigma^* i_2$. $\Diamond$

The above example can be generalised to give the free product of more than two groupoids.

We use the free product in the following proposition and the proof is taken from Brown [4]. We include the proof as it shows how an element of a connected groupoid can be written uniquely as a conjugate of a vertex group element and an element of a spanning tree.

**Proposition 1.2.20** *If $\mathcal{G}$ is a connected groupoid and $T$ a spanning tree in $\mathcal{G}$ then $\mathcal{G} \simeq G * \mathcal{F}$ where $G$ is an arbitrary vertex group and $\mathcal{F}$ is the free groupoid generated by $T$.*

**Proof** Let $i_1 : \mathcal{G}(u) \to \mathcal{G}$ for some $u \in \mathrm{Ob}(\mathcal{G})$ and $i_2 : T \to \mathcal{G}$ be inclusion maps. Every element of $\mathcal{G}(x, y)$ can be written uniquely as $t_y a' t_x^{-1}$ for $a' \in \mathcal{G}(u)$ and $t_x$, $t_y \in T$ where $t_x$ is the unique element of $T(u, x)$.

The morphisms $f_1 : \mathcal{G}(u) \to \mathcal{K}$ and $f_2 : T \to \mathcal{K}$ which agree on $u$ define a morphism $f : \mathcal{G} \to \mathcal{K}$ by $f(a) = f_2(t_y) f_1(a') f_2(t_x^{-1})$ and $f$ is the only morphism such that $fi_1 = f_1$ and $fi_2 = f_2$. $\square$

### 1.2.6 Groupoid Pushouts and Presentations

In this subsection we will show how group constructions such as free products with amalgamation can be constructed using universal groupoids and pushouts.

The following construction is taken from Higgins [16] and shows that coequalisers exist in the category of groupoids. This construction is used to construct pushouts of groupoids.

Let $\theta, \phi : \mathcal{G} \to \mathcal{H}$ be groupoid morphisms with object maps $\theta_{\mathrm{Ob}}, \phi_{\mathrm{Ob}} : \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{H})$, and let $\sigma_{\mathrm{Ob}} : \mathrm{Ob}(\mathcal{G}) \to X$ be the coequaliser of $\theta_{\mathrm{Ob}}$ and $\phi_{\mathrm{Ob}}$ in the category $\mathcal{S}et$. Then $\sigma_{\mathrm{Ob}}$ induces a universal groupoid morphism $\sigma : \mathcal{H} \to \mathcal{H}' = \mathcal{U}_{\sigma_{\mathrm{Ob}}}(\mathcal{H})$, and if $\gamma : \mathcal{H} \to \mathcal{K}$ is any groupoid morphism such that $\gamma\theta = \gamma\phi$, then $\gamma$ has the form $\gamma = \gamma^*\sigma$ where $\gamma^* : \mathcal{U}_{\sigma_{\mathrm{Ob}}}(\mathcal{H}) \to \mathcal{K}$ is the groupoid morphisms uniquely determined by $\gamma$.

The groupoid morphism $\gamma$ is the coequaliser in $\mathcal{G}pd$ of $\theta$ and $\phi$ if and only if $\gamma^*$ is the coequaliser of $\theta' = \sigma\theta$ and $\sigma' = \sigma\phi$; this follows from the definition of a coequaliser.

If $g$ is any arrow of $\mathcal{G}$, the arrows $\theta'(g) = \phi'(g)$ of $\mathcal{H}'$ have the same source and target. For arrows $p$ and $q$ of $\mathcal{H}'$ we write $p \sim q$ whenever $p = h_1(\theta' g)h_2$, $q = h_1(\phi' g)h_2$ for some $g \in \mathcal{G}$ and $h_1,\ h_2 \in \mathcal{H}'$. This relation on $\mathcal{H}'$ generates an equivalence relation $\equiv$ and the equivalence classes form a graph $\Gamma$ with the same vertex set $K$ as $\mathcal{H}'$.

Since $p \equiv q$ implies $hp \equiv hq$, it is clear $p \equiv q$ implies $hp \equiv hq$ and $ph \equiv qh$. Hence $pp' \equiv qp' \equiv qq'$ whenever $pp'$ is defined. Thus $\Gamma$ inherits a groupoid structure from $\mathcal{H}'$ and the canonical map $\pi : \mathcal{H}' \to \Gamma$ is the coequaliser in $\mathcal{G}pd$ of $\theta'$ and $\phi'$. It follows that $\theta'$ and $\phi'$ have coequaliser $\pi\sigma : \mathcal{H} \to \Gamma$. Thus $\mathcal{G}pd$ admits coequalisers. By Proposition 1.1.9 since the category of groupoids admits coproducts and coequalisers, the category of groupoids admits colimits.

The presentation of groupoids by generators and relations is a special case of the above. Since free groupoids are defined on graphs a groupoid presentation is given by a graph and relations.

Suppose we are given a graph $X$, and a set $R$ whose elements are ordered pairs of arrows $(r_1, r_2)$ where $r_1,\ r_2 \in \mathcal{F}(X)$. Let $\mathcal{D}$ be the disjoint union of unit groupoids one for each element of $R$. Then there are unique groupoid morphisms $\theta_1, \theta_2 : \mathcal{D} \to \mathcal{F}(X)$ given by $\theta_1(r_1, r_2) = r_1$ and $\theta_2(r_1, r_2) = r_2$.

If $\phi : \mathcal{F}(X) \to \mathcal{G}$ is the coequaliser in $\mathcal{G}pd$ of $\theta_1,\ \theta_2$ we write $\mathcal{G} = \langle X \mid R \rangle$ and say $\mathcal{G}$ is the groupoid with generators $X$ and relations $r_1 = r_2$ where $(r_1, r_2) \in R$. The triple $(X, R, \phi_X)$ where $\phi : X \to \mathcal{G}$ is the restriction of $\phi$ on $X$ is a *presentation* of $\mathcal{G}$ in the category $\mathcal{G}pd$.

To construct pushouts of groupoids explicitly we use universal groupoids. Consider the following pushout of groupoids:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathcal{A} \arrow[d, "i"] \\ \mathcal{B} \arrow[r, "j"'] & \mathcal{P}. \end{tikzcd}$$

To obtain $\mathcal{P}$ we first form the pushout of object sets to obtain the objects of $\mathcal{P}$.

$$\begin{tikzcd} \mathrm{Ob}(\mathcal{C}) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathrm{Ob}(\mathcal{A}) \arrow[d, "i"] \\ \mathrm{Ob}(\mathcal{B}) \arrow[r, "j"'] & \mathrm{Ob}(\mathcal{P}). \end{tikzcd}$$

We recall from subsection 1.1.2 that a pushout in the category $\mathcal{S}et$ is a set of equivalence classes. Hence

$$\mathrm{Ob}(\mathcal{P}) = (\mathrm{Ob}(\mathcal{A}) \sqcup \mathrm{Ob}(\mathcal{B}))\big/(\alpha(v) \equiv \beta(v))$$

where $v \in \mathrm{Ob}(\mathcal{C})$. We then use the universal groupoid methods to get the following pushout:

$$\begin{tikzcd} \mathrm{Ob}(\mathcal{A}) \sqcup \mathrm{Ob}(\mathcal{B}) \arrow[r, "\sigma"] \arrow[d] & \mathrm{Ob}(\mathcal{P}) \arrow[d] \\ \mathcal{A} \sqcup \mathcal{B} \arrow[r, "\sigma'"'] & \mathcal{U}_\sigma(\mathcal{A} \sqcup \mathcal{B}). \end{tikzcd}$$

The groupoid $\mathcal{U}_\sigma(\mathcal{A} \sqcup \mathcal{B})$ has the correct object set for $\mathcal{P}$ but no identifications have been made on the arrows. The arrows of $\mathcal{P}$ are given by the coequaliser

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\alpha'"] \arrow[r, shift right, "\beta'"'] & \mathcal{U}_\sigma(\mathcal{A} \sqcup \mathcal{B}) \end{tikzcd}$$

as explained above.

We now show how this construction of a pushout of groupoids can be applied to the trefoil group.

**Example 1.2.21** *Trefoil Group Pushout*
The trefoil group can be given by the pushout of groups

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\ B \arrow[r] & T \end{tikzcd}$$

where $C$, $A$ and $B$ are free groups on one generator $c$, $a$ and $b$ respectively and $\alpha(c) = a^3$ and $\beta(c) = b^2$.

We first form the pushout of object sets. Since $C$, $A$ and $B$ are all groups they each have one object which we denote $\cdot_C$, $\cdot_A$ and $\cdot_B$ respectively. Hence $T$ has one object which we denote $\cdot_T$

We then construct the pushout

$$\begin{tikzcd} \{\cdot_A, \cdot_B\} \arrow[r, "\sigma"] \arrow[d, "i"'] & \{\cdot_T\} \arrow[d, "i'"] \\ A \sqcup B \arrow[r, "\sigma'"'] & \mathcal{U}_\sigma(A \sqcup B). \end{tikzcd}$$

Let $\mathcal{U} := \mathcal{U}_\sigma(A \sqcup B)$. The maps $i$ and $i'$ are the identity maps on objects and $\sigma$ maps $\cdot_A$ and $\cdot_B$ to $\cdot_T$. The groupoid map $\sigma'$ is $\sigma$ on objects and $\sigma'(g) = g \in \mathcal{U}(\sigma(sg), \sigma(tg))$. The groupoid $\mathcal{U}$ consists of words of elements which are not composable in $A \sqcup B$. Hence $\mathcal{U} = A * B$ which is the coproduct of $A$ and $B$.

To obtain the pushout of $A \xleftarrow{\alpha} C \xrightarrow{\beta} B$ we need to construct the coequaliser of

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\alpha'"] \arrow[r, shift right, "\beta'"'] & \mathcal{U} \end{tikzcd}$$

where $\alpha' = \sigma'\alpha$ and $\beta' = \sigma'\beta$.

We give some examples of words which are equivalent. From the construction of the coequaliser at the beginning of this subsection arrows $p$ and $q$ of $\mathcal{U}$ are related if $p = h_1\alpha(c)h_2$ and $q = h_1\beta(c)h_2$ for some $c \in C$ and $h_1,\ h_2 \in \mathcal{U}$.

If $c = c^j$, $h_1 = a^i$ and $h_2 = b^k$ then $h_1\alpha(c)h_2 = a^i a^{3j} b^k = a^{i+3j}b^k$ and $h_1\beta(c)h_2 = a^i b^{2j} b^k = a^i b^{2j+k}$. Hence $a^{i+3j}b^k \sim a^i b^{2j+k}$. If $i = 0,\ j = 1$ and $k = -2$ then $a^3b^{-2} \sim e$ where $e$ is the identity element of $\mathcal{U}$.

If $c = c^j$ and $h_1$, $h_2$ are word in $A * B$ then $h_1 a^{3j} h_2 \sim h_1 b^{2j} h_2$. Hence $a^{3j} \sim b^{2j}$ which we expect since the trefoil group has relation $a^3 = b^2$.

The elements of $T$ are words in $A{*}B$ where no $a^3b^{-2}$ occurs. We pick a representative from each class and these representatives are the elements of $T$. $\Diamond$

We now give the general construction of an HNN-extension of groups using pushouts in the category of groupoids.

**Example 1.2.22** *HNN-extension Pushout*

Let $A$ and $B$ be subgroups of a group $C$ and $\phi : A \to B$ be an isomorphism, then $C$ is a subgroup of $C*_\phi$ (the HNN-extension of $C$)with an extra generator $t$ so that $\phi$ is given by conjugation by $t$.

$$C*_\phi = (C * \langle t \rangle) \diagup \{t^{-1}at = \phi a\} \mid a \in A\}$$

This HNN-extension can be obtained by the following pushout of groupoids.

$$\begin{tikzcd} A \sqcup A \arrow[r, "(i,\phi)"] \arrow[d] & C \arrow[d] \\ A \times \mathcal{I} \arrow[r] & C*_\phi \end{tikzcd}$$

# Chapter 2

# Graphs of Groups and Normal Forms

In this chapter graphs of groups, fundamental groupoids and normal forms are considered. This work is based upon Higgins' paper, *The Fundamental Groupoid of a Graph of Groups* [16] which is a modification of the work of Serre in *Trees* [23] on the fundamental group of a graph of groups. Higgins' work establishes a normal form for elements of the fundamental groupoid of a graph of groups.

In the first section we use the material of Higgins and Serre and apply the theory to the motivating examples of this exposition. We also give a different proof of the uniqueness of the normal form using a diamond lemma type argument.

The second section includes Knuth-Bendix methods which are used to formulate a GAP4 [14] program showing how the normal form of elements can be computed.

## 2.1 Fundamental Groupoid of a Graph of Groups

In this section we define a graph of groups for computational purposes, the fundamental groupoid of a graph of groups and hence the fundamental group of a graph of groups.

We show how Higgins' work on the fundamental groupoid gives Serre's classical results on fundamental groups of graphs of groups.

A normal form theorem using the fundamental groupoid of a graph of groups is given and applied to free products with amalgamation, HNN-extensions and a group which combines free products and HNN-extensions.

### 2.1.1 Graph of Groups

Graphs of groups are most widely used in Bass-Serre theory which provides connections between trees and amalgams. For a full account we refer the reader to Serre [23] where Bass-Serre theory first appeared.

Serre shows how a fundamental group of a graph of groups can be obtained by choosing a vertex or a tree in the graph.

Haataja et al [15] in *Bass-Serre Theory for Groupoids and the Structure of Full Regular Semigroup Amalgams* apply Bass-Serre theory to groupoids to obtain results on semigroups. This paper also includes results on normal forms for graphs of groupoids.

Scott and Wall [22] in *Topological Methods in Group Theory* replace graphs of groups by graphs of spaces and relate the two by the fundamental group. We use results from this paper in Chapter 3 which will then apply to graphs of free crossed resolutions and the fundamental groupoid of a graph of free crossed resolutions.

A graph of groups is classically defined in Serre [23], for computational purposes of this chapter we will give an equivalent definition.

**Definition 2.1.1** A *Graph of Groups* $\Gamma_{\mathrm{G}} := (\Gamma, \mathrm{G}, \mathrm{H}, \Phi)$ consists of the following:

1. a graph $\Gamma$ with involution,
2. a family of groups: $\mathrm{G} := \{G_u \mid u \in V(\Gamma)\}$;
3. a family of subgroups $\mathrm{H} := \{H_y \subseteq G_{s(y)} \mid y \in E(\Gamma)\}$; and
4. a family of isomorphisms $\Phi := \{\phi_y : H_y \to H_{\overline{y}} \mid y \in E(\Gamma)\}$ such that ${\phi_y}^{-1} = \phi_{\overline{y}}$.

Since the normal form is established for elements of the fundamental groupoid we aim to construct graphs of groups whose fundamental groupoid contains the group(oid) we are interested in.

We give for completeness the definition of a graph of groups morphism.

**Definition 2.1.2** A *graph of groups morphism* $g : \Gamma_{\mathrm{G}} \to \Gamma'_{\mathrm{G}'}$ is given by a 4-tuple $(\gamma, g, h, \psi)$:

1. $\gamma$ is a graph map which preserves the involution:$\overline{\gamma y} = \gamma \overline{y}$;

2. g is a family of group homomorphisms $g_v : G_v \to G'_{\gamma(v)}$;

3. h is a family of group homomorphisms $h_y : H_y \to H'_{\gamma(y)}$; namely the restrictions $h_y = g_{s(y)}|_{H_y}$

such that the following diagram commutes for each $y \in E(\Gamma)$.

$$\begin{tikzcd}
G_{sy} \arrow[d, "g_{sy}"'] & H_y \arrow[l, "1"'] \arrow[r, "\phi_y"] \arrow[d, "h_y"'] & H_{\overline{y}} \arrow[r, "1"] \arrow[d, "h_{\overline{y}}"] & G_{ty} \arrow[d, "g_{ty}"] \\
G'_{\gamma sy} & H'_{\gamma y} \arrow[l, "1"] \arrow[r, "\phi_{\gamma y}"'] & H'_{\overline{y}} \arrow[r, "1"'] & G_{\gamma ty}
\end{tikzcd}$$

We refer the reader to page 534 of [3] for details of complexes of groups which are a generalisation of graphs of groups, morphisms of complexes of groups and homotopies of complexes of groups morphisms.

### 2.1.2 Fundamental Groupoid

In Chapter 1 we defined the free groupoid of a graph and constructed universal groupoids. We now combine these two types of groupoids to obtain the fundamental groupoid of a graph of groups.

Let $\Gamma_{\mathrm{G}} := (\Gamma, \mathrm{G}, \mathrm{H}, \Phi)$ be a graph of groups and let $\mathcal{F}(\Gamma)$ be the free groupoid on the underlying graph $U\Gamma$ with no involution with the relations $y^{-1} = \overline{y}$ for all $y \in E(\Gamma)$. We use the same letters $y$ and $\overline{y}$ for elements of $E(\Gamma)$ and the corresponding elements of $\mathcal{F}(\Gamma)$.

Let $\overline{\mathrm{G}}$ be the disjoint union of the groups $G_u$ for all $u \in V(\Gamma)$. If we consider $V(\Gamma)$ as a discrete groupoid then we can set $V(\Gamma) = \mathrm{Ob}(\mathcal{F}(\Gamma)) = \mathrm{Ob}(\overline{\mathrm{G}})$.

Let $A(\Gamma_{\mathrm{G}})$ be the free product groupoid $\mathcal{F}(\Gamma) * \overline{\mathrm{G}}$ of the graph of groups amalgamated over $V(\Gamma)$ which is given by the universal groupoid construction where $\rho_1$ :

$V(\Gamma) \to \mathrm{Ob}(\overline{\mathrm{G}})$ and $\rho_2 : V(\Gamma) \to \mathrm{Ob}(\mathcal{F}(\Gamma))$, namely the pushout groupoid

$$\begin{tikzcd}
V(\Gamma) \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] & \overline{\mathrm{G}} \arrow[d] \\
\mathcal{F}(\Gamma) \arrow[r] & A(\Gamma_{\mathrm{G}}).
\end{tikzcd}$$

An element of $A(\Gamma_{\mathrm{G}})$ which we call a *graph of groups word* is represented either by $()_u$ where $u \in V(\Gamma)$ or by a word

$$w = (g_1, y_1, g_2, \ldots, g_n, y_n, g_{n+1})$$

$$\begin{tikzcd}
\bullet \arrow[loop above, "g_1"] \arrow[r, "y_1"] & \bullet \arrow[loop above, "g_2"] \arrow[r, "y_2"] & \bullet \arrow[loop above, "g_3"] \arrow[r, dashed] & \bullet \arrow[loop above, "g_{n-1}"] \arrow[r, "y_{n-1}"] & \bullet \arrow[loop above, "g_n"] \arrow[r, "y_n"] & \bullet \arrow[loop right, "g_{n+1}"]
\end{tikzcd}$$

where $y_i \in E(\Gamma)$, $v_1 = s(y_1)$, $v_i = s(y_i) = t(y_{i-1})$ for $1 < i < n$, $v_{n+1} = t(y_n)$ and $g_i \in G_{v_i}$ for $i = 1, \ldots, n+1$. Such a word is said to be of *type* $p = (y_1, \ldots, y_n)$, where $p$ is a directed path in $\Gamma$, and represents an element of $\mathcal{F}(\Gamma)$. The *length* of a word of type $p$ is $n$. A *subword* of $w = (w_1, \ldots, w_{2n-1})$ is $(w_i, \ldots w_{i+j})$ where $i \geqslant 1$, $j \geqslant 0$ and $i + j \leqslant 2n - 1$. A normal form exists for words of the groupoid $A(\Gamma_{\mathrm{G}})$ given by theorem 1.2.16.

The *fundamental groupoid* $\pi_1(\Gamma_{\mathrm{G}})$ *of a graph of groups* $\Gamma_{\mathrm{G}}$ is the quotient of the groupoid $A(\Gamma_{\mathrm{G}})$ by the relations $\phi_y(h) = \overline{y}hy$ (or equivalently, $hy = y\phi_y(h)$), for all $y \in E(\Gamma)$ and $h \in H_y$.

The $\pi_1$-*value* of a word $w = (g_1, y_1, g_2, \ldots, g_n, y_n, g_{n+1})$ is the element $|w| = g_1 y_1 g_2 \ldots g_n y_n g_{n+1}$ of $\pi_1(\Gamma_{\mathrm{G}})$, where we use the same letters $g_i$ and $y_j$ for elements of $E(\Gamma)$ and G respectively and their projections into $\pi_1(\Gamma_{\mathrm{G}})$.

We now construct a normal form for elements of the fundamental groupoid of a graph of groups. For each edge $y \in E(\Gamma)$ we choose a left transversal $T_y$ of $H_y$ in $G_{s(y)}$, containing the identity element $1_{G_{s(y)}}$ of $G_{s(y)}$. Thus each $g \in G_{s(y)}$ can be written uniquely as $\tau_y(g)h_y(g)$ where $\tau_y(g) \in T_y$, $h_y(g) \in H_y$ and when $g \in H_y$, $\tau_y(g) = 1 \in G_{s(y)}$, $h_y(g) = g$.

The *normal form* of a graph of groups word $w$ is the $\pi_1$-value of the $\pi$-reduced word of $w$.

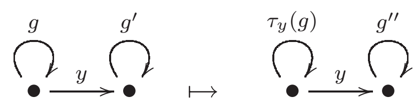
**Definition 2.1.3** A word $w = (g_1, y_1, g_2, \ldots g_n, y_n, g_{n+1})$ is a $\pi$-*reduced word* if

(i) $g_i \in T_{y_i}$ for $i = 1, \ldots, n$ and $g_{n+1} \in G_{t(y_n)}$, and

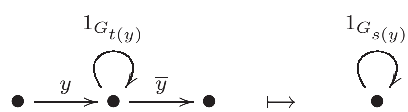
(ii) if $y_i = \overline{y}_{i-1}$ for some $2 \leqslant i \leqslant n$ then $g_i \neq 1_{G_{s(y)}}$.

We define the *reduction* of words by using the relation $\phi_y(h) = \overline{y}hy$. We have two types of reduction: coset and length.

**Definition 2.1.4** If $u = (g, y, g')$ is a subword of a word $w \in A(\Gamma_{\mathrm{G}})$ then the *coset reduction* $(u, w)$ is the word obtained from $w$ by replacing $u$ by $v = (\tau_y(g), y, g'')$ where $g'' := \phi_y(h_y(g))g'$.




**Definition 2.1.5** If $u = (y, 1_{G_{t(y)}}, \overline{y})$ is a subword of a word $w \in A(\Gamma_{\mathrm{G}})$ then the *length reduction* $(u, w)$ is the word obtained from $u$ by replacing $u$ by $1_{G_{s(y)}}$. The length of the word is decreased by two.




Hence a $\pi$-reduced word is a word in which no coset and length reductions can be applied. The reduction process removes subwords $(y, 1, \overline{y})$ of words and moves $h$ in $\tau h$ to the right.

To show that after a finite number of steps reduction will produce a $\pi$-reduced word we outline an algorithm of the process.

**Algorithm 2.1.6** *Reduction algorithm*
Given a word $w = (g_1, y_1, g_2, \ldots g_n, y_n, g_{n+1}) \in A(\Gamma_{\mathrm{G}})$ we relabel the $i$th letter $w_k$.
Set $k := 1$, $len := n + 1$.
If $len = 1$ return $w := (w_1)$.
While $k < len$ do

$c_k := (w_{2k-1}, w_{2k}, w_{2k+1})$

$w :=$ CosetReduction$(c_k, w)$

While $(1 < k < len)$ and $(w_{2k} = 1_{G_{t(y_{2k-2})}})$ and $(w_{2k-2} = w_{2k}^{-1})$ do

$l_k := (w_{2k-2}, w_{2k-1}, w_{2k+1})$

$w :=$ LengthReduction$(l_k, w)$

od

od

return $w$.

In Section 2.2 the algorithm will be used to compute the normal form of a graph of groups word using GAP4.

### 2.1.3 Fundamental Group

In this subsection we give details of Serre's fundamental groups of a graph of groups and how they relate to Higgins' fundamental groupoid of a graph of groups. Serre [23] defines the fundamental group of a graph of groups in two ways, by a vertex and by a tree, whereas Higgins' [16] fundamental groupoid of a graph of groups involves no choices.

We begin by defining Serre's fundamental groups of a graph of groups, then show how these fundamental groups can be obtained from the fundamental groupoid.

Let $F(\Gamma, \mathrm{G})$ be the quotient of the free product of groups $G_u$ and $F(\Gamma)$ the free group on the edge set $E(\Gamma)$ by the normal subgroup generated by elements $y\overline{y}$ and $y\phi_y(a)y^{-1}a$ for all $a \in H_y$ and $y \in E(\Gamma)$. Elements of $G(\Gamma_{\mathrm{G}})$ are words $g_1y_1g_2\ldots g_ny_ng_{n+1}$ where $y_i \in E(\Gamma)$, $g_i \in H_y$, $g_{n+1} \in G_{t(y_n)}$ and $s(y_{i-1}) = t(y_i)$ for $i = 1, \ldots, n$.

The *fundamental group* $\pi_1(\Gamma, \mathrm{G})(u)$ *at a vertex* $u$ is the set of elements of $F(\Gamma, \mathrm{G})$ where $s(y_1) = t(y_n) = u$, this set forms a subgroup of $F(\Gamma, \mathrm{G})$.

The *fundamental group* $\pi_1(\Gamma, \mathrm{G})(T)$ *of a graph of groups at a maximal tree* $T$ *of* $\Gamma$ is the quotient of $F(\Gamma, \mathrm{G})$ by the normal subgroup generated by elements $y$ of $E(T)$. If $g_y$ denotes the image of $y$ in $\pi_1(\Gamma, \mathrm{G})(T)$ the group $\pi_1(\Gamma, \mathrm{G})(T)$ is generated by groups $G_u$ and elements $g_y$ subject to the relations $a = g_y^{-1}\phi_y(a)g_y$, $g_{\overline{y}} = {g_y}^{-1}$ if $y \in E(\Gamma)$, $a \in G_{s(y)}$ and $g_y = 1$ if $y \in E(T)$.

By taking the universal group $\mathcal{U}(\pi_1(\Gamma_{\mathrm{G}}))$ of Higgins' fundamental groupoid $\pi_1(\Gamma_{\mathrm{G}})$

we obtain $F(\Gamma, \mathrm{G})$. The group $\pi_1(\Gamma, \mathrm{G})(u)$ is identified with the image of the vertex group $\pi_1(\Gamma_{\mathrm{G}})(u)$ under the canonical morphism $\pi_1(\Gamma_{\mathrm{G}}) \to \mathcal{U}(\pi_1(\Gamma_{\mathrm{G}}))$. We note that all vertex groups are embedded in $\mathcal{U}(\pi_1(\Gamma_{\mathrm{G}}))$ by this morphism.

By proposition 1.2.20 $\pi_1(\Gamma_{\mathrm{G}})$ is the free product of the vertex group $\pi_1(\Gamma_{\mathrm{G}})(u)$ and the free groupoid generated by $T$. Hence $\mathcal{U}(\pi_1(\Gamma_{\mathrm{G}}))$ is the free product of the isomorphic image of $\pi_1(\Gamma_{\mathrm{G}})(u)$ and a free group generated by the image of $T$.

The group obtained from $\mathcal{U}_\sigma(\pi_1(\Gamma_{\mathrm{G}})$ by adding extra relations to kill the image of $T$ is isomorphic with $\pi_1(\Gamma, \mathrm{G})(u)$.

We will use these relationships between fundamental groups and the fundamental groupoid to state corollaries to theorem 2.1.7.

In this work we will use *the* fundamental groupoid as it involves no choices whereas with fundamental groups we are working with *a* fundamental group given by a vertex or a tree.

### 2.1.4 Normal Form

In this subsection we give an alternative proof of Higgins' normal form theorem [16]. Higgins uses van der Waerden's method in proving the normal form for graph of groups. In this exposition we use Cohen's account of normal forms and the diamond lemma method in *Combinatorial Group Theory*[11] to prove the uniqueness of the normal form. The diamond lemma approach is also used in Knuth-Bendix proofs as shown in subsection 2.2.1.

**Theorem 2.1.7** Normal Form Theorem (P.J. Higgins 1976)
*The $\pi$-reduced form of a word $w$ is unique.*

**Proof** We use a diamond lemma type argument to show that distinct $\pi$-reduced words have distinct $\pi_1$-values. To show this we prove that if two $\pi$-reduced words have the same $\pi_1$-value then they are equal.

We look at two different sequences of reduction for a word $w$. We assume that the first steps in the sequences are different. If the two reductions in the first steps occur to disjoint parts of the word $w$ (the non-overlapping case) it is clear from the diagram below that the resulting words $w'$ and $w''$ can by a sequence of reductions be reduced to a common $\pi$-reduced word $w^*$.

Given a word $w = (\alpha, \beta, \gamma, \delta, \varepsilon)$ where $\beta$, $\delta$ are subwords and $\alpha$, $\gamma$ and $\varepsilon$ are substrings we can reduce $w$ by reduction $r_1$ which reduces $\beta$ to $\beta'$ and by reduction $r_2$ which reduces $\delta$ to $\delta'$. The reductions $r_1$ and $r_2$ are the four combinations of coset and length reductions. We get the following commutative diamond.

$$\begin{tikzcd}
& \boxed{w = (\alpha, \beta, \gamma, \delta, \varepsilon)} \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
\boxed{w' = (\alpha, \beta', \gamma, \delta, \varepsilon)} \arrow[dr, "r_2"'] & & \boxed{w'' = (\alpha, \beta, \gamma, \delta', \varepsilon)} \arrow[dl, "r_1"] \\
& \boxed{w^* = (\alpha, \beta', \gamma, \delta', \varepsilon)} &
\end{tikzcd}$$

Now we consider two reductions in the first steps of the sequence to occur to the same part of a word $w$. When a part of a word can be reduced in two ways we call this an overlap. There are four possible types of overlap.

We denote the coset reduction $(g, y, g') \mapsto (\tau, y, \phi_y(h)g'))$ by $c_y$ and the length reduction $(y, 1_{G_v}, \overline{y}) \mapsto (1_{G_v})$ by $l_y$ in the following diagrams.

(1) If $w = (\alpha, g, y, 1_{G_v}, \overline{y}, \tau h, y', g'\varepsilon)$ we can use coset and length reduction on $w$ to get $w'$ and $w''$ respectively.

$$\begin{tikzcd}
& \boxed{w = (\alpha, g, y, 1_{G_v}, \overline{y}, \tau h, y', g', \varepsilon)} \arrow[dl, "l_y"'] \arrow[dr, "c_{y'}"] & \\
\boxed{w' = (\alpha, g\tau h, y', g', \varepsilon)} \arrow[d, "=_1"'] & & \boxed{w'' = (\alpha, g, y, 1_{G_v}, \overline{y}, \tau, y', \phi_{y'}(h)g', \varepsilon)} \arrow[d, "l_y"] \\
\boxed{(\alpha, \tau''h''h, y', g', \varepsilon)} \arrow[dddr, "c_{y'}"'] & & \boxed{(\alpha, g\tau, y', \phi_{y'}(h)g', \varepsilon)} \arrow[d, "=_1"] \\
& & \boxed{(\alpha, \tau''h'', y', \phi_{y'}(h)g', \varepsilon)} \arrow[d, "c_{y'}"] \\
& & \boxed{(\alpha, \tau'', y', \phi_{y'}(h'')\phi_{y'}(h)g', \varepsilon)} \arrow[dl, "=_2"] \\
& \boxed{w^* = (\alpha, \tau'', y', \phi_{y'}(h''h)g', \varepsilon)} &
\end{tikzcd}$$

(2) If $w = (\alpha, \tau h, y, 1_{G_v}, \overline{y}, g'\varepsilon)$ we can use coset and length reduction on $w$ to get $w'$ and $w''$ respectively.

$$\begin{tikzcd}
 & \boxed{w = (\alpha, \tau h, y, 1_{G_v}, \overline{y}, g', \varepsilon)} \arrow[dl, "c_y"'] \arrow[dr, "l_y"] & \\
\boxed{w' = (\alpha, \tau, y, \phi_y(h), \overline{y}, g', \varepsilon)} \arrow[d, "c_{\overline{y}}"'] & & \boxed{w'' = (\alpha, \tau h g', \varepsilon)} \arrow[ddddl, "="] \\
\boxed{(\alpha, \tau, y, 1_{G_v}, \overline{y}, \phi_{\overline{y}}(\phi_y(h))g', \varepsilon)} \arrow[d, "="'] & & \\
\boxed{(\alpha, \tau, y, 1_{G_v}, \overline{y}, hg', \varepsilon)} \arrow[ddr, "l_y"'] & & \\
 & & \\
 & \boxed{w^* = (\alpha, \tau h g', \varepsilon)} &
\end{tikzcd}$$

(3) If $w = (\alpha, \tau h, y, \tau' h', y', g', \varepsilon)$ we can use two coset reductions to get $w'$ and $w''$.

$$\begin{tikzcd}
 & \boxed{w = (\alpha, \tau h, y, \tau' h', y', g', \varepsilon)} \arrow[dl, "c_y"'] \arrow[dr, "c_{y'}"] & \\
\boxed{w' = (\alpha, \tau, y, \phi_y(h)\tau' h', y', g', \varepsilon)} \arrow[d, "=_1"'] & & \boxed{w'' = (\alpha, \tau h, y, \tau', y', \phi_{y'}(h')g', \varepsilon)} \arrow[d, "c_y"] \\
\boxed{(\alpha, \tau, y, \tau'' h'', y', g', \varepsilon)} \arrow[d, "c_{y'}"'] & & \boxed{(\alpha, \tau, y, \phi_y(h)\tau', y', \phi_{y'}(h')g', \varepsilon)} \arrow[d, "=_2"] \\
\boxed{(\alpha, \tau, y, \tau'', y', \phi_{y'}(h'')g', \varepsilon)} \arrow[ddr, "="'] & & \boxed{(\alpha, \tau, y, \tau''' h''', y', \phi_{y'}(h')g', \varepsilon)} \arrow[d, "c_{y'}"] \\
 & & \boxed{(\alpha, \tau, y, \tau''', y', \phi_{y'}(h''')\phi_{y'}(h')g', \varepsilon)} \arrow[dl, "=_3"] \\
 & \boxed{w^* = (\alpha, \tau, y, \tau'', y', \phi_{y'}(h'')g', \varepsilon)} &
\end{tikzcd}$$

To show that $(\alpha, \tau, y, \tau''', y', \phi_{y'}(h''')\phi_y(h')g', \varepsilon) =_3 w^*$ we need to verify that $\tau'' = \tau'''$ and $\phi_{y'}(h'') = \phi_{y'}(h''')\phi_{y'}(h')$. We use the equalities $=_1$ and $=_2$ to obtain $\phi_y(h)\tau' h' = \tau'' h''$ and $\phi_y(h)\tau' = \tau''' h'''$. Substituting the second equation into the first gives $(\tau''' h''')h' = \tau'' h''$. Hence $\tau''' = \tau''$ and $h''' h' = h''$ which gives the result.

(4) If $w = (\alpha, g, y, 1_{G_v}, \overline{y}, 1_{G_u}, y, g', \varepsilon)$ then we can use two length reductions on the overlapping part to obtain $w'$ and $w''$.

$$\begin{tikzcd}
 & \boxed{w = (\alpha, g, y, 1_{G_v}, \overline{y}, 1_{G_u}, y, g', \varepsilon)} \arrow[dl, "l_y"'] \arrow[dr, "l_{\overline{y}}"] & \\
\boxed{w' = (\alpha, g, 1_{G_u}, y, g'\varepsilon)} \arrow[dr, "="'] & & \boxed{w'' = (\alpha, g, y, g', \varepsilon)} \arrow[dl, "="] \\
 & \boxed{w^* = (\alpha, g, y, g'\varepsilon)} &
\end{tikzcd}$$

Now take two equivalent words $w$ and $w'$ and consider a list $w_1, \ldots, w_n$ such that $w_1 = w$ and $w_n = w'$ and for each $i$, one of $w_{i+1}$ and $w_i$ comes from simple reduction from the other. Suppose there is some $r$ such that $w_{i+1}$ comes by simple reduction from $w_i$ for $i < r$ and $w_i$ comes by simple reduction from both $w_{i+1}$ for $i \geqslant r$. Then $w_r$ comes form both $w$ and $w'$ by reduction.

If there is no such $r$ then there must be some $k$ such that both $w_{k-1}$ and $w_{k+1}$ come from $w_k$ by simple reduction. We then obtain a new list which shows the equivalence of $w$ and $w'$ by deleting $w_k$ and $w_{k+1}$ or by replacing $w_k$ by a word $w^\star$ such that both $w_{k-1}$ and $w_{k+1}$ reduce to $w^\star$.

Since the length of the list decreases there must be a word $w^*$ that comes from $w$ and $w'$ by reduction. $\square$

**Remark 2.1.8** From the sequences of reduction we get relations of reductions which correspond to braid relations. Given $n$-strings from a lower to an upper bar a general $n$-braid is constructed iteratively by applying $b_i$ to $i = 1, \ldots, n-1$. The application of $b_i$ switches the lower endpoints of the $i$th and $(i+1)$th strings keeping their upper endpoints fixed with the $(i+1)$th string brought over the $i$th string. If the $(i+1)$th string passes below the $i$th string it is denoted $b_i^{-1}$. Topological equivalence of a braid word generated by $b_i$ for $i = 1, \ldots, n-1$ is given by the following relations

$$\begin{aligned} b_j b_k &= b_k b_j \text{ for } 1 \leqslant j < k-1 < n-1 \\ b_i b_j b_i &= b_{j+1} b_i b_{j+1} \text{ for } 1 \leqslant j \leqslant n-1. \end{aligned}$$

We can compare these braid relations with the relations of reductions. In case 3 of the above proof we have $c'_y c_y = c'_y c_y c'_y$. Applying $c_y$ to the left hand side does not affect the reduction so we get $c_y c'_y c_y = c'_y c_y c'_y$. From the non-overlapping case we get $c_{y_i} c_{y_j} = c_{y_j} c_{y_i}$. The operation $b_i$ of passing the $i$th string over corresponds to mapping an element $h_i$ "over" $y_i$ to $\phi_{y_i}(h_i)$ by the coset reduction $c_i$ in a graph of groups word.

The scheme of the above proof is similar to methods used in Subsection 2.2.1 on *Knuth-Bendix* procedures which try to create a set of rules to reduce algebraic expressions to a normal form. In many results the set of rules needs to be Noetherian and confluent to obtain a normal form. Noetherian is a finiteness condition similar to the condition that a word can be reduced to a $\pi$-reduced word. Confluent means if there are two rules that can be applied to an algebraic expression it will still lead to the same unique normal form. This can be shown by Newmans' lemma [2] page 176 which is similar to the diamond lemma [11] page 6.

The following corollaries and proofs are taken from Higgins [17].

**Corollary 2.1.9** *The maps $G_u \to \pi_1(\Gamma_{\mathrm{G}})$ are injective.*

**Proof** If $w$ and $w'$ are distinct elements of $G_u$ then they are distinct reduced words. Hence they lie in different equivalence classes. □

**Corollary 2.1.10** *The elements of the fundamental group $\pi_1(\Gamma_{\mathrm{G}})(x_0)$ at $x_0$ are uniquely expressible as values of $\pi$-reduced words $w$ with $s(w) = t(w) = x_0$*

**Proof** This result is a special case of theorem 2.1.7. □

The HNN-extension is such a case and is given below as example 2.1.17.

**Corollary 2.1.11** *Each non-identity element of the universal group $\mathcal{U} = \mathcal{U}(\pi_1(\Gamma_{\mathrm{G}}))$ is uniquely represented by a $\pi$-reduced word.*

**Proof** The universal group $\mathcal{U}$ of $\pi_1(\Gamma_{\mathrm{G}})$ is obtained under the mapping that identifies all the objects of $\pi_1(\Gamma_{\mathrm{G}})$ to a single object. Recall that each edge of a universal groupoid is represented by exactly one $\sigma$-reduced path by theorem 1.2.16 and that if two distinct edges of the groupoid $\mathcal{G}$ have exactly the same image in the universal groupoid $\mathcal{U}_\sigma(\mathcal{G})$ then they are identity elements at vertices $v$ and $v'$ such that $\sigma(v) = \sigma(v')$ by corollary 1.2.17. So combining these two results, if two arrows have the same image under the map $\pi_1(\Gamma_{\mathcal{G}}) \to \mathcal{U}_\sigma(\mathcal{G})$ then they are both identities of $\pi_1(\Gamma_{\mathrm{G}})$ giving the result. $\square$

**Corollary 2.1.12** *The generating set $E(\Gamma) \cup \overline{\mathrm{G}}$ is embedded in the groupoid $\pi_1(\Gamma_{\mathrm{G}})$; hence the set $E(\Gamma) \cup \overline{\mathrm{G}}_*$ is embedded in the group $\mathcal{U} = \mathcal{U}(\pi_1(\Gamma_{\mathrm{G}}))$, where $\overline{\mathrm{G}}_*$ denotes the disjoint union of the vertex groups with all the identity elements removed.*

**Proof** The elements of $E(\Gamma) \cup \overline{\mathrm{G}}$ are distinct $\pi$-reduced words, so represent distinct elements of $\pi_1(\Gamma_{\mathrm{G}})$. The elements of $E(\Gamma) \cup \overline{\mathrm{G}}_*$ remain distinct in $\mathcal{U}$. $\square$

**Corollary 2.1.13** *Let $\Gamma$ be connected and choose a vertex $v$ and a spanning tree $T$. Let $\rho : \pi_1(\Gamma_{\mathrm{G}}) \to \pi_1(\Gamma_{\mathrm{G}})(v)$ be the morphism obtained by killing $T$. Then each $G_u$ for $u \in V(\Gamma)$ is embedded in $\pi_1(\Gamma_{\mathrm{G}})(v)$ by $\rho$.*

**Proof** The group $G_u$ can be identified with a subgroup of the vertex group of $\pi_1(\Gamma_{\mathrm{G}})$ at $u$, by corollary 2.1.12. The morphism $\rho$ is a deformation retraction by theorem 6(ii), page 92 in [16] and therefore maps $G_u$ isomorphically to a conjugate subgroup of the vertex group at $v$. $\square$

For a graph of groups word $w = (g_1, y_1, \ldots, g_n, y_n, g_{n+1})$ to be reduced in the sense of Serre, if $n = 0$ then $g_1 \neq 1$ and if $n > 0$ and $y_i = \overline{y}_{i-1}$ for some $i$ then $g_i \notin H_{y_i}$.

**Corollary 2.1.14** *Let $w$ be any word which is reduced in the sense of Serre [23]. Then the $\pi$-value of $w$ is not an identity arrow in $\pi_1(\Gamma_{\mathrm{G}})$.*

**Proof** We show that the normal form of $w$ is of length exactly $n$ and therefore its value is not an identity.

The case $n = 0$ is trivial, so suppose that $n \geqslant 1$ and define $\tau_1, \ldots \tau_n, h_1, \ldots, h_n$ inductively by the equations

$$\begin{aligned} g_1 &= \tau_1 h_1,\ \tau_1 \in T_{y_1},\ h_1 \in G_{sy_1} \\ h_i g_{i+1} &= \tau_{i+1} h_{i+1},\ \tau_{i+1} \in T_{y_{i+1}},\ h_{i+1} \in G_{s_{y_{i+1}}} \end{aligned}$$

Then $w = (\tau_1, y_1, \ldots, \tau_n, y_n, s)$ where $\tau_i \in T_{y_i}$ and $s \in G_{t(y_n)}$. If $y_i = \overline{y}_{i-1}$ for some $i = 2, 3, \ldots, n$ then $h_{i-1}$ and $h_i$ lie in $H_{y_i}$ that is $\tau_i \neq 1$. This shows that $(\tau_1, y_1, \ldots, \tau_n, y_n, s)$

is a normal word. $\square$

The last corollary contains theorem 11 of Serre [23] and Britton's Lemma see corollary 2.1.15 below.

Let $G^* := \langle G, y \mid y^{-1}ay = \phi(a) \text{ for all } a \in A\rangle$ be an HNN-extension, the letter $g_i$ will denote an element of $G$ and $\varepsilon = \pm 1$. A sequence $(g_0, y^\varepsilon, g_1, \ldots g_n, y^\varepsilon, g_n)$ for $n \geqslant 0$ is said to be reduced if there is no consecutive sequence $(y^{-1}, g_i, y)$ with $g_i \in A$ or $(y, g_j, y^{-1})$ with $g_j \in B$.

**Corollary 2.1.15** Britton's Lemma
*If the sequence* $(g_0 y^\varepsilon g_1 \ldots g_{n-1} y^\varepsilon g_n)$ *is reduced and* $n \geqslant 1$ *then* $(g_0 y^\varepsilon g_1 \ldots g_{n-1} y^\varepsilon g_n) \neq 1$ *in* $G^*$.

**Proof** Britton's Lemma is a special case of corollary 2.1.14 where $\Gamma$ has only one vertex. $\square$

### 2.1.5 Examples

In this subsection we give examples of graphs of groups and normal forms for free products with or without amalgamation, HNN-extensions and groups which combine free products and HNN-extensions. Normal forms already exist for free products and HNN-extensions, we refer the reader to Lyndon and Schupp's *Combinatorial Group Theory* [19].

We aim to show that the fundamental groupoid of a graph of groups is a more powerful method for obtaining normal forms as it allows smaller rewriting processes at the vertices. This process of obtaining a larger process from small processes is an example of local to global methods. In the following example we use graphs of groups to obtain a normal form for the trefoil group, the motivating example of this work.

**Example 2.1.16** *Normal Form for the Trefoil Groupoid*
If $\Gamma := \begin{tikzcd} u \arrow[r, shift left, "y"] & v \arrow[l, shift left, "\overline{y}"] \end{tikzcd}$ we can assign two groups $G_u$ and $G_v$ having subgroups $H_y$ and $H_{\overline{y}}$ respectively and isomorphisms $\phi_y : H_y \to H_{\overline{y}}$ and $\phi_{\overline{y}} : H_{\overline{y}} \to H_y$ to the vertices and edges of $\Gamma$. This graph of groups will provide a model for the free product with amalgamation $G_u *_{\phi_y} G_v$.

For the trefoil groupoid let $G_u := \langle a \rangle$ and $G_v := \langle b \rangle$ having subgroups $H_y := \langle a^3 \rangle$ and $H_{\overline{y}} := \langle b^2 \rangle$. The isomorphisms are given by $\phi_y(a^3) = b^2$ and $\phi_{\overline{y}} = {\phi_y}^{-1}$.

We choose left transversals $T_y = \{1, a, a^2\}$ of $H_y$ and $T_{\overline{y}} = \{1, b\}$ of $H_{\overline{y}}$. To calculate the normal form in the fundamental groupoid we use the relation $(a^3) = (y, \phi_y(a^3), \overline{y})$ to obtain the following reductions: $(a^{3i}, y)$ reduces to $(y, b^{2i})$ and $(b^{2i}, \overline{y})$ reduces to $(\overline{y}, a^{3i})$ by coset reductions and $(\overline{y}, 1_{G_v}, y)$ reduces to $(1_{G_u})$ by length reduction. If we consider the subword $(a^7, y, b^{-5})$ then $a^7 = aa^6$ as a product of a coset and subgroup element. By coset reduction, $(a^7, y, b^{-5})$ reduces to $(a, y, \phi_y(a^6)b^{-5}) = (a, y, b^4, b^{-5}) = (a, y, b^{-1})$.

If $w = (a^7, y, b^{-6}, \overline{y}, a^{-11}, y, b^9, \overline{y}, a^7)$ then using algorithm 2.1.6 we have the following sequence of reductions:

$$\begin{aligned}
w &= (a^7, y, b^{-6}, \overline{y}, a^{-11}, y, b^9, \overline{y}, a^7) && \\
&= (a, y, b^{-2}, \overline{y}, a^{-11}, y, b^9, \overline{y}, a^7) && \text{by coset reduction on } (a^7, y, b^2) \\
&= (a, y, 1, \overline{y}, a^{-14}, y, b^9, \overline{y}, a^7) && \text{by coset reduction } (b^{-2}, \overline{y}, a^{-11}) \\
&= (a^{-13}, y, b^9, \overline{y}, a^7) && \text{by length reduction } (y, 1, \overline{y}) \\
&= (a^2, y, b^{-1}, \overline{y}, a^7) && \text{by coset reduction } (a^{-13}, y, b^9) \\
&= (a^2, y, b, \overline{y}, a^4) && \text{by coset reduction } (b^{-1}, \overline{y}, a^7).
\end{aligned}$$

The result is a $\pi$-reduced word $w = (a^2, y, b, \overline{y}, a^4)$.

In the trefoil groupoid we have two isomorphic vertex groups $\pi_1(\Gamma_{\mathrm{G}})(u)$ and $\pi_1(\Gamma_{\mathrm{G}})(v)$. The first is generated by $(a)$ and $(y, b, \overline{y})$ with $(a^3) = (y, b^2, \overline{y})$. If we relabel $(y, b^2, \overline{y})$ by $(c)$ then we obtain the familiar trefoil group presentation $\langle a, c \mid a^3 = c^2 \rangle$. The normal form $(a^2, y, b, \overline{y}, a^4)$ above becomes $a^2ca^4$ in the trefoil group.

A $\pi$-reduced word $w = (g_1, y_1, \ldots, g_n, y_n, g_{n+1})$ in the trefoil groupoid is reduced if for $i = 1, \ldots, n$, $g_i \in \tau_y$ or $\tau_{\overline{y}}$, $g_{n+1} \in G_{t(y_n)}$ and if $y_i = \overline{y}_{i+1}$ then $g_{i+1} \neq 1$. $\Diamond$

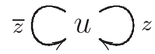
More generally, if we have the free product with amalgamation $\langle a, b \mid a^n = b^m \rangle$, we obtain a graph of groups as defined in the above example but with subgroups $H_y = \langle a^n \rangle$ and $H_{\overline{y}} = \langle b^m \rangle$ and isomorphisms $\phi_y(a^n) = b^m$ and $\phi_{\overline{y}} = \phi_y^{-1}$.

Using the normal form theorem, a normal word in a graph of groups is a $\pi$-reduced word. A $\pi$-reduced word of $\pi(\Gamma_{\mathrm{G}})(u)$ is a word $(\tau_1, y, \ldots, \tau_n, \overline{y}, r)$ where $\tau_i \in T_y$ for $i$ odd, $\tau_i \in T_{\overline{y}}$ for $i$ even, $r \in G_u$ and if $y_i = \overline{y}_i$ for some $i$ then $\tau_i \neq 1$.

Relabelling $(y, \tau_i, \overline{y})$ by $(\tau_i)$ then we obtain a normal form $(\tau_1, \ldots, \tau_n, r)$ where $\tau_i \in T_y$ or $T_{\overline{y}}$, $r \in G_u$ and no $\tau_i$ is in the same transversal as $\tau_{i+1}$.

Hence the normal form of an element of the group $\pi_1(\Gamma_{\mathrm{G}})(u)$ with relabelling for a free product with amalgamation graph of groups corresponds to the normal form theorems for free products with amalgamation as given in Lyndon and Schupp [19].

We now use graph of groups to obtain a normal form for an HNN-extension.

**Example 2.1.17** *Normal Form for an HNN-extension*
Let $\Gamma_{\mathrm{G}}$ be the graph of groups given by the graph $\Gamma :=$ 

and assigning a group $G_u$ and two isomorphic subgroups $H_z$ and $H_{\overline{z}}$ of $G_u$.

Let $G_u := \langle a, b \rangle$ be the free group on two generators having subgroups $H_z := \langle a^3 \rangle$ and $H_{\overline{z}} := \langle b^2 \rangle$. We have isomorphisms $\phi_z(a^3) = b^2$ and $\phi_{\overline{z}} = {\phi_z}^{-1}$.

We choose the left transversal $T_z$ of $H_z$ to contain all freely reduced words on $a$ and $b$ which do not end with a power of $a$ except possibly $a$ and $a^2$ and the left transversal $T_{\overline{z}}$ of $H_{\overline{z}}$ contains all freely reduced words which do not end in powers of $b$ except possibly $b^1$.

To calculate the normal form we use the relation $(z, b^2, \overline{z}) = (a^3)$ to obtain the reductions $(b^{2i}, \overline{z})$ reduces to $(\overline{z}, a^{3i})$ and $(z, a^{3i})$ reduces to $(z, b^{2i})$ by coset reductions.

If $w = (a^3b^2, \overline{z}, aba, z, b^9, \overline{z}, a^3b)$ we have the reductions;

$$\begin{aligned}
w &= (a^3b^2, \overline{z}, aba, z, b^9, \overline{z}, a^3b) \\
&= (a^3.b^2, \overline{z}, aba, z, b^9, \overline{z}, a^3b) && \text{by writing } g_1 \text{ as } \tau_1.h_1 \\
&= (a^3, \overline{z}, a^4ba, z, b^9, \overline{z}, a^3b) && \text{by coset reduction} \\
&= (a^3, \overline{z}, a^4ba, z, b.b^8, \overline{z}, a^3b) && \text{the element } a^4ba \text{ is in } T_z \text{ so move to } g_3 \\
&= (a^3, \overline{z}, a^4ba, z, b, \overline{z}, a^{15}b) && \text{by coset reduction.}
\end{aligned}$$

to obtain the normal word $(a^3, \overline{z}, a^4ba, z, b, \overline{z}, a^{15}b)$.

Since the fundamental groupoid of $\Gamma_{\mathrm{G}}$ for this example has one vertex it is a fundamental group with generators $a$, $b$ and $z$ and relations $\phi_z(h) = \overline{z}hz$ for each $h \in H_z$. If we relabel $\overline{z}$ by $z^{-1}$ then we have a normal form for elements of the group $\langle a, b, z \mid \phi_z(h) = z^{-1}hz \text{ for all } h \in H_z \rangle$ which is a HNN-extension of the group $G_u$. The letter $z$ is called a stable letter in the literature and acts as a conjugator. $\Diamond$

We now use graphs of groups to define free products of groups.

**Example 2.1.18** *Free Product*
Let $\Gamma_{\mathrm{G}}$ be the graph of groups given by the graph $\Gamma$;

$$\Gamma := \ u \underset{\overline{y}}{\overset{y}{\rightleftarrows}} v \underset{\overline{z}}{\overset{z}{\rightleftarrows}} x$$

groups $G_u := \langle a \rangle$, $G_v := \langle b \rangle$ and $G_x := \langle c \rangle$; subgroups the identity subgroups of the respective groups and the isomorphisms are given by the identity maps of the identity elements.

The transversals contain all the freely reduced elements of the respective groups, for example $T_z$ of $H_z$ is $\{b^i \mid i \in \mathbb{Z}\}$.

A $\pi$-reduced word starting and finishing at $v$ has the form

$$(b^{\beta_1}, k_1, b^{\beta_2}, k_2, \ldots, b^{\beta_n}, k_n, b^r)$$

where $k_i = (\overline{y}, a^p, y)$ or $(z, c^q, \overline{z})$, $p, q, \beta_i \in \mathbb{Z}$ and $r \in \mathbb{Z}$.

If we relabel $(\overline{y}, b^p, y)$ by $(b^p)$ and $(z, c^q, \overline{z})$ by $(c^q)$ we get words which are combinations of powers of $a, b$ and $c$. The fundamental group at $v$ with this relabelling is a group with generators $a$, $b$ and $c$ and no relations which is the free product of the groups $G_u$, $G_v$ and $G_x$. Since the groups $G_u$, $G_v$ and $G_x$ are each free on one generator the fundamental group at $v$ is the free group on three generators. $\Diamond$

We now consider a graph of groups for a free product of three groups and amalgamations of their subgroups to obtain the group with presentation $\langle b, m, n \mid b^2 = m^3 = n^5 \rangle$.

**Example 2.1.19** *Free Product with Amalgamation*
Given the graph $\Gamma$, groups $G_u := \langle a \rangle$, $G_v := \langle b \rangle$ and $G_x := \langle c \rangle$.

$$\Gamma := \ u \underset{\overline{y}}{\overset{y}{\rightleftarrows}} v \underset{\overline{z}}{\overset{z}{\rightleftarrows}} x$$

We choose subgroups $H_y := \langle a^3 \rangle$, $H_{\overline{y}} := \langle b^2 \rangle$, $H_z := \langle b^2 \rangle$ and $H_{\overline{z}} := \langle c^5 \rangle$, and isomorphisms $\phi_y(a^3) = b^2$, $\phi_z(b^2) = c^5$ where $\phi_{\overline{y}}$ and $\phi_{\overline{z}}$ are the respective inverse isomorphisms.

We have the following left transversals:

$$\begin{aligned} T_y &= \{1, a, a^2\} \qquad & T_{\overline{y}} &= \{1, b\} \\ T_z &= \{1, b\} \qquad & T_{\overline{z}} &= \{1, c^2, c^3, c^4\}. \end{aligned}$$

The reduction relations are $(y, b^2, \overline{y}) = (a^3)$ and $(z, c^5, \overline{z}) = (b^2)$. If we consider the group $\pi_1(\Gamma_{\mathrm{G}})(v)$ and relabel $(\overline{y}, a, y)$ by $m$ and $(z, c, \overline{z})$ by $n$ we obtain the group with presentation $\langle b, m, n \mid b^2 = m^3 = n^5 \rangle$. $\Diamond$

We now combine graphs of groups of free products with amalgamation and HNN-extensions to construct a group which we will call a "mixed amalgam."

**Example 2.1.20** *Mixed Amalgam*
Consider the graph $\Gamma$ which is a combination of the graphs given in examples 2.1.16 and 2.1.17.

$$\begin{tikzcd} \Gamma := u \arrow[loop above, "z"] \arrow[loop below, "\overline{z}"] \arrow[r, shift left, "y"] & v \arrow[l, shift left, "\overline{y}"] \end{tikzcd}$$

Let the graph of groups have graph $\Gamma$ as above and groups $G_u := \langle a \rangle$ and $G_v := \langle b \rangle$. For the pair of edges $\{y, \overline{y}\}$ we have the same subgroups and isomorphisms as for the trefoil groupoid; let $H_y := \langle a^3 \rangle$, $H_{\overline{y}} := \langle b^2 \rangle$ with $\phi_y(a^3) = b^2$ and $\phi_{\overline{y}} = \phi_y^{-1}$. For the pair of edges $\{z, \overline{z}\}$ we associate subgroups $H_z := \langle a^7 \rangle$, $H_{\overline{z}} := \langle a^5 \rangle$ and isomorphisms $\phi_z(a^7) = a^5$ and $\phi_{\overline{z}} = \phi_z^{-1}$.

We choose left transversals

$$\begin{aligned} T_y &= \{1, a^{-1}, a^{-2}\} \qquad & T_{\overline{y}} &= \{1, b^{-1}\} \\ T_z &= \{1, a^{-1}, a^{-2}, a^{-3}, a^{-4}, a^{-5}, a^{-6}\} \qquad & T_{\overline{z}} &= \{1, a^{-1}, a^{-2}, a^{-3}, a^{-4}\}. \end{aligned}$$

To calculate the normal form of elements we use the following relations $(a^3) = (y, \phi_y(a^3), \overline{y})$ and $(a^7) = (z, \phi_z(a^7), \overline{z})$ to obtain the reductions $(a^{3i}, y) = (y, b^{2i})$ and $(a^{7i}, z) = (z, a^{5i})$ by coset reductions.

If $w = (a^5, y, b^4, \overline{y}, a^2, z, a^4, \overline{z}, a, y, b, \overline{y}, a)$ then by applying the reduction algorithm we get the following sequence of reductions.

$$\begin{aligned}
w &= (a^{-1}, y, b^8, \overline{y}, a^2, z, a^4, \overline{z}, a, y, b, \overline{y}, a) && \text{by } (a^5, y) = (a^{-1}, y, \phi_y(a^6))\\
&= (a^{-1}, y, 1, \overline{y}, a^{14}, z, a^4, \overline{z}, a, y, b, \overline{y}, a) && \text{by } (b^8, \overline{y}) = (1, \overline{y}, \phi_{\overline{y}}(b^8))\\
&= (a^{13}, z, a^4, \overline{z}, a, y, b, \overline{y}, a) && \text{since } (y, 1, \overline{y}) = (1)\\
&= (a^{-1}, z, a^{14}, \overline{z}, a, y, b, \overline{y}, a) && \text{by } (a^{13}, z) = (a^{-1}, z, \phi_z(a^{14}))\\
&= (a^{-1}, z, a^{-1}, \overline{z}, a^{22}, y, b, \overline{y}, a) && \text{by } (a^{14}, \overline{z}) = (a^{-1}, \overline{z}, \phi_{\overline{z}}(a^{15}))\\
&= (a^{-1}, z, a^{-1}, \overline{z}, a^{-2}, y, b^{17}, \overline{y}, a) && \text{by } (a^{22}, y) = (a^{-2}, y, \phi_y(a^{24}))\\
&= (a^{-1}, z, a^{-1}, \overline{z}, a^{-2}, y, b^{-1}, \overline{y}a^{28}) && \text{by } (b^{17}, \overline{y}) = (b^{-1}, \overline{y}, \phi_{\overline{y}}(b^{18})).
\end{aligned}$$

If we relabel $(y, b, \overline{y})$ by $m$ and $\overline{z}$ by $z^{-1}$ we obtain the group, $\pi_1(\Gamma_{\mathrm{G}})(u)$ with presentation

$$\langle a, m, z \mid a^3 = m^2,\ a^7 = za^5z^{-1}\rangle.$$

In subsection 2.2.2 we give output from a GAP4 session for the above group. $\Diamond$

**Example 2.1.21** *Identity Graphs of Groups*

If we choose graphs of groups where the groups, subgroups and isomorphisms are all identity groups and identity morphisms then we get the following four particular cases.

1. When $\Gamma :=\ u \underset{\overline{y}}{\overset{y}{\rightleftarrows}} v$ then graph of groups words have the following forms

$$\begin{aligned}
(1, y, 1, \overline{y}, 1, \ldots, 1, y, 1, \overline{y}, 1) &\in A(\Gamma_{\mathrm{G}})(u)\\
(1, y, 1, \overline{y}, 1, \ldots, 1, \overline{y}, 1, y, 1) &\in A(\Gamma_{\mathrm{G}})(u, v)\\
(1, \overline{y}, 1, y, 1, \ldots, 1, y, 1, \overline{y}, 1) &\in A(\Gamma_{\mathrm{G}})(u, v)\\
(1, \overline{y}, 1, y, 1, \ldots, 1, \overline{y}, 1, y, 1) &\in A(\Gamma_{\mathrm{G}})(v)
\end{aligned}$$

   Since we have length reductions $(y, 1, \overline{y}) = (1)$ and $(\overline{y}, 1, y) = (1)$, elements of the fundamental groupoid of a graph of groups have the following forms $\{1_u, 1_u y 1_v, 1_v \overline{y} 1_u, 1_v\}$. Hence the fundamental groupoid is isomorphic to the unit groupoid.

2. When $\Gamma :=\ {}_{\overline{z}}\circlearrowleft u \circlearrowright_{z}$ then graph of groups words have the form

$$(1, z^\varepsilon, 1, z^\varepsilon, 1, \ldots, 1, z^\varepsilon, 1, z^\varepsilon, 1) \in A(\Gamma_{\mathrm{G}})(u) \text{ where } \varepsilon = \pm.$$

We have length reductions $(z, 1, z^{-1}) = (1)$ and $(z^{-1}, 1, z) = (1)$, so elements of the fundamental groupoid are $\{1, 1z1z \ldots 1z1, 1z^{-1}1z^{-1} \ldots 1z^{-1}1\}$. The fundamental groupoid is isomorphic to the free group on one generator.

3. If $\Gamma$ has one vertex and $2n$ loops $\{z_i, z_i^{-1} \mid 1 \leqslant i \leqslant n\}$ then the fundamental groupoid of the graph of groups is isomorphic to the free group on $n$ generators.

4. Let $T$ be an undirected tree and let $\Gamma$ be obtained from $T$ by adding an identity loop at each vertex and replacing each edge by a pair of directed edges $\{y_i, \overline{y}_i\}$. The fundamental groupoid of the graph of groups is the connected groupoid on $V(\Gamma)$ with trivial object groups. $\Diamond$

The methods used in the examples above can be adapted easily to obtain normal forms for groups obtained from graphs with more edges and loops and normal forms can be computed in GAP4.

### 2.1.6 Graph of Groupoids

In this subsection we define graphs of groupoids, and graphs of groupoids words. We then give a normal form theorem for graphs of groupoids words and an example of a groupoid HNN-extension as a graph of groupoids.

Haataja et al [15] study an analogue of Bass-Serre theory for groupoids. In their paper a normal form theorem is given for a free product of groupoids with amalgamation. The normal form theorem we give in this subsection is more general and includes the free product with amalgamation as a special case.

A *Graph of Groupoids* $\Gamma_{\mathfrak{G}} := (\Gamma, \mathfrak{G}, \mathfrak{H}, \Phi)$ consists of the following: a directed graph $\Gamma$ with involution; a family of groupoids $\mathfrak{G} = \{\mathcal{G}_u \mid u \in V(\Gamma)\}$; a family of wide subgroupoids $\mathfrak{H} := \{\mathcal{H}_y \subseteq \mathcal{G}_{s(y)} \mid y \in E(\Gamma)\}$; and a family of isomorphisms $\Phi := \{\phi_y : \mathcal{H}_y \to \mathcal{H}_{\overline{y}} \mid y \in E(\Gamma)\}$ such that ${\phi_y}^{-1} = \phi_{\overline{y}}$.

We impose the condition that the subgroupoids are wide so we can write the groupoid elements as products of a transversal and subgroupoid element.

For each edge $y \in E(\Gamma)$ we choose a left transversal $\mathcal{T}_y$ of $\mathcal{H}_y$ in $\mathcal{G}_{s(y)}$, containing the identity elements of $\mathcal{G}_{s(y)}$. Thus each $g \in \mathcal{G}_{s(y)}$ can be written uniquely as $\tau_y(g)h_y(g)$ where $\tau_y(g) \in \mathcal{T}_y$, $h_y(g) \in \mathcal{H}_y$.

We adapt the ideas for graphs of groups and let $\Gamma_{\mathfrak{G}}$ be a graph of groupoids and $\mathcal{F}(\Gamma)$ be the free groupoid on the graph $\Gamma$ as defined for graph of groups. Let $\overline{\mathfrak{G}}$ be the disjoint union of the groupoids $\mathcal{G}_u$ for all $u \in V(\Gamma)$. Let $\mathcal{A}(\Gamma_{\mathfrak{G}})$ be the free product $\mathcal{F}(\Gamma) * \overline{\mathfrak{G}}$ of groupoids. The normal form follows by adapting the methods of graphs of groups in Subsection 2.1.2.

An element of $\mathcal{A}(\Gamma_{\mathfrak{G}})$ called a *graph of groupoids word* is represented either by $()_u$ where $u \in V(\Gamma)$ or by

$$w = (g_1, y_1, g_2, \ldots, g_n, y_n, g_{n+1})$$

where $y_i \in E(\Gamma)$, $v_1 = s(y_1)$, $v_i = s(y_i) = t(y_{i-1})$ for $1 < i < n$, $v_{n+1} = t(y_n)$ and $g_i \in \mathcal{G}_{v_i}$ for $i = 1, \ldots, n+1$ and subwords $(g_i, y_i, g_{i+1}) = (\tau_i h_i, y_i, \tau_{i+1} h_{i+1})$ must satisfy $t(\phi_{y_i}(h_i)) = s(\tau_{i+1})$ for $i = 1, \ldots n$.

**Definition 2.1.22** A graph of groupoids word $w = (g_1, y_1, g_2, \ldots g_n, y_n, g_{n+1})$ is $\pi$-*reduced word* if

(i) $g_i \in \mathcal{T}_{y_i}$ for $i = 1, \ldots, n$ and $g_{n+1} \in \mathcal{G}_{t(y_n)}$;

(ii) if $y_{i-1} = \overline{y}_i$ for some $2 \leqslant i \leqslant n$ then $g_i \neq 1$.

We define the *reduction* of words using the relation $\phi_y(h) = \overline{y} h y$ and have two types of reduction: coset and length as for graphs of groups.

Adapting the proof of Theorem 2.1.7 to groupoids we obtain the following result.

**Theorem 2.1.23** *The $\pi$-reduced form of a graph of groupoids word is unique.* $\square$

We now give an example of a groupoid HNN-extension.

**Example 2.1.24** *Groupoid HNN-extension*
Let an HNN-extension graph of groupoids be given by the graph $\Gamma := {}_{\overline{z}}\circlearrowleft u \circlearrowright_{z}$. We assign the groupoid $G_u := S_3 \times \mathcal{I}$ where $S_3$ has presentation $\langle a, b \mid a^3 = b^2 = abab = 1 \rangle$ to the vertex $u$, and subgroupoids $\mathcal{H}_z$ and $\mathcal{H}_{\overline{z}}$ to the edges $z$ and $\overline{z}$. The subgroupoid $\mathcal{H}_z$ has elements $\{(1, 1_0), (a, 1_0), (a^2, 1_0), (1, 1_1), (b, 1_1)\}$ and $\mathcal{H}_{\overline{z}}$ is an isomorphic subgroupoid to $\mathcal{H}_z$ with elements $\{(1, 1_0), (b, 1_0), (1, 1_1), (a, 1_1), (a^2, 1_1)\}$. The isomorphism $\phi_z$ is defined by

$$\phi_z(g, 1_i) = \begin{cases} (g, 1_1) & \text{if } i = 0 \\ (g, 1_0) & \text{if } i = 1 \end{cases}$$

and $\phi_{\overline{z}} = \phi_z^{-1}$.

We choose left transversals $\mathcal{T}_z$ and $\mathcal{T}_{\overline{z}}$ in $\mathcal{H}_z$ and $\mathcal{H}_{\overline{z}}$ respectively.

$$\begin{aligned}
\mathcal{T}_z &:= \{(1,1_0),(1,\iota^{-1}),(b,1_0),(b,\iota^{-1}),(1,1_1),(a,1_1),(a^2,1_1),(1,\iota),(a,\iota),(a^2,\iota)\} \\
\mathcal{T}_{\overline{z}} &:= \{(1,1_0),(1,\iota^{-1}),(a,1_0),(a,\iota^{-1}),(a^2,1_0),(a^2,\iota^{-1}),(1,1_1),(b,1_1),(1,\iota),(b,\iota)\}
\end{aligned}$$

If we apply the reduction algorithm to the graph of groupoids word $w := ((1,1_1), z, (b,1_0), \overline{z}, (a,\iota^{-1})$ then we get the following sequence of reductions:

$$\begin{aligned}
w &= (1,1_1), z, (1,1_0)(b,1_0), \overline{z}, (a,\iota^{-1}), z, (a,1_1) && \text{by } (b,1_0) = (1,1_0)(b,1_0) \\
&= (1,1_1), z, (1,1_0), \overline{z}, (b,1_1)(a,\iota^{-1}), z, (a,1_1) && \text{by } \phi_z((b,1_0)) = (b,1_1) \\
&= (1,1_1), z, (1,1_0), \overline{z}, (ba,\iota^{-1}), z, (a,1_1) && \text{by } (b,1_0)(a,\iota^{-1}) = (ba,\iota^{-1}) \\
&= (ba,\iota^{-1}), z, (a,1_1) && \text{by } (z,(1,1_0),\overline{z}) = (1,1_1) \\
&= (a^2,\iota^{-1})(b,1_0), z, (a,1_1) && \text{by } (ba,\iota^{-1}) = (a^2,\iota^{-1})(b,1_0) \\
&= (a^2,\iota^{-1}), z, (b,1_1)(a,1_1) && \text{by } \phi_z((b,1_0)) = (b,1_1) \\
&= (a^2,\iota^{-1}), z, (a^2 b,1_1) && \text{by } (b,1_1)(a,1_1) = (a^2 b,1_1)
\end{aligned}$$

to obtain the reduced graph of groupoids word $(a^2,\iota^{-1}), z, (a^2 b, 1_1)$.

This HNN-groupoid $\mathcal{HNN}$ has groupoid presentation

$$\mathcal{HNN} = \langle (a,1_0),(b,1_0),(1,\iota) \mid z\phi_z(h)\overline{z} = h \ : \ \text{for } h \in H_{\overline{z}} \rangle$$

with three generators and five relations. $\Diamond$

## 2.2 Implementation

In this section we give details of the implementation of graphs of groups and the reduction process of a graph of groups word in GAP4.

To obtain a function that returns a unique normal form for a graph of groups word we need: a normal form function for elements of the groups associated to the vertices of the graph of the graph of groups; a representation for the structure of a graph of groups and the reduction process for a graph of groups word.

### 2.2.1 Normal Form and Knuth Bendix Methods

The normal form of a group element can be obtained using Knuth-Bendix methods which is a procedure that attempts to obtain a set of rules to reduce elements to a normal form. For more details refer to Epstein et al *Word Processing in Groups* [12] and Becker and Weispfenning *Gröbner Bases* [2].

We give details here as the Knuth-Bendix methods model the scheme of the proof of theorem 2.1.7.

For computational purposes it is more convenient to use semigroups. Given a group $G$ with presentation $\langle A \mid R\rangle$ we can obtain a semigroup presentation $\langle A' \mid R'\rangle$ of $G$. The set of semigroup generators $A'$ is obtained from $A$ by adding formal inverses. The set of semigroup relations $R'$ contains pairs $(\varepsilon, w)$ where $w \in R$ is a word of $A$ and pairs $(\varepsilon, aa^{-1})$ and $(\varepsilon, a^{-1}a)$ for all $a \in A$.

A group presentation is given by a quotient of a free group by a normal subgroup. For semigroup presentations we have free semigroups factored by congruences.

Let $S$ be a semigroup and $R$ a symmetric relation on $S$. The quotient of $S$ by $R$ is the set of equivalence classes of $S$ under the equivalence relation generated by the equivalences $s_1 a s_2 \sim s_1 b s_2$ for all $a, b, s_1, s_2 \in S$ with $(a, b) \in R$.

To obtain unique normal forms we use ordering of elements. There are many different orders that can be used, the most common is *length-lexicographic* order. Lexicographic order ranks the set $A^*$ of strings over the set $A'$ by comparing the letters in the first position where the strings differ. In length-lexicographic order $w < w'$ if and only if $w$ is shorter than $w'$ or they both have the same length and $w$ comes before $w'$ in lexicographic order.

We use ordering to determine reduction rules. Given a set $A'$, a *reduction rule* over $A'$ is denoted $l \to r$ where $l$ and $r$ are strings over $A$ and $l > r$ in the length-lexicographic order. We call $l$ and $r$ the left and right hand sides of the rule.

Let $\mathfrak{R}$ be a set of reduction rules over $A'$. Let $a$ and $b$ be strings over $A'$. We write $a \xrightarrow[\mathfrak{R}]{} b$ if there exists strings $w_1$ and $w_2$ over $A$ and a rule $l \to r$ in $\mathfrak{R}$ such that $a = w_1 l w_2$ and $b = w_1 r w_2$.

We let $\xrightarrow[\mathfrak{R}]{t}$ denote the transitive closure ($a \xrightarrow[\mathfrak{R}]{t} b$ if and only if $b$ can be obtained from $a$ by repeated application of rules in $\mathfrak{R}$) of $\xrightarrow[\mathfrak{R}]{}$. Let $\xleftrightarrow[\mathfrak{R}]{*}$ denote the reflexive, symmetric and transitive closure of $\mathfrak{R}$. The set of equivalence classes can be identified with the semigroup $\langle A' \mid \mathfrak{R}\rangle$ where $\mathfrak{R}$ is the set of unordered pairs $(l, r)$. The set $\mathfrak{R}$ is a *rewrite system* for the group $G$ on $A$ if $G \simeq A'/ \underset{\mathfrak{R}}{\overset{*}{\longleftrightarrow}}$ .

All sequences of reduction in length-lexicographic order terminate. A string $w$ is either irreducible or can be reduced to some irreducible string $w'$ which cannot be reduced further. We say that $w'$ is a $\mathfrak{R}$-*residue* of $w$. The set of irreducible strings under $\mathfrak{R}$ is denoted $Irr_{\mathfrak{R}}$.

A set of reduction rules $\mathfrak{R}$ is *complete* if all strings have unique $\mathfrak{R}$-residues and $k$-complete if all strings of length at most $k$ have unique $\mathfrak{R}$-residues. A necessary and sufficient condition for completeness is that if $u \to v$ and $u \to v'$ then there is a $w$ such that $v \underset{\mathfrak{R}}{\overset{t.r}{\longleftrightarrow}}$ and $v' \underset{\mathfrak{R}}{\overset{t.r}{\longleftrightarrow}} w$ where $\underset{\mathfrak{R}}{\overset{t.r}{\longrightarrow}}$ denotes the transitive, reflexive closure of $\underset{\mathfrak{R}}{\longrightarrow}$ . This is called *confluence* of the rules.

**Lemma 2.2.1** *If $\mathfrak{R}$ is a complete set of rules, the set $Irr_{\mathfrak{R}}$ is a semigroup with multiplication defined by concatenation followed by taking the residues. The map $Irr_{\mathfrak{R}} \to (A'/\mathfrak{R})$ is a semigroup isomorphism.*

**Proof** We refer the reader to the proof of lemma 6.2.1 in [12]. □

The confluence of the rules ensures that the normal form of a word $w$ is the same even if there are two different reduction rules that can be applied.

The proof of the following lemma is taken from Lemma 6.2.4 [12] and uses a diamond lemma type argument. We note the similarities to the proof of theorem 2.1.7.

**Lemma 2.2.2** *Let $k$ be a positive integer or infinity, and let $\mathfrak{R}$ be a set of rules over $A'$. Then $\mathfrak{R}$ is $k$-complete if and only if, for all $a, b, c \in A^*$ with $|abc| \leqslant k$ (where $|abc|$ denotes the length of the string $abc$) the following conditions are satisfied:*

*(i) Suppose that $b \neq \varepsilon$ and that $ab \longrightarrow a'$ and $bc \longrightarrow c'$ are reduction rules of $\mathfrak{R}$. Then there exists a string $s$ and reductions $abc \longrightarrow ac' \underset{\mathfrak{R}}{\overset{t.r}{\longrightarrow}} s$ and $abc \longrightarrow a'c \underset{\mathfrak{R}}{\overset{t.r}{\longrightarrow}} s$ .*

*(ii) Let $b \longrightarrow b'$ and $abc \longrightarrow b''$ be reduction rules of $\mathfrak{R}$. Then there exists a string $t$ and reductions $abc \longrightarrow ab'c \underset{\mathfrak{R}}{\overset{t.r}{\longrightarrow}} t$ and $abc \longrightarrow b'' \underset{\mathfrak{R}}{\overset{t.r}{\longrightarrow}} t$.*

**Proof** The two conditions are implied by $k$-completeness. Conversely, if the two conditions are satisfied, we prove by induction on the ordering that each string has a unique residue.

We look at two different sequences of reduction. We assume that the first steps in the sequences of reduction are different. If the two reductions in the first steps occur to disjoint parts of the word $w$ it is clear that the resulting word words $w'$ and $w''$ can be reduced to a common word $s$.

$$\begin{tikzcd}
 & \boxed{w = \alpha a\beta c\varepsilon} \arrow[dl, "a\to a'"'] \arrow[dr, "c\to c'"] & \\
\boxed{w' = \alpha a'\beta c\varepsilon} \arrow[dr, "c\to c'"'] & & \boxed{w'' = \alpha a\beta c'\varepsilon} \arrow[dl, "a\to a'"] \\
 & \boxed{w^* = \alpha a'\beta c'\varepsilon} &
\end{tikzcd}$$

The word $w*$ is either irreducible or can be reduced to an irreducible word.

There are two overlapping cases where two different reductions can be applied to the same part of a word. The first overlapping case uses the first condition of the lemma.

$$\begin{tikzcd}
 & \boxed{w = \alpha abc\varepsilon} \arrow[dl, "ab\to a'"'] \arrow[dr, "bc\to c'"] & \\
\boxed{w' = \alpha a'c\varepsilon} \arrow[dr, "a'c\to t"'] & & \boxed{w'' = \alpha ac'\varepsilon} \arrow[dl, "ac'\to t"] \\
 & \boxed{w^* = \alpha t\varepsilon} &
\end{tikzcd}$$

The second overlap case uses the reduction rules of the second condition.

$$\begin{tikzcd}
 & \boxed{w = \alpha abc\varepsilon} \arrow[dl, "b\to b'"'] \arrow[dr, "abc\to b''"] & \\
\boxed{w' = \alpha ab'c\varepsilon} \arrow[dr, "ab'c\to t"'] & & \boxed{w'' = \alpha b''\varepsilon} \arrow[dl, "b''\to t"] \\
 & \boxed{w^* = \alpha t\varepsilon} &
\end{tikzcd}$$

Again both words are either irreducible or can be reduced to an irreducible word by repeated application of reduction rules. □

The lemma checks that all pairs of reduction rules and their overlaps will reduce to irreducible strings $s$ and $s'$. If the strings are the same then $\mathfrak{R}$ is complete. If not then $\mathfrak{R}$ is not complete but the lemma gives a method to make $\mathfrak{R}$ complete. If $s$ and $s'$ as described above are not the same irreducible word then we adjoin to $\mathfrak{R}$ the rule $s \to s'$ or $s' \to s$ depending on the length-lexicographic order. We keep repeating this process until there are no non-confluent rules.

We can now give the Knuth-Bendix algorithm. For full details refer to [12].

**Algorithm 2.2.3** *Knuth-Bendix Algorithm*
Let $S$ be a semigroup and $A'$ a finite ordered set of generators. Given a finite set $\mathfrak{R}_0$ of reduction rules for $S$, we construct a sequence of finite sets of rules $\mathfrak{R}_i$ for each $i \geqslant 0$ by induction. To obtain $\mathfrak{R}_{i+1}$ from $\mathfrak{R}_i$ we add rules to make up for failures of the two confluence conditions of lemma 2.2.2.

The algorithm can be refined more by omitting *redundant rules* which are rules that are reductions of rules.

All the results in this section hold for semigroups and groups. We now give an example.

**Example 2.2.4** *Normal Form on $S_3$ using Knuth-Bendix*
Given the symmetric group $S_3$ on three symbols with presentation $\langle x, y \mid x^3 = y^2 = xyxy = 1\rangle$ we have the following relations:

$$\mathfrak{R}_0 := \{x^3 \to 1,\ y^2 \to 1,\ xyxy \to 1\}.$$

Now we look for overlaps with these rules. The dashed arrows denote the reduction rule determined by the length-lexicographic ordering.

$$\begin{tikzcd} & \boxed{xxxyxy} \arrow[dl, "x^3\to 1"'] \arrow[dr, "xyxy\to 1"] & \\ \boxed{yxy} \arrow[rr, dashed] & & \boxed{x^2} \end{tikzcd} \qquad \begin{tikzcd} & \boxed{xyxyy} \arrow[dl, "xyxy\to 1"'] \arrow[dr, "y^2\to 1"] & \\ \boxed{y} & & \boxed{xyx} \arrow[ll, dashed] \end{tikzcd}$$

We add these rules to $\mathfrak{R}_0$

$$\mathfrak{R}_1 := \{x^3 \to 1,\ y^2 \to 1,\ xyxy \to 1,\ yxy \to x^2,\ xyx \to y\}$$

and repeat the process of checking for overlapping cases.

$$
\begin{tikzcd}
 & \boxed{yxyy} \arrow[dl, "yxy\to x^2"'] \arrow[dr, "y^2\to 1"] & \\
\boxed{x^2y} \arrow[rr, dashed] & & \boxed{yx}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \boxed{yyxy} \arrow[dl, "y^2\to 1"'] \arrow[dr, "yxy\to x^2"] & \\
\boxed{xy} & & \boxed{yx^2} \arrow[ll, dashed]
\end{tikzcd}
$$

We add these rules to $\mathfrak{R}_1$ to get

$$\mathfrak{R}_2 := \{x^3 \to 1,\ y^2 \to 1,\ xyxy \to 1,\ yxy \to x^2,\ xyx \to y,\ x^2y \to yx,\ yx^2 \to xy\}.$$

Checking for overlaps all the pairs now resolve to the same word. The rule $xyxy \to 1$ is the only redundant rule.

$$
\begin{tikzcd}
 & \boxed{xyxy} \arrow[dl, "xyx\to y"'] \arrow[dr, "yxy\to x^2"] & \\
\boxed{y^2} \arrow[dr, "y^2\to 1"'] & & \boxed{x^3} \arrow[dl, "x^3\to 1"] \\
 & \boxed{1} &
\end{tikzcd}
$$

So we omit $xyxy \to 1$ from $\mathfrak{R}_2$ to obtain a rewrite system $\mathfrak{R}$ for $S_3$. If we enumerate elements $w$ of the free group on $x$ and $y$ under the rules $\mathfrak{R}$ we get elements $w_{\mathfrak{R}}$:

| $w$ | $w_{\mathfrak{R}}$ | $x^2$ | | $x^3$ | $1$ | $yx^2$ | $xy$ |
|---|---|---|---|---|---|---|---|
| $1$ | | $xy$ | | $x^2y$ | $yx$ | $yxy$ | $x^2$ |
| $x$ | | $yx$ | | $xyx$ | $y$ | $y^2x$ | $x$ |
| $y$ | | $y^2$ | $1$ | $xy^2$ | $x$ | $y^3$ | $y$ |

All words of length 3 reduce so all elements of length greater than 3 reduce so the rewrite system is 3-complete. The group has elements $\{1,\ x,\ y,\ x^2,\ xy,\ yx\}$ which are unique normal forms. $\Diamond$

We now have a method for obtaining normal forms for elements of groups and semigroups.

### 2.2.2 Implementation and GAP4 Output

GAP4 [14] is a computational discrete algebra program in which new structures may be implemented as objects with attributes and properties. We have developed a collection of GAP4 functions to implement graphs of groups and reduction of words to normal forms. The input data for a graph of groups is a weighted graph with involution $\Gamma$ given by the ordered sets vertices $V(\Gamma) := \{v_1, \ldots, v_n\}$ and edges $E(\Gamma) := \{y_1, \ldots, y_m\}$ where $y_i := [s(y_i), y_i, t(y_i)]$; an ordered list of groups $\mathrm{G} := \{G_{v_1}, \ldots, G_{v_n}\}$ where $G_{v_i}$ is the group associated to the vertex $v_i$; an ordered list of subgroups $\mathrm{H} := \{H_{y_1}, \ldots, H_{y_m}\}$ where $H_{y_i}$ is a subgroup of $G_{s(y_i)}$; and an ordered list of isomorphisms $\Phi : \{\phi_{y_1}, \ldots, \phi_{y_m}\}$ where $\phi_{y_i}$ is the isomorphism from $H_{s(y_i)} \to H_{t(y_i)}$.

The weighted graph with involution, `FpWeightedDigraph` is given by two lists; vertices and edges. The edges are triples $[s(y), y, t(y)]$ where $s(y)$ and $t(y)$ are vertices and $y$ the edge. We choose the edge $y$ to be represented in GAP4 by a generator of a free group so that for an edge $[s(y), y, t(y)]$ the involution edge is $[t(y), y^{-1}, s(y)]$.

We implement a graph of groups $\Gamma_{\mathrm{G}}$ as an object `GraphOfGroups` with representation having attributes:

| | |
|---|---|
| `DigraphOfGraphOfGroups`, | the weighted directed graph with involution |
| `GroupsOfGraphOfGroups`, | the family of groups |
| `SubgroupsOfGraphOfGroups`, | the family of subgroups |
| `IsomorphismsOfGraphOfGroups`, | the family of isomorphisms. |

A graph of groups word is given by a triple $(\Gamma_{\mathrm{G}}, v, w)$ and is implemented as an object `GraphOfGroupsWord` whose representation has attributes:

| | |
|---|---|
| `GraphOfGroupsOfWord`, | the graph of groups $\Gamma_{\mathrm{G}}$ |
| `Source`, | the source vertex $v$ of the word $w$ |
| `WordOfGraphOfGroupsWord`, | the word in $A(\Gamma_{\mathrm{G}})$. |

For a graph of groups word we write the position of the edge in the list of edges instead of the weighted label of the edge. This relabelling simplifies the code and computation.

**Example 2.2.5** Let $\Gamma$ have $V = \{5, 6\}$ and $E = \{\{5, y, 6\}, \{6, y^{-1}, 5\}\}$, with $\mathrm{G} = \{G_5 = \langle a \rangle,\ G_6 = \langle b \rangle\}$, $\mathrm{H} = \{H_y = \langle a^3 \rangle,\ H_{y^{-1}} = \langle b^3 \rangle\}$ and $\Phi = \{\phi_y,\ \phi_{y^{-1}}\}$ where $\phi_y(a^3) = b^2$. Then the word $(b, y^{-1}, a, y, b, y^{-1}, a)$ is represented by the list $[b, 2, a, 1, b, 2, a]$. $\Diamond$

The operation `ReducedGraphOfGroupsWord` determines the reduced word of a graph of groups word using Algorithm 2.1.6. The operations `IsGraphOfGroups`, `IsGraphOfGroupsWord` and `IsReducedGraphOfGroupsWord` are used to test that an object is a graph of groups, a graph of groups word or a reduced graph of groups word respectively.

When calculating the reduced word of a graph of groups word we use the functions `NormalFormKBRWS` and `LeftTransversalsOfGraphOfGroups` which we outline below.

`NormalFormKBRWS` is a function that gives a normal form for a group element. It uses standard Knuth-Bendix rewriting techniques on semigroups. Given a group $G = \langle X \mid R \rangle$ and a word $w$ in the free group on $X$, the function `NormalFormKBRWS` first constructs the free semigroup $S$ on $X$, then creates a Knuth-Bendix rewriting system $KBRWS$ which is confluent on $S$. We then map $w$ to its corresponding element $w'$ in $S$, using $KBRWS$ we get the reduced word $rw'$. We then map $rw'$ to the group $G$ to obtain the normal form of the group element.

GAP4 has a function that determines the right cosets of a subgroup in a group. In `RightTransversalsOfGraphOfGroups`, right cosets for the subgroups $H_y$ in the group $G_{s(y)}$ are determined and a coset representative is chosen by the representative function in GAP4. We then apply `NormalFormKBRWS` to get the normal form of the coset representatives.

The function `LeftTransversalsOfGraphOfGroups` is used to obtain left coset representatives. For each coset representative $t$ in the right transversals the function `LeftTransversalsOfGraphOfGroups` computes `NormalFormKBRWS` of $t^{-1}$ in the appropriate group.

Given a graph of groups $\Gamma_{\mathrm{G}} := (\Gamma, \mathrm{G}, \mathrm{H}, \Phi)$ a word $w = (g_1, y_1, \ldots, g_n, y_n, g_{n+1})$ and the vertex $v$ of the group $G_v$ to which the element $g_1$ belongs, a normal form can be obtained in GAP4.

We now give an example of the implementation of graphs of groups in GAP4. Let $\Gamma_{\mathrm{G}}$ be the graph of groups as defined for the trefoil group in Example 2.1.16 on page 41.

**Example 2.2.6** *Trefoil Groupoid implemented in* GAP4
With GAP4 running the package is accessed in the usual way..

```
gap> RequirePackage("xres");
```

```
#I ---------- The XRES share package -----------
#I -- For graphs of groups and crossed resolutions --
true
```

We first define the graph `G1` with 2 vertices and 2 edges.

```
gap> f1 := FreeGroup("y");; y := GeneratorsOfGroup(f1)[1];;
gap> V1 := [5,6];;
gap> E1 := [ [5,y,6], [6,y^-1,5] ];;
gap> G1 := FpWeightedDigraph( V1, E1 );;
```

We now define the lists of groups, subgroups and isomorphisms.

```
gap> za := FreeGroup( 1, "a" );; gza := GeneratorsOfGroup( za );;
gap> SetName( za, "za" );; a := gza[1];;
gap> hy := Subgroup( za, [a^3] );;
gap> zb := FreeGroup( 1, "b" );; gzb := GeneratorsOfGroup( zb );;
gap> SetName( zb, "zb" );; b := gzb[1];;
gap> hybar := Subgroup( zb, [b^2] );;
gap> homy := GroupHomomorphismByImagesNC(hy,hybar,[a^3],[b^2]);;
gap> homybar := GroupHomomorphismByImagesNC(hybar,hy,[b^2],[a^3]);;
gap> gps := [ za, zb ];;
gap> sgps := [ hy, hybar ];;
gap> isos := [ homy, homybar ];;
```

We then combine the lists to form a graph of groups `GG1`.

```
gap> GG1 := GraphOfGroups( G1, gps, sgps, isos );
Graph of Groups: 2 vertices; 2 edges; groups [ za, zb ]
```

A graph of groups word in `GG1` is a triple `(G1,v,w)` where `w` is a word with source `v`.

```
gap> L1 := [ a^7, 1, b^-6, 2, a^-11, 1, b^9, 2,a^7 ];;
gap> gw1 := GraphOfGroupsWord( GG1, 5, L1 );
(5)a1^7.z.b1^-6.z^-1.a1^-11.z.b1^9.z^-1.a1^7(5)
```

To calculate the normal form we use `ReducedGraphOfGroupsWord`.

```
gap> ReducedGraphOfGroupsWord(gw1);
(5)a1^-1.z.b1^-1.z^-1.a1^10(5)
```

We note that this reduced word is different from the reduced word in Example 2.1.16 since the set of transversal elements are not those we made in the example.

```
gap> LeftTransversalsOfGraphOfGroups( GG1 );
[ [ <identity ...>, a1^-1, a1^-2 ], [ <identity ...>, b1^-1 ] ]
```

The following are interesting examples of normal forms for graph of groups words for the trefoil group as a free product with amalgamation. If we set the `InfoLevel` to 2 the steps in the reduction process are printed.

```
gap> SetInfoLevel( InfoXRes, 2 );
gap> L2 := [ b^6, 2, a, 1, b, 2, a ];;
gap> gw2 := GraphOfGroupsWord( GG1, 6, L2 );
(6)b1^6.z^-1.a1.z.b1.z^-1.a1(5)
gap> ReducedGraphOfGroupsWord( gw2 );
#I  w = [ <identity ...>, 2, a1^10, 1, b1, 2, a1 ]
#I  w = [ <identity ...>, 2, a1^-2, 1, b1^9, 2, a1 ]
#I  w = [ <identity ...>, 2, a1^-2, 1, b1^-1, 2, a1^16 ]
(6)<identity ...>.z^-1.a1^-2.z.b1^-1.z^-1.a1^16(5)


gap> L3:=[ a^3, 1, b^-2 ];;
gap> gw3 := GraphOfGroupsWord( GG1, 5, L3 );
(5)a1^3.z.b1^-2(6)
gap> ReducedGraphOfGroupsWord( gw3 );
#I  w = [ <identity ...>, 1, <identity ...> ]
(5)<identity ...>.z.<identity ...>(6)
```

The following reduction of a graph of groups word illustrates length reduction.

```
L5 := [ b^2, 2, a^0, 1, b^0, 2, a^0, 1, b^0, 2, a^0 ];
[ b1^2, 2, <identity ...>, 1, <identity ...>, 2, <identity ...>, 1,
  <identity ...>, 2, <identity ...> ]
```

```
gap> gw5:=GraphOfGroupsWord(GG1,6,L5);
(6)b1^2.z^-1.<identity ...>.z.<identity ...>.z^
-1.<identity ...>.z.<identity ...>.z^-1.<identity ...>(5)
gap> ReducedGraphOfGroupsWord(gw5);
#I  w = [ <identity ...>, 2, a1^3, 1, <identity ...>, 2,
  <identity ...>, 1, <identity ...>, 2, <identity ...> ]
#I  w = [ <identity ...>, 2, <identity ...>, 1, b1^2, 2,
  <identity ...>, 1, <identity ...>, 2, <identity ...> ]
#I  shorter w = [ <identity ...>, 2, a1^3, 1, <identity ...>, 2,
  <identity ...> ]
#I  w = [ <identity ...>, 2, <identity ...>, 1, b1^2, 2,
  <identity ...> ]
#I  shorter w = [ <identity ...>, 2, a1^3 ]

(6)<identity ...>.z^-1.a1^3(5)
```
◊

**Example 2.2.7** *Mixed Amalgam implemented in* GAP4
We now define the graph of groups for the mixed amalgam of groups in example 2.1.20 and apply the reduction of the graph of groups word.

```
gap> GG2;
Graph of Groups: 2 vertices; 4 edges; groups [ za, zb ]
gap> Display(GG2);
Graph of Groups with :-
    vertices: [ 5, 6 ]
       edges: [ [ 5, z, 5 ], [ 5, z^-1, 5 ], [ 5, y, 6 ],
                [ 6, y^-1, 5 ] ]
      groups: [ za, zb ]
   subgroups: [ Group( [ a1^7 ] ), Group( [ a1^5 ] ),
                Group( [ a1^3 ] ), Group( [ b1^2 ] ) ]
isomorphisms: [ GroupHomomorphismByImages( Group( [ a1^7 ] ),
                Group( [ a1^5 ] ), [ a1^7 ], [ a1^5 ] ),
                GroupHomomorphismByImages( Group( [ a1^5 ] ),
```

```
                  Group( [ a1^7 ] ), [ a1^5 ], [ a1^7 ] ),
                  GroupHomomorphismByImages( Group( [ a1^3 ] ),
                  Group( [ b1^2 ] ), [ a1^3 ], [ b1^2 ] ),
                  GroupHomomorphismByImages( Group( [ b1^2 ] ),
                  Group( [ a1^3 ] ), [ b1^2 ], [ a1^3 ] ) ]
```

Apply the reduction algorithm to the word $w$ as given in Example 2.1.20.

```
gap> w := [ a^2, 3, b^6, 4, a^2, 1, a^4, 2, a, 3, b, 4, a ];;
gap> gw:=GraphOfGroupsWord(GG2,5,L);
(5)a1^2.y.b1^6.y^-1.a1^2.z.a1^4.z^-1.a1.y.b1.y^-1.a1(5)
gap> ReducedGraphOfGroupsWord(gw);
#I  w = [ a1^-1, 3, b1^8, 4, a1^2, 1, a1^4, 2, a1, 3, b1, 4, a1 ]
#I  w = [ a1^-1, 3, <identity ...>, 4, a1^14, 1, a1^4, 2, a1, 3,
  b1, 4, a1 ]
#I  shorter w = [ a1^-1, 1, a1^14, 2, a1, 3, b1, 4, a1 ]
#I  w = [ a1^-1, 1, a1^-1, 2, a1^22, 3, b1, 4, a1 ]
#I  w = [ a1^-1, 1, a1^-1, 2, a1^-2, 3, b1^17, 4, a1 ]
#I  w = [ a1^-1, 1, a1^-1, 2, a1^-2, 3, b1^-1, 4, a1^28 ]
(5)a1^-1.z.a1^-1.z^-1.a1^-2.y.b1^-1.y^-1.a1^28(5)
```

The word of the reduced graph of group word has the same normal form as given in example 2.1.20 with the following transversal.

```
gap> LeftTransversalsOfGraphOfGroups( GG2 );
[ [ <identity ...>, a1^-1, a1^-2, a1^-3, a1^-4, a1^-5, a1^-6 ],
  [ <identity ...>, a1^-1, a1^-2, a1^-3, a1^-4 ],

  [ <identity ...>, a1^-1, a1^-2 ], [ <identity ...>, b1^-1 ] ]
```

◊

# Chapter 3

# Total Groupoids and Total Spaces

In the first section of this chapter we consider graphs of groups and how they relate to cylinder constructions in the category of groupoids. We define the total groupoid for a graph of groups and show how this relates to the fundamental groupoid of a graph of groups.

The second section defines CW-complexes and graphs of CW-complexes which will then be adapted to graphs of crossed complexes in Chapter 4. Given a group we can construct a CW-complex such that the fundamental group of the CW-complex is isomorphic to the group, this isomorphism provides the connection between graphs of groups and graphs of CW-complexes.

## 3.1 Cylinders

This section involves graphs of groups and techniques used to fit it into a groupoid setting involving cylinders from abstract homotopy theory. The notion of a total groupoid of a graph of groups is explored and its connection to the fundamental groupoid of a graph of groups.

We refer the reader to Kamps and Porter, *Abstract Homotopy Theory and Simple Homotopy Theory* [18] which derives abstract homotopy theory from the structure of cylinders.

### 3.1.1 Mapping Cylinders

In this subsection we define mapping cylinders which will be used later in this section to define total groupoids for graphs of groups and give examples of mapping cylinder constructions in the category of groupoids which will model free products with amalgamations of groups.

We begin by defining a cylinder which is given by a functor, mapping cylinders and double mapping cylinders which are objects in a small category.

**Definition 3.1.1** A *cylinder*, $\mathrm{M} = (M, i_0, i_1, \sigma)$, in a category $\mathcal{C}$ consists of a functor, $M : \mathcal{C} \to \mathcal{C}$ called the cylinder functor, together with three natural transformations

$$\begin{aligned} i_0, i_1 : Id_{\mathcal{C}} &\to M \\ \sigma : M &\to Id_{\mathcal{C}} \end{aligned}$$

such that $\sigma i_0 = \sigma i_1 = Id_{\mathcal{C}}$.

If we apply the cylinder functor $M$ of $\mathrm{M}$ to an object $C$ of $\mathcal{C}$ we write $M(C)$ for the cylinder. As the name "cylinder" suggests we can represent a cylinder by the following picture.

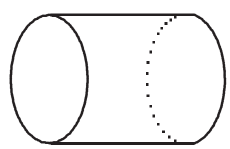


Figure 3.1: Cylinder

We now give examples of cylinders in the categories of topological spaces and groupoids which will be used to define the total space of a graph of spaces of a graph of spaces and to obtain total groupoids for graphs of groups respectively.

**Example 3.1.2** *Topological Space Cylinder*
A cylinder on the category of topological spaces can be given by a cylinder functor

$$M : X \to X \times [0, 1]$$

where $X$ is a topological space and $[0,1]$ the unit interval, together with the three natural transformations

$$\begin{aligned} i_0 : X \to X \times [0,1] \qquad & i_0(x) = (x,0) \\ i_1 : X \to X \times [0,1] \qquad & i_1(x) = (x,1) \\ \sigma : X \times [0,1] \to X \qquad & \sigma(x,t) = x \end{aligned}$$

where $t \in [0,1]$. $\Diamond$

**Example 3.1.3** *Groupoid Cylinder*
A cylinder on $\mathcal{G}pd$ can be defined by a cylinder functor

$$M : \mathcal{G} \to \mathcal{G} \times \mathcal{I}$$

where $\mathcal{G}$ is a groupoid, $\mathcal{I}$ is the unit groupoid, together with the natural transformations

$$\begin{aligned} i_0 : \mathcal{G} \to \mathcal{G} \times \mathcal{I} \qquad & i_0(g) = (g, 1_0) \\ i_1 : \mathcal{G} \to \mathcal{G} \times \mathcal{I} \qquad & i_1(g) = (g, 1_1) \end{aligned}$$

and $\sigma : \mathcal{G} \times \mathcal{I} \to \mathcal{G}$ is the projection onto $\mathcal{G}$. $\Diamond$

The above examples are the canonical cylinders for the respective categories and throughout this exposition the term cylinder will refer to the canonical cylinder.

In Subsection 1.2.1 we defined homotopy of two groupoid morphisms. We use cylinders to define homotopies of arrows in arbitrary categories.

**Definition 3.1.4** A *homotopy* between arrows $f, g : X \to Y$ of $\mathcal{C}$ exists if there is an arrow $\phi : M(X) \to Y$ where $M$ is a cylinder of $\mathcal{C}$ such that $\phi i_0 = f$ and $\phi i_1 = g$.

If a homotopy $X \to Y$ exists, then we say $X$ and $Y$ are *homotopy equivalent* (or of the same *homotopy type*). A map from $X \to Y$ is *nullhomotopic* if it is homotopic to some constant map.

We now define mapping cylinders and double mapping cylinders which can be thought of pictorially as gluing objects of a category to the ends of a cylinder.

**Definition 3.1.5** A *mapping cylinder* of $f$, $M_f$, where $f : C \to A$ is an arrow in $\mathcal{C}$ is given by a pushout in $\mathcal{C}$

$$\begin{tikzcd} C \arrow[r, "i_0"] \arrow[d, "f"'] & M(C) \arrow[d, "k_f"] \\ A \arrow[r, "j_f"'] & M_f \end{tikzcd}$$

where $M(C)$ is a cylinder in $\mathcal{C}$.

A mapping cylinder can be represented by the following picture where $A$ is "glued" to one end of a cylinder $M(C)$.

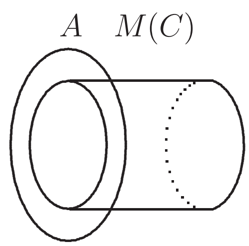



Figure 3.2: Mapping Cylinder

We now define a double mapping cylinder of two arrows $f$ and $g$ of a category $\mathcal{C}$ as a colimit. We relate this construction to a total groupoid of a graph of groups with one pair of involutary edges in Subsection 3.1.3.

**Definition 3.1.6** A *double mapping cylinder* of arrows $f : C \to A$ and $g : C \to B$ in a category $\mathcal{C}$ is the colimit of the diagram

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "i_0"] & & C \arrow[dl, "i_1"'] \arrow[dr, "g"] & \\ A & & M(C) & & B. \end{tikzcd}$$

This means we have a diagram

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[dr, "i_0"] & & C \arrow[dl, "i_1"'] \arrow[dr, "g"] & \\
A \arrow[drr, "j_A"'] & & M(C) \arrow[d, "k"] & & B \arrow[dll, "j_B"] \\
& & M_{f,g} & &
\end{tikzcd}$$

such that the following universal property holds:

(i) $j_A f = k i_0$ and $j_B g = k i_1$

(ii) If $j'_A : A \to D$, $j'_B : B \to D$ and $k' : M(C) \to D$ are any maps such that $j'_A f = k' i_0$ and $j'_B g = k' i_1$ then there exists a unique map $l : M_{f,g} \to D$ such that $l j_A = j'_A$, $l j_B = j'_B$ and $lk = k'$.

The colimit of the above diagram can be constructed by repeated pushouts.

$$\begin{tikzcd}
& C \arrow[r, "g"] \arrow[d, "i_1"'] & B \arrow[d, "j_g"] \\
C \arrow[r, "i_0"] \arrow[d, "f"'] & M(C) \arrow[r, "k_g"] \arrow[d, "k_f"'] & M'_g \arrow[d] \\
A \arrow[r, "j_f"'] & M_f \arrow[r] & M_{f,g}
\end{tikzcd}$$

We can construct the double mapping cylinder as a two stage pushout where we construct a pushout of a mapping cylinder which itself is a pushout. Given a mapping cylinder

$$\begin{tikzcd}
C \arrow[r, "i_0"] \arrow[d, "f"'] & M(C) \arrow[d, "k_f"] \\
A \arrow[r, "j_f"'] & M_f
\end{tikzcd}$$

we construct the pushout

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "i_f"'] & B \arrow[d, "j_B"] \\ M_f \arrow[r, "g'"'] & M_{f,g} \end{tikzcd}$$

where we define $i_f = k_f i_1 : C \to M_f$. Then $M_{f,g}$ is a double mapping cylinder with $j_A = g' j_f$, $k = g' k_f$ and $j_B$. We note that the above pushout is the composite of the top right and bottom right pushout squares in the repeated pushout construction.

We can represent a double mapping cylinder by the following picture.

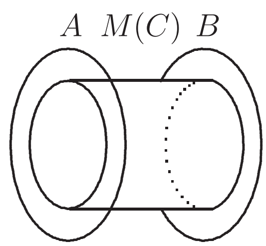



Figure 3.3: Double Mapping Cylinder

The double mapping cylinder is also a homotopy colimit of the diagram

$$A \xleftarrow{f} C \xrightarrow{g} B$$

since $M_{f,g}$ is a colimit and $k$ is a homotopy $k : j_A f \simeq j_B g$.

We now give a definition and result which hold in the categories $\mathcal{T}op$, $\mathcal{G}pd$ and $\mathcal{FC}rs$, the category of free crossed complexes. We refer the reader to Kamps and Porter [18] for further detail and more general results.

**Definition 3.1.7** An arrow $i : A \to X$ of $\mathcal{C}$ is a *cofibration* if and only if the diagram

$$\begin{tikzcd} A \arrow[r, "i_0(A)"] \arrow[d, "i"'] & M(A) \arrow[d, "i \times I"] \\ X \arrow[r, "i_0(X)"'] & M(X) \end{tikzcd}$$

is a pushout.

The following result is used to give an explicit construction of a mapping cylinder in the category $\mathcal{G}pd$. We refer the reader to the proof of Proposition 7.4 in [18].

**Proposition 3.1.8** *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

*be a pushout in which $i$ is a cofibration and $f$ is a homotopy equivalence. Then $g$ is a homotopy equivalence.*

### 3.1.2 Groupoid Mapping Cylinders

In this subsection we use the cylinder functor for groupoids to construct mapping cylinders and double mapping cylinders in the category of groupoids.

Let $f : \mathcal{G} \to \mathcal{H}$, $g : \mathcal{G} \to \mathcal{K}$ be morphisms of groupoids and suppose we have pushout diagrams

$$\begin{tikzcd} \mathcal{G} \arrow[r, "i_0"] \arrow[d, "f"'] & \mathcal{G} \times \mathcal{I} \arrow[d, "k_f"] \\ \mathcal{H} \arrow[r, "j_f"'] & M_f \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \arrow[r, "k_f i_1"] \arrow[d, "g"'] & M_f \arrow[d] \\ \mathcal{K} \arrow[r] & M_{f,g} \end{tikzcd}$$

where $M_f$ is a mapping cylinder of $f$ and $M_{f,g}$ is a double mapping cylinder of $f$ and $g$ in the category of groupoids.

We now give examples of a mapping and a double mapping cylinder which are used to model the trefoil group.

**Example 3.1.9** *Trefoil Double Mapping Cylinder*
The trefoil mapping cylinder is given by the pushout of groupoids

$$\begin{tikzcd} C \arrow[r, "i_0"] \arrow[d, "f"'] & C \times \mathcal{I} \arrow[d, "k_f"] \\ A \arrow[r, "j_f"'] & M_f \end{tikzcd}$$

where $C$ and $A$ are free groups on one generator $c$ and $a$ respectively and $f(c) = a^3$ and $i_0(c) = (c, 1_0)$.

By proposition 3.1.8 the groupoid morphism $A \to M_f$ is a homotopy equivalence and the fundamental groups of $M_f$ are homotopy equivalent to $A$.

$$\begin{array}{ll} M_f(0) = \{a^n : n \in \mathbb{Z}\} & M_f(0,1) = \{a^n\iota : n \in \mathbb{Z}\} \\ M_f(1) = \{\iota^{-1}a^n\iota : n \in \mathbb{Z}\} & M_f(1,0) = \{\iota^{-1}a^n : n \in \mathbb{Z}\} \end{array}$$

where $\iota$ is abbreviated from $(1,\iota)$.

The group $C$ "disappears" although there is a groupoid presentation of $M_f$ with $C$ present.

$$\langle a, (c, 1_0), (1, \iota) \mid (c, 1_0) = a^3 \rangle$$

We use this presentation since $C$ needs to be present for "gluing" at the other end of the cylinder.

Let $B$ be the free group on $b$ and define the maps $g : C \to B$ by $g(c) = b$ and $i_f : C \to M_f$ by composition of two maps $k_f$ and $i_1$. This map takes an arrow of $C$ and maps it to the end of the cylinder where no identifications have been made.

We construct the following pushout.

$$\begin{tikzcd} C \arrow[r, "i_f"] \arrow[d, "g"'] & M_f \arrow[d] \\ B \arrow[r] & M_{f,g} \end{tikzcd}$$

So $i_f = k_f i_1 : C \to M_f$ gives $c \mapsto (c, 1_1)$ and $g : C \to B$ is defined to be the injective map $g(c) = b^2$.

Using the construction of pushouts in Subsection 1.2.6 $M_{f,g}$ can be given by a universal groupoid factored by an equivalence relation. Hence $M_{f,g}$ is a groupoid with presentation

$$\langle a, b, \iota \mid a^3 = \iota b^2 \iota^{-1} \rangle.$$

which is also the presentation of the trefoil groupoid of Example 2.1.16. $\Diamond$

A double mapping cylinder for the diagram $A \xleftarrow{f} C \xrightarrow{g} B$ where $A$, $B$ and $C$ have group presentations $\langle X|R\rangle$, $\langle Y|S\rangle$ and $\langle Z|T\rangle$ respectively is a two object groupoid with presentation

$$\langle X, Y, Z, \iota \mid f(c) = \iota g(c) \iota^{-1} : \text{ for all } c \in C \rangle.$$

### 3.1.3 Total Groupoids for Graphs of Groups

In this subsection we use ideas from abstract homotopy theory and apply it to graphs of groups using cylinder constructions.

**Definition 3.1.10** A *graph of groups* $\Gamma_{\mathrm{G}}$ consists of a graph $\Gamma$ with involution, a group $G_v$ for each vertex $v \in V(\Gamma)$ and a group $G_y$ for each edge $y \in E(\Gamma)$ such that $G_y = G_{\overline{y}}$ together with an injective homomorphism $\mu_y : G_y \to G_{t(y)}$.

The total groupoid of a graph of groups $\Gamma_{\mathrm{G}}$ is the groupoid obtained by taking the disjoint union of groupoids associated to the vertices and edges and then factoring by an equivalence relation generated by relations, as follows.

**Definition 3.1.11** Given a graph of groups, $\Gamma_{\mathrm{G}}$, the *total groupoid* $\mathrm{Tot}(\Gamma_{\mathrm{G}})$ is defined as the quotient of

$$\Big(\sqcup \{G_v : v \in V(\Gamma)\}\Big) \sqcup \Big(\sqcup \{G_y \times \mathcal{I} : y \in E(\Gamma)\}\Big)$$

by the relations

$$\begin{aligned} G_y \times \mathcal{I} &\rightarrow G_{\overline{y}} \times \mathcal{I} \text{ by } (g, \iota) \rightarrow (g, \iota^{-1}) \\ G_y \times 1_0 &\rightarrow G_{t(y)} \text{ by } (g, 1_0) \rightarrow \mu_y(g). \end{aligned}$$

For the graph of groups with one pair of involutary edges the total groupoid is homotopy equivalent to a double mapping cylinder. Hence the total groupoid for the graph of groups for the trefoil groupoid is the double mapping cylinder of example 3.1.9 given by taking the homotopy colimit of

$$\begin{tikzcd} & G_y \arrow[dl, "\mu_{\overline{y}}"'] \arrow[dr, "i_0"] & & G_y \arrow[dl, "i_1"'] \arrow[dr, "\mu_y"] & \\ G_{s(y)} & & G_y \times \mathcal{I} & & G_{t(y)} \end{tikzcd} .$$

The following theorem provides a connection between the fundamental groupoid and total groupoid of a graph of groups.

**Theorem 3.1.12** *The total groupoid of a graph of groups with one pair of involutary edges is homotopy equivalent to the fundamental groupoid of the given graph of groups.*

In Sections 3.2 and 4.2 the total space and the total crossed complex which are similar to the total groupoid construction are connected to the fundamental groupoid.

## 3.2 Graphs of CW-complexes

The study of algebraic topology is a combination of algebra and topology as the name suggests. It is a method of assigning algebraic structures such as groups to topological spaces and homomorphisms to continuous maps. An example of this is calculating the fundamental group of a space which associates a group to a space with a base point. In this exposition we associate a group to a CW-complex and use the theory of algebraic topology to get more information about the group.

In this section we define CW-complexes, graphs of CW-complexes and give a result proved by Scott and Wall in *Topological Methods in Group Theory* [22].

In Chapter 4 we associate a crossed complex to a CW-complex and adapting Scott and Wall's result to crossed complexes.

### 3.2.1 CW-complexes

An important class of spaces are CW-complexes which are spaces constructed in stages by attaching cells. Whitehead formally defined CW-complexes by adding combinatorial structure to spaces which provided a better understanding of homotopy groups. We refer the reader to Fritsch and Piccinini *Cellular Structures in Topology* [13] for more details.

The main result of CW-complexes is Whitehead's Theorem 3.2.9 below. This theorem is the key to the results in this subsection and used to prove Proposition 3.2.19.

We begin by defining some spaces. Let $D^{n+1} = \{x : |x| \leqslant 1\} \subseteq \mathbb{R}^{n+1}$ be the unit disk and $S^n$ its boundary.

**Definition 3.2.1** A *CW-complex* $X$ is a space $X$ which is the union of an expanding sequence of subspaces $X^n$ such that inductively $X^0$ is a discrete set of points and $X^{n+1}$ is the pushout obtained from $X^n$ by attaching disks $D^{n+1}$ along attaching maps

$j_n : S^n \to X^n$

$$\begin{tikzcd} J_{n+1} \sqcup S^n \arrow[r] \arrow[d] & J_{n+1} \times D^{n+1} \arrow[d] \\ X^n \arrow[r] & X^{n+1}. \end{tikzcd}$$

Thus $X^{n+1}$ is the quotient space obtained from $X^n \cup (J_{n+1} \times D^{n+1})$ by identifying $(j, x)$ with $j(x)$ for $x \in S^n$ where $J_{n+1}$ is the discrete set of such attaching maps $j$.

Each resulting map $D^{n+1} \to X$ is called a *cell*. The subspace $X^n$ is called the $n$-*skeleton* of $X$. A continuous map $f : X \to Y$ of CW-complexes is *cellular* if $f(X^n) \subseteq Y^n$ for all $n \geqslant 0$.

We define the fundamental groupoid and homotopy groups of spaces since CW-complexes provide information on homotopy groups. We first consider fundamental groupoids and homotopy groups which are used to define the fundamental crossed complex of a CW-complex in Subsection 4.1.4.

We have defined the fundamental groupoid of a graph, we can also define the fundamental groupoid of a space. Instead of paths in a graph we have paths in a space. Let $X$ be a space. The category of paths $PX$ on $X$ has object set, $\mathrm{Ob}(PX) = X$ and for any $x, y \in X$ the set of arrows $\mathrm{Arr}(PX)$ is the set of paths from $x$ to $y$. The fundamental groupoid $\pi_1(X)$ of a space $X$ will be a groupoid such that $\pi_1(X)(x, y)$ is the set of equivalence classes of $PX(x, y)$. We let $\pi_1(X)(x)$ denote the fundamental groupoid which has object set $x$ and has arrows, equivalence classes of $PX(x, x)$ paths starting and ending at $x$. This fundamental groupoid is called the fundamental group at $x$. We refer to [4] for more details.

We also have $n$th homotopy groups $\pi_n(X, A, x_0)$. Let $(X, x_0)$ be a based space. A based pair $(X, A, x_0)$ is a pair of spaces $X$, $A$ with base point $x_0$ in which $A$ is a subspace of $X$ and contains $x_0$. If $(X, A, x_0)$ and $(Y, B, y_0)$ are based pairs then $[(X, A, x_0), (Y, B, y_0)]$ is the set of homotopy classes of based pair maps $\beta : (X, A, x_0) \to (Y, B, y_0)$. Let $s_n = (1, 0, \ldots, 0) \in S^n$ be the common base point of $S^n$ and $D^{n+1}$ for $n \geqslant 1$. The $n$th homotopy group of a based pair $(X, A, x_0)$ is $\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_{n-1}), (X, A, x_0)]$. We note that the based pair maps correspond to the characteristic maps for CW-complexes.

Given a group $G$ we can construct a CW-complex such that the fundamental group of the CW-complex is isomorphic to $G$. Let $G$ be a group with presentation $\langle A \mid R \rangle$. We can construct a CW-complex $X$ such that $\pi_1(X) \simeq G$.

Let $X_0 = \{*\}$ a singleton point. Now consider the generators of the group $a_1, \dots, a_n$. We associate each $a_i$ to a 1-cell which is a map from the boundary of $D^{n+1}$ to $X$. The space $X^1 = \bigvee_{a \in A} S^1_a$ is a wedge of 1-spheres based at $*$ one for each generator $a \in A$. The fundamental group of $X^1$ is isomorphic to the free group on $A$.

To each relation we associate a 2-cell which is a map from a relator circle $S^1_r$ to $X^1$. Then $X^2$ is the pushout as described in Definition 3.2.1. We now have a 2-dimensional CW-complex with fundamental group isomorphic to $G$. Adding cells of dimension greater than two does not affect the fundamental group. We refer to [22] for details of this result.

**Definition 3.2.2** A *filtered space* $X^*$ is a space $X$ which consists of a sequence of subspaces.

$$X^* : X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n \subseteq \cdots \subseteq X.$$

**Definition 3.2.3** A filtered space is *connected* if the following conditions hold for each $m \geqslant 0$:

(i) $\phi(X^*, 0)$: If $j > 0$, the map $\pi_0 \to \pi_0 X^j$ induced by inclusion is surjective.

(ii) $\phi(X^*, 0)(m \geqslant 1)$: If $j > m$ and $v \in X^0$, then the map $\pi_m(X^m, X^{m-1}, v) \to \pi_0(X^j, X^{m-1}, v)$ induced by inclusion is surjective.

A CW-complex filtered by its skeleta is a filtered space. A CW-complex with skeleta filtration is a connected filtered space.

We now define properties and attributes of topological spaces and hence CW-complexes. A topological space $X$ is *contractible* if it is homotopy equivalent to a point. A topological space is *simply connected* if it is path connected and $\pi_1(X)(x) = 1$ for some $x \in X$. Thus contractible spaces are simply connected.

A subset $A$ of a topological space $X$ is a *retract* of $X$ if there is a continuous map $r : X \to A$ such that $ri = 1_A$ where $i : A \to X$ is the inclusion map. The map $r$ is a *retraction*. A subset $A$ of $X$ is called a *deformation retract* if there is a retraction $r : X \to A$ such that $ir \simeq 1_X$ where $i : A \to X$ is the inclusion.

**Definition 3.2.4** A space is *aspherical* if for every $n \geqslant 2$ and every continuous map $f : S^n \to X$, there exists a continuous map $g : D^{n+1} \to X$ with the restriction to the subspace $S^n$ equal to $f$.

If a $CW$-complex $X$ is aspherical then we denote it $K(G,1)$ where $G$ is the group isomorphic to the fundamental group of $X$.

The *weak topology* on $X$ is determined by $\{U_\lambda\}$ where $\lambda \in \Lambda$ is the topology whose closed sets are those subsets $V$ for which $V \cap U_\lambda$ is closed for every $\lambda \in \Lambda$.

We define coverings and universal coverings of spaces which we use in the proof of Proposition 3.2.19.

Let $X$ and $\widetilde{X}$ be topological spaces and let $p : \widetilde{X} \to X$ be continuous. An open set $U$ in $X$ is *evenly covered* by $p$ if $p^{-1}(U)$ is a disjoint union of open sets $S_i$ in $\widetilde{X}$, called sheets, with $p|_{S_i} : S_i \to U$ a homeomorphism for every $i$.

**Definition 3.2.5** An ordered pair $(\widetilde{X}, p)$ is a *covering space* of $X$ where $X$ is a topological space if

(i) $\widetilde{X}$ is a path connected space,

(ii) $p : \widetilde{X} \to X$ is a continuous map,

(iii) each $x \in X$ has an open neighbourhood $U_x$ that is evenly covered by $p$.

**Definition 3.2.6** A *universal covering space* of $X$ is a covering space $(\widetilde{X}, p)$ with $\widetilde{X}$ simply connected.

We state the following theorems, corollary and proposition without proof. We refer the reader to Rotman, *An Introduction to Algebraic Topology* [21] for the proofs.

**Theorem 3.2.7** *Every connected CW-complex has a universal covering space.*

**Theorem 3.2.8** *If $(\widetilde{X}, p)$ is a covering space of $X$, then $p_* : \pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism for all $n \geqslant 2$.*

**Theorem 3.2.9** Whitehead's Theorem
*If $X$ and $Y$ are connected CW-complexes, and if $f : X \to Y$ is a continuous map such*

*that $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $n$, then $f$ is a homotopy equivalence (so that $X$ and $Y$ have the same homotopy type).*

**Corollary 3.2.10** *A connected CW-complex is contractible if and only if $\pi_n(X) = 0$ for all $n$.*

**Proposition 3.2.11** *The universal covering space of a $K(G, 1)$ space is contractible.*

Pullbacks in the categories of topological spaces and groupoids induce exact sequences giving information on the arrows in the category. Given two maps of spaces $f : A \to X$ and $p : Y \to X$ the pullback of $f$ and $p$

$$\begin{array}{ccc} B & \xrightarrow{\underline{f}} & Y \\ {\scriptstyle \underline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & X \end{array}$$

is the subspace $B$ of $A \times Y$ given by

$$B = A \times_X Y = \{(a, y) \in A \times Y : f(a) = p(y)\}.$$

We now state a result on induced covering maps using pullbacks and refer the reader to result 9.7.2 page 368 of [4] for a proof.

**Proposition 3.2.12** *If $p : Y \to X$ is a covering map, then for any map $f : A \to X$, the induced map $\underline{p} : A \times_X Y \to A$ is also a covering map. Further if $p$ is an $n$-fold covering map, so also is $\underline{p}$.*

We also have analogous results for groupoids. Let $f : \mathcal{L} \to \mathcal{G}$, $p : \mathcal{H} \to \mathcal{G}$ be groupoid morphisms. The pullback of $f$ and $p$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\underline{f}} & \mathcal{H} \\ {\scriptstyle \underline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{L} & \xrightarrow[f]{} & \mathcal{G} \end{array}$$

is the subgroupoid $\mathcal{M}$ of $\mathcal{L} \times \mathcal{H}$ given by

$$\mathcal{M} = \mathcal{L} \times_{\mathcal{G}} \mathcal{H} = \{(l, h) \in \mathcal{L} \times \mathcal{H} : f(l) = p(h)\}.$$

If $f : \mathcal{L} \to \mathcal{G}$ and $p : \mathcal{H} \to \mathcal{G}$ are morphisms of groupoids, and $p$ is a covering morphism then $\underline{p} : \mathcal{L} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{L}$ is a covering morphism.

We now show how pullbacks of spaces and groupoids are linked, by the following result. We refer the reader to result 9.7.5 page 369 of [4] for a proof.

**Proposition 3.2.13** *Suppose given a pullback of spaces. Then there is an induced morphism of groupoids*

$$\theta : \pi(A \times_X Y) \to \pi A \times_{\pi X} \pi Y$$

*which is the identity on objects. Further, if $p$ is a covering morphism of spaces, then $\theta$ is an isomorphism.*

The following proposition is used to prove Proposition 3.2.19.

**Proposition 3.2.14** *Suppose that we have a pullback of groupoids as given above and that $p$ is a covering morphism. Let $(l, y) \in \mathrm{Ob}(\mathcal{M})$, so that $f(l) = p(y) = x$, say. Then there is a sequence*

$$\mathcal{M}(l, y) \xrightarrow{i} \mathcal{L}(l) \times \mathcal{H}(y) \xrightarrow{\partial} \mathcal{G}(x) \xrightarrow{\Delta} \pi_0\mathcal{L} \times_{\pi_0\mathcal{G}} \pi_0\mathcal{H}$$

*which is exact.*

For details of the exactness of the sequence refer to [4] page 370.

### 3.2.2 Total Spaces and Aspherical CW-complexes

We now define a graph of spaces and give the result that we will adapt to crossed complexes to determine the asphericity of groups. We will concentrate on the case of graphs of $CW$-complexes as this is sufficient for the results that we will use.

**Definition 3.2.15** A *graph of $CW$-complexes* $\Gamma_{\mathrm{X}}$ is given by a graph $\Gamma$ with involution and an assignment of $CW$-complexes $X_v$, $X_y$ to each vertex $v$ of $V(\Gamma)$ and to each edge $y$ of $E(\Gamma)$ respectively; and satisfying $X_y = X_{\overline{y}}$, and a cellular map $f_y : X_y \to X_{t(y)}$ for each edge $y \in E(\Gamma)$.

We now define the total space of a graph of CW-complexes.

**Definition 3.2.16** Given a graph of $CW$-complexes, $\Gamma_{\mathrm{X}}$, the *total space* $\mathrm{Tot}(\Gamma_{\mathrm{X}})$ is defined as the quotient of

$$\Big(\cup\{X_v : v \in V(\Gamma)\}\Big) \cup \Big(\cup\{X_y \times I : y \in E(\Gamma)\}\Big)$$

by the identifications

$$\begin{aligned} X_y \times I &\rightarrow X_{\overline{y}} \times I \text{ by } (x,t) \rightarrow (x, 1-t) \\ X_y \times 0 &\rightarrow X_{t(y)} \text{ by } (x,0) \rightarrow f_y(x). \end{aligned}$$

To relate the total space and to the fundamental groupoid $\pi_1(\Gamma_{\mathrm{G}})$ of a graph of groups we recall the construction of $\pi_1(\Gamma_{\mathrm{G}})$.

We have the free product groupoid $A(\Gamma_{\mathrm{G}}) = \underset{v\in V(\Gamma)}{\sqcup} G_v * \mathcal{F}(\Gamma)$ which is the amalgamation over vertex set $V(\Gamma)$ of $\underset{v\in V(\Gamma)}{\sqcup} G_v$ and $\mathcal{F}(\Gamma)$. We then factor by an equivalence relation to obtain $\pi_1(\Gamma_{\mathrm{G}})$.

For spaces, we let $A(\Gamma_{\mathrm{X}}) = \underset{v\in V(\Gamma)}{\sqcup} X_v * \Gamma$ where $*$ is the amalgamation of spaces over vertices.

If $(X, X_0)$ and $(Y, Y_0)$ are spaces with base points then $X*Y$ is given by the pushout

$$\begin{tikzcd} X_0 \sqcup Y_0 \arrow[r] \arrow[d] & X_0 \cup Y_0 \arrow[d] \\ X \sqcup Y \arrow[r] & X * Y \end{tikzcd} .$$

For $A(\Gamma_{\mathrm{X}})$ we have the spaces $(\overline{X}, V(\Gamma))$ where $\overline{X} = \underset{v}{\sqcup} X_v$ and $(\Gamma, V(\Gamma))$. The space $\overline{X} * \Gamma$ is the space amalgamated over $V(\Gamma)$.

The total space $\mathrm{Tot}(\Gamma_{\mathrm{X}})$ can then be given by the quotient of

$$A(\Gamma_{\mathrm{X}}) \cup \Big(\cup\{X_y \times I : y \in E(\Gamma)\}\Big)$$

by the identifications

$$\begin{aligned} X_y \times I &\rightarrow X_{\overline{y}} \times I \text{ by } (x,t) \rightarrow (x, 1-t) \\ X_y \times 0 &\rightarrow X_{t(y)} \text{ by } (x,0) \rightarrow f_y(x) \\ X_y \times I &\rightarrow \Gamma \text{ by } (1,\iota) \rightarrow y. \end{aligned}$$

which can be thought of as "gluing" cylinders onto $A(\Gamma_{\mathrm{X}})$.

We choose spaces such that $\pi_1(\overline{X}, V(\Gamma)) \simeq \overline{G}$ and $\pi_1(\Gamma, V(\Gamma)) \simeq \mathcal{F}(\Gamma)$.

The following example relates to the free product with amalgamation of groups.

**Example 3.2.17** If given a graph $\Gamma := \; u \underset{\overline{y}}{\overset{y}{\rightleftarrows}} v$ then the total space $\mathrm{Tot}(\Gamma_{\mathcal{X}})$ is given by the disjoint union of the spaces $X_u,\; X_v,\; X_y \times I$ and $X_{\overline{y}} \times I$ associated to the vertices and edges of $\Gamma$ and the following identifications;

$$\begin{aligned} X_y \times I &\;\rightarrow\; X_{\overline{y}} \times I \text{ by } (x_y, t) \rightarrow (x_{\overline{y}}, 1-t) \\ X_y \times 0 &\;\rightarrow\; X_v \text{ by } (x_y, 0) \rightarrow f_y(x_y) = x_v \\ X_{\overline{y}} \times 0 &\;\rightarrow\; X_u \text{ by } (x_{\overline{y}}, 0) \rightarrow f_{\overline{y}}(x_{\overline{y}}) = x_u \end{aligned}$$

where the suffix of $x$ denotes what space the element belongs to. We then have the following classes of elements;

$$\begin{aligned} &[x_v,\; (x_{\overline{y}}, 0),\; (x_y, 1)] \\ &[x_u,\; (x_y, 0),\; (x_{\overline{y}}, 1)] \\ &[(x_y, t),\; (x_{\overline{y}}, 1-t)] \qquad \text{for } t \neq 0, 1 \end{aligned}$$

We note that these identifications are related to adjunction mappings and mapping cylinders. $\Diamond$

Graphs of groups, $CW$-complexes and aspherical $CW$-complexes are used in this thesis to determine whether a group has an aspherical presentation. We need a connection between these structures.

Given a graph of groups $\Gamma_{\mathrm{G}}$ (defined classically), we can associate connected 2-dimensional $CW$-complexes $X_y$ and $X_v$ to the vertex and edge groups with $\pi_1(X_v, *) \simeq G_v$ and $\pi_1(X_y, *) \simeq G_y$. The injective group homomorphisms $i_y : G_y \rightarrow G_{t(y)}$ induce a homomorphism $\pi_1(X_y, *) \rightarrow \pi_1(X_{t(y)}, *)$ and a continuous map $(X_y, *) \rightarrow (X_{t(y)}, *)$ by the following result.

**Lemma 3.2.18** *For any homomorphism $\phi : \pi_1(X^2, x) \rightarrow \pi_1(Y, y)$ there is a map $\alpha : X \rightarrow Y$ with $\alpha_* = \phi$ (where $\alpha_*$ is the induced morphism of fundamental groups.*

**Proof** The cells of $X^2$ give a presentation $\pi_1(X^2) = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle = \langle A \mid R \rangle$. The image of the generator $a_i$ by $\phi$ is an element of $\pi_1(Y)$ represented by a map

$(S^1, x) \to (Y, y)$. The family of maps for all $a_i$ is used to define $\alpha^1 : X^1 \to Y$ and $\alpha^1_*$ is the map of fundamental groups induced by $\alpha^1$.

We then have the following diagram.

$$\begin{tikzcd}
F(A) \simeq \pi_1(X^1, x) \arrow[r, "\alpha^1_*"] \arrow[d] & \pi_1(Y, y) \\
\langle A \mid R \rangle \simeq \pi_1(X^2, x) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

which commutes, by construction of $\alpha^1$. Hence $\alpha^1_*(r_j) = 1$. For each 2-cell of $X$, with characteristic map $\chi : (D^2, S^1) \to (X^2, X^1, x)$ the class of $\chi|_{S^1}$ is $r_j$ and the class of $\alpha^1 \cdot \chi|_{S^1}$ is $\alpha^1_*(r_j) = 1$. Thus $\alpha^1 \cdot \chi_j$ is nullhomotopic, so there is a continuous extension $\psi_j : D^2 \to Y$ with $\psi|_{S^1} = \alpha^1 \cdot (\chi_j|_{S^1})$. By definition of $X$ as an identification space, the diagram

$$\begin{tikzcd}
X^1 \cup (\underset{\cup}{j}(D^2 \times j)) \arrow[r, "\alpha^1 \cup \{\psi_j\}"] \arrow[d, "\mathrm{inc} \cup \{\chi_j\}"'] & Y \\
X^2 \arrow[ur] &
\end{tikzcd}$$

defines a map $\alpha : X \to Y$ such that $\alpha|_{X^1} = \alpha^1$ and $\alpha \cdot \chi_j = \psi_j$. Since $\pi_1(X, x)$ is a quotient of $\pi_1(X^1, x)$ it follows that $\alpha_* = \phi$. $\square$

So we can now define a graph of $CW$-complexes where we can recover the original graph of groups. For any graph of two dimensional $CW$-complexes we can attach cells greater than or equal to three to each $X_v$ and $X_y$ to obtain aspherical $CW$-complexes $K_v$ and $K_y$ with the same fundamental groups. Adding cells of dimension greater than two does not affect the fundamental groups.

The map $f_y : X_y \to X_{t(y)}$ extends to a map $k_y : K_y \to K_{t(y)}$ so we have a graph of aspherical $CW$-complexes still inducing $\Gamma_{\mathcal{G}}$. The total space $\mathrm{Tot}(\Gamma_{\mathrm{K}})$ is obtained from $\Gamma_{\mathrm{X}}$ by adding cells of dimension greater than or equal to three so it has the same fundamental group.

The aspherical $CW$-complex $K_v$ is a space of type $K(G_v, 1)$ its homotopy type is determined by $G_v$ (similarly $K(G_y, 1)$). Also the map $k_y$ is determined up to homotopy by $i_y : G_y \to G_{t(y)}$. Thus $\Gamma_{\mathrm{K}}$ is determined up to homotopy, and its fundamental group is unique up to isomorphism.

**Proposition 3.2.19** Scott and Wall (1979)
*(i) If* $\Gamma_{\mathrm{G}}$ *is a graph of groups, each map* $G_v$ *to the fundamental group of* $\Gamma_{\mathrm{G}}$ *is injective.*
*(ii) If* $\Gamma_{\mathrm{K}}$ *is a graph of aspherical spaces, the total space* $\mathrm{Tot}(\Gamma_{\mathrm{K}})$ *is aspherical.*

**Proof** Given a graph of aspherical spaces, for each vertex $v$ of $\Gamma$, the space

$$L_v = K_v \cup \{\bigcup_{t(y)=v} K_y \times I\}$$

admits $K_v$ as a deformation retract as the cylinders $K_y \times I$ glued to $K_v$ collapse down to $K_v$. Hence $L_v$ is a contractible space.

By the results that every connected CW-complex has a universal cover and if the CW-complex is aspherical then its universal cover is aspherical, the universal cover $\widetilde{L}_v$ of $L_v$ is contractible.

Further as $G_y \to G_v$ is injective the sequence induced by the pullback of spaces

$$\begin{tikzcd} \widetilde{K}_y \arrow[r] \arrow[d] & \widetilde{K}_v \arrow[d] \\ K_y \arrow[r, "i_y"'] & K_v \end{tikzcd}$$

where $i_y$ is injective on $\pi_1$ of $K_y$ and $K_v$ shows that $\widetilde{K}_y$ is the universal covering space of $K_y$ and hence contractible. So $\widetilde{L}_v$ can be obtained by attaching copies of $\widetilde{K}_y \times I$ to $\widetilde{K}_v$.

We now construct a space $Y = \cup Y^n$ by induction. Choose any vertex $v_0$ of $\Gamma$ and set $\widetilde{L}_{v_0}$. For any $n \geqslant 1$, the space $Y^{n-1}$ will have had a number of copies of $\widetilde{K}_y \times I$ attached each along $\widetilde{K}_y \times 0$ for various edges $y$.

We define $Y^n$ to be the union of $Y^{n-1}$ with a copy of $\widetilde{L}_{t(y)}$ for each such copy of $\widetilde{K}_y \times I$ in $Y^{n-1}$ identified along $\widetilde{K}_y \times I$. For $Y^1$ we glue the $\widetilde{L}_v$'s to $Y^0$ for those $v$ of the form $t(y) = v$ for $y$ in $\widetilde{L}_{v_0}$.

Since we are attaching contractible sets along contractible subsets each $Y^{n-1}$ is contractible.

We set $Y = \cup Y^n$ with the weak topology then $Y$ is also contractible. There is a projection $Y \to \mathrm{Tot}(\Gamma_{\mathrm{K}})$. By construction of $Y$, $\mathrm{Tot}(\Gamma_{\mathrm{K}})$ is evenly covered by $Y$ which proves (ii). Since for each $K_v \subset \mathrm{Tot}(\Gamma_{\mathrm{K}})$ the induced covering of $K_v$ contains the universal covering by the sequence obtained from the pullback of spaces

$$\begin{tikzcd} \widetilde{K}_v \arrow[r] \arrow[d] & Y \arrow[d] \\ K_v \arrow[r] & \mathrm{Tot}(\Gamma_{\mathrm{K}}) \end{tikzcd}$$

and $\pi_1(K_v, x) \to \pi_1(\mathrm{Tot}(\Gamma_{\mathrm{K}}))$ is injective. □

We note that we have already proved part (i) of proposition 3.2.19 by theorem 2.1.7. In the above proposition $\pi_1(\Gamma_{\mathrm{G}})$ is defined as the fundamental group of the total space of the graph of groups and hence different methods are employed.

# Chapter 4

# Crossed Complexes

This chapter defines crossed complexes of groupoids which are analogous to chain complexes but contain non-abelian information in dimensions 1 and 2. The motivating example of a crossed complex is obtained by applying the fundamental crossed complex functor to a filtered space. This connection between filtered spaces and crossed complexes allows modelling topological proofs with crossed complexes. The fundamental crossed complex provides an algebraic model of a filtered space in which we can carry out calculations.

After defining a crossed complex, we show how to construct a free crossed resolution of a finite cyclic group and we define the tensor product of crossed complexes. In Section 4.2 we will apply these resolutions to the results on crossed complexes obtained from CW-complexes.

For the motivation and history of crossed complexes we refer the reader to Brown and Higgins, *Crossed Complexes and Non-Abelian Extensions* [5].

## 4.1 Crossed Complexes over Groupoids

This section gives the definitions of different types of crossed complex; explicit examples and morphisms of small free crossed resolutions. We then show how CW-complexes and crossed complexes are related by the fundamental crossed complex functor.

### 4.1.1 Groupoid Modules and Crossed Modules

In this subsection we define groupoid actions, groupoid modules and crossed groupoid modules, which are used to define crossed complexes over groupoids in subsection 4.1.2.

A *right action* of a group $H$ on a set $S$ is a function $S \times H \to X$, $(s, h) \mapsto s^h$ such that $s^1 = s$ and $(s^h)^{h'} = s^{(hh')}$ for all $h,\ h' \in H$. The set $S$ is called a *H-Set*. The right action of a group on a set is the same as a contravariant functor $F : H \to \mathcal{S}ets$ where $H$ is viewed as a groupoid with one object denoted $*$, and $F(*) = S$ for some $S$ in $\mathcal{S}ets$, the value of $F(h)$ is a bijection of $S$, and $s^h = F(h)(s)$.

We can adapt the idea of *H-Sets* to $\mathcal{H}$*-Sets* where $\mathcal{H}$ is a groupoid. A right action of a groupoid on sets is equivalent to a contravariant functor $F : \mathcal{H} \to \mathcal{S}ets$. So if $h \in \mathcal{H}(u, v)$, $F(u) = X_u$ and $F(v) = X_v$, then $F(h)$ is a bijection from $X_u$ to $X_v$

$$u \xrightarrow{h} v \xrightarrow{h'} w \in \mathcal{H} \qquad\qquad F(u) \xrightarrow{F(h)} F(v) \xrightarrow{F(h')} F(w) \in \mathcal{S}ets.$$

For $x \in X_u$, $\left(x^h\right)^{h'} = x^{hh'}$ provided $h,\ h'$ are composable in $\mathcal{H}$. For each object group $\mathcal{H}(u)$ of $\mathcal{H}$, $F$ restricts to an action of $\pi_1(\mathcal{H}, u)$ on $X_u = F(u)$.

Further, we extend the notion of $\mathcal{H}$*-Sets* to $\mathcal{H}$*-Gpds*. A right action of $\mathcal{H}$ on groupoids is a contravariant functor $F : \mathcal{H} \to \mathcal{G}pds$ where for $u \in \mathrm{Ob}(\mathcal{H})$, $F(u) = X_u$ a groupoid, $F(1_u)$ is the identity morphism on $X_u$ and for $h \in \mathcal{H}(u, v)$, $F(h) : X_u \to X_v$ is a groupoid isomorphism and we write $g^h$ for $F(h)(g)$. The action of $u \xrightarrow{h} v \in \mathcal{H}$ on $x \xrightarrow{g} y \in F(u)$ is $x^h \xrightarrow{g^h} y^h \in F(v)$. For each object $u$ of $\mathcal{H}$, $F$ restricts to an action of $\pi_1(\mathcal{H}, u)$ on $F(u)$. We shall only be interested in groupoid actions on totally disconnected groupoids, since these actions occur for crossed modules of groupoids.

**Definition 4.1.1** Suppose $\mathcal{G}$ and $\mathcal{H}$ are groupoids over the same object set and $\mathcal{G}$ is totally disconnected. Then a *groupoid action* of $\mathcal{H}$ on $\mathcal{G}$ is given by a partially defined function

$$\begin{array}{rcl} \mathrm{Arr}(\mathcal{G}) \times \mathrm{Arr}(\mathcal{H}) & \to & \mathrm{Arr}(\mathcal{G}) \\ (g, h) & \mapsto & g^h \end{array}$$

which satisfies, for all $g \in \mathcal{G}(u),\ h \in \mathcal{H}(u, v)$ and $h_1 \in \mathcal{H}(v, w)$,

(i) $g^h \in \mathcal{G}(v)$, and $(1_u)^h = 1_v$,

(ii) $(gg_1)^h = g^h {g_1}^h$, and

(iii) $g^{(hh_1)} = (g^h)^{h_1}$ and $g^1 = g$.

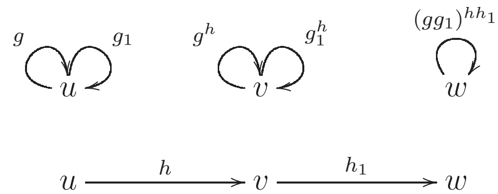



Figure 4.1: Action of $\mathcal{H}$ on $\mathcal{G}$

We call a groupoid action of $\mathcal{H}$ on $\mathcal{G}$ an $\mathcal{H}$-*action* on $\mathcal{G}$. The following two examples are well known group actions and are used to construct a free crossed resolution of a group.

**Example 4.1.2** Given a group $G$ and subgroup $H$ of $G$ the *trivial action* of $H$ on $G$ is given by the function $(g, h) \mapsto g$. $\Diamond$

**Example 4.1.3** Given a normal subgroup $H$ of a group $G$, the *conjugation action* of $H$ on $G$ is given by the function $(g, h) \mapsto h^{-1}gh$. $\Diamond$

The following example is the corresponding conjugation action for groupoids.

**Example 4.1.4** If $\mathcal{G}$ is the largest totally disconnected subgroupoid of a groupoid $\mathcal{H}$ then $\mathcal{H}$ acts on $\mathcal{G}$ by $g^h = h^{-1}gh$. $\Diamond$

We now define $\mathcal{H}$-modules and free $\mathcal{H}$-modules which we will use to construct a free crossed resolution.

**Definition 4.1.5** An $\mathcal{H}$-*module* $(\mathcal{M}, \mathcal{H})$ is a pair of groupoids, where $\mathcal{M}$ is a family of abelian groups $\mathcal{M}(u)$, $u \in \mathrm{Ob}(\mathcal{H})$, together with a specified action of $\mathcal{H}$ on $\mathcal{M}$.

A morphism of groupoid modules is a pair $(\theta, \phi) : (\mathcal{M}, \mathcal{H}) \to (\mathcal{N}, \mathcal{K})$ where $\phi : \mathcal{H} \to \mathcal{K}$ is a morphism of groupoids and $\theta$ is a family of morphisms of abelian groups

$\theta(u) := \mathcal{M}(u) \to N(\phi(u))$ preserving the actions, that is $\theta(v)(m^h) = (\theta(u)(m))^{\phi(h)}$ when $m \in \mathcal{M}(u)$ and $h \in \mathcal{H}(u, v)$.

**Definition 4.1.6** An $\mathcal{H}$-*module* $(\mathcal{F}, \mathcal{H})$ is *free* on generators $X(u)$, $u \in \mathrm{Ob}(\mathcal{H})$ if there exists a map $X(u) \to \mathcal{F}(u)$ for all $u$ and given any $\mathcal{H}$-*module* $(\mathcal{M}, \mathcal{H})$ and maps $f(u) : X(u) \to \mathcal{M}(u)$ for all $u \in \mathrm{Ob}(\mathcal{H})$ there exists a unique morphism $(\mathcal{F}, \mathcal{H}) \to (\mathcal{M}, \mathcal{H})$.

The following example is used in subsection 4.1.3 to give a free crossed resolution of finite cyclic groups.

**Example 4.1.7** If $C_r$ is a cyclic group of order $r$ with generator $x$ then a free $C_r$-module $(F, C_r)$ is a free abelian group of rank $r$. If $F$ is a free group with generator $y$ then $(F, C_r)$ is additively generated by $y \cdot 1, y \cdot x, \ldots, y \cdot x^{r-1}$. We say that $(F, C_r)$ is free on one generator $y$ together with a circular $C_r$-action. $\Diamond$

We recall that for any group $G$, the set $\mathbb{Z}[G]$ denotes all formal sums of the form $\sum\limits_{g \in G} a_g g$ where $a_g \in \mathbb{Z}$ and only finitely many of the $a_g$ are non-zero. The set $\mathbb{Z}[G]$ has a natural structure of a free abelian group with basis the set $|G|$ of elements of $G$. Right multiplication gives an action of $G$ on $\mathbb{Z}[G]$: $(\sum a_g g) \cdot g' = \sum a_g (gg')$. We show there is an isomorphism between a free $C_r$-module and $\mathbb{Z}[C_r]$.

**Proposition 4.1.8** *A free $C_r$-module $(\mathcal{F}, C_r)$, given by a $C_r$-action on a free group $F$ on one generator where $y$ is the free generator of $(\mathcal{F}, C_r)$, is isomorphic to $\mathbb{Z}[C_r]$.*

**Proof** The isomorphism $(\mathcal{F}, C_r) \simeq \mathbb{Z}[C_r]$ is determined by $\sum\limits_{i=0}^{r-1} a_i y \cdot x^i \mapsto \sum\limits_{i=0}^{r-1} a_i x^i$. $\square$

We now define two special types of groupoid module, crossed modules and free crossed modules. For more details on free crossed modules and their construction see Brown and Huebschmann, *Identities among relations* [9].

**Definition 4.1.9** A *crossed module* $\mathfrak{X} = (\mu : \mathcal{G} \to \mathcal{H})$ consists of an $\mathcal{H}$-action on $\mathcal{G}$ and a morphism of groupoids $\mu : \mathcal{G} \to \mathcal{H}$, satisfying

(i) $\mu(g^h) = h^{-1}(\mu g)h$,

(ii) $g^{\mu g_1} = {g_1}^{-1} g g_1$

for all $g, g_1 \in \mathcal{G}$, $h \in \mathcal{H}$ whenever the terms are defined.

**Definition 4.1.10** A morphism of crossed modules $\mathfrak{X}_1 = (\mu_1 : \mathcal{G}_1 \to \mathcal{H}_1)$ to $\mathfrak{X}_2 = (\mu_2 : \mathcal{G}_2 \to \mathcal{H}_2)$ is a pair of groupoid morphisms $(\theta : \mathcal{G}_1 \to \mathcal{G}_2,\ \phi : \mathcal{H}_1 \to \mathcal{H}_2)$ such that $\phi\mu_1 = \mu_2\theta$ and the actions are preserved, $\theta(g^h) = \theta(g)^{\phi(h)}$

Crossed modules allow for free models of the inclusion map $N \to F$ of a normal subgroup of a group and give non-abelian information. We now give examples of crossed modules for groups and groupoids.

**Example 4.1.11** There is a crossed module of groups $(\partial : F \to C_r)$ where $F$ is the free group on the generator $y$, $\partial$ is given by $\partial(y) = x$, and the action of $C_r$ on $F$ is the trivial action $y^x = y$. More generally, any morphism of abelian groups $(\mu : A \to A')$ can be regarded as a crossed module where the action of $A$ on $A'$ is trivial. $\Diamond$

**Example 4.1.12** A conjugation crossed module of groups $(i : N \to G)$ is given by $N$ a normal subgroup of a group $G$, $i$ the inclusion map, and $n^g = g^{-1}ng$.

This generalises as follows. Suppose $\mathcal{H}$ is a connected groupoid and $\mathcal{N}$ is a totally disconnected normal subgroupoid of $\mathcal{H}$. Then the inclusion map $i : \mathcal{N} \to \mathcal{H}$ and the action $n^h = h^{-1}nh$ gives a crossed module of groupoids. $\Diamond$

We now define a free crossed module using graphs and a universal property.

**Definition 4.1.13** Let $\mathcal{H}$ be a groupoid, and $\Gamma$ a totally disconnected graph where $V(\Gamma) = \mathrm{Ob}(\mathcal{H})$. The path groupoid $P\Gamma$ is a totally disconnected union of free groups and the rank of $P\Gamma(u)$ is the number of loops at $u$ in $\Gamma$. A graph morphism $\gamma : \Gamma \to \Gamma\mathcal{H}$ which is the identity on vertices determines an unique groupoid morphism $\gamma : P\Gamma \to \mathcal{H}$ which is the identity on objects. We define the *free crossed $\mathcal{H}$-module* on $\gamma$ to be a crossed module $\mathfrak{X}(\gamma) = (\partial : \mathcal{C}(\gamma) \to \mathcal{H})$, together with a groupoid morphism $\overline{\gamma} : P\Gamma \to \mathcal{C}(\gamma)$ such that

(i) $\partial\overline{\gamma} = \gamma$,

(ii) if $\mathfrak{X} = (\mu : \mathcal{G} \to \mathcal{H})$ is a crossed module and $f : P\Gamma \to \mathcal{G}$ is a groupoid morphism which is the identity on objects such that $\mu f = \gamma$, then there is a unique morphism

$(f', 1) : (X)(\gamma) \to \mathfrak{X}$ of crossed $\mathcal{H}$-modules such that $f'\overline{\gamma} = f$.

$$\begin{tikzcd}
\mathcal{C}(\gamma) \arrow[rr, "f'"] \arrow[dr, "\partial"'] & & \mathcal{G} \arrow[dl, "\mu"] \\
& P\Gamma \arrow[ul, "\overline{\gamma}"'] \arrow[ur, "f"] \arrow[d, "\gamma"] & \\
& \mathcal{H} &
\end{tikzcd}$$

We now give examples of free crossed modules for groups and groupoids.

**Example 4.1.14** If $H$ is a free group on $Y$ then the identity crossed module $(1 : H \to H)$ is a free crossed module on $Y$. The action is essentially conjugation. $\diamondsuit$

**Example 4.1.15** If $(\delta : F \to G)$ is a free crossed G-module on $X \to G$ where $G$ is a group then $(\delta' : F \times \{0,1\} \to G \times \mathcal{I})$ is a free crossed $G \times \mathcal{I}$-module on $X \times \Gamma \to G \times \mathcal{I}$ where $\Gamma$ consists of two identity loops. $\diamondsuit$

**Example 4.1.16** If $\langle X : R\rangle$ presents the group $G$ then we get a free crossed $F(X)$-module $(\delta_2 : C(R) \to F(X))$. The group $C(R)$ is generated as an $F(X)$-group by the set $R$ and its elements are of the form $c = \prod_{i=1}^{n} (r_i^{\varepsilon_i})^{u_i}$ where $n \geqslant 0$, $r_i \in R$, and $\varepsilon = \{+,-\}$. The morphism $\delta_2$ is defined by $\delta_2(r^{\varepsilon})^u = u^{-1}\phi(r^{\varepsilon})u$ subject to the crossed module rule $c^{\delta_2(c_1)} = c_1^{-1}cc_1$ for $c,\ c_1 \in C(R)$. $\diamondsuit$

The following result is used in Subsection 4.1.3 defining a small free crossed resolution which then makes computations neater.

**Lemma 4.1.17** *If $(\mu : M \to P)$ is a free crossed $P$-module on one generator $m$ and $P$ is abelian, then $M$ is also abelian and is acted on trivially by $\mu(M)$.*

**Proof** As a group, $M$ is generated by elements $m^p$ for all $p \in P$. But $m^p m^q = m^{(p+q)} = m^{(q+p)} = m^q m^p$. Hence $M$ is abelian. So if $n \in M$, then $m^{\mu(n)} = n^{-1}mn = m$ and $\mu(n)$ acts trivially on $m$ and so on $M$. $\square$

### 4.1.2 Crossed Complexes

Crossed complexes are similar to chain complexes but can hold non-abelian information. The concepts of morphisms and homotopies of crossed complexes are modelled on the chain complex analogues. For more details on connections between chain and crossed complexes we refer the reader to Brown and Higgins, *Crossed Complexes and Chain Complexes with Operators* [7].

In this subsection we give the definition of a crossed complex and properties of crossed complexes. We also define the tensor product of crossed complexes which will be used in for constructing the total crossed complex of a graph of crossed complexes in section 4.2.

**Definition 4.1.18** A *crossed complex* $(\mathcal{C}, \chi)$ (over a groupoid) is a sequence

$$\cdots \xrightarrow{\chi_{n+1}} \mathcal{C}_n \xrightarrow{\chi_n} \mathcal{C}_{n-1} \xrightarrow{\chi_{n-1}} \cdots \xrightarrow{\chi_3} \mathcal{C}_2 \xrightarrow{\chi_2} \mathcal{C}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C}_0$$

given by

1. a crossed module of groupoids $(\chi_2 : \mathcal{C}_2 \to \mathcal{C}_1)$ with object set $\mathcal{C}_0$, and
2. for $n \geqslant 3$, $\mathcal{C}_1$-modules $\mathcal{C}_n$ such that the image of $\chi_2$ acts trivially on $\mathcal{C}_n$, and
3. for $n \geqslant 3$ the $\mathcal{C}_1$-morphisms $\chi_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ are $\mathcal{C}_1$-operator morphisms which satisfy $\chi_{n-1}\chi_n = 0$.

The arrows of the groupoid $\mathcal{C}_n$ are said to be in *dimension* $n$ and we write $\chi$ for $\chi_n$ when the dimension is clear.

For $n \geqslant 2$, $\mathcal{C}_n$ is a family of groups $\{\mathcal{C}_n(u)\}_{u \in \mathcal{C}_0}$ and for $n \geqslant 3$ the groups $\mathcal{C}_n(u)$ are abelian. We use additive notation for all groups $\mathcal{C}_n(u)$ and the groupoid $\mathcal{C}_1$ and denote the action of $h \in \mathcal{C}_1$ on $g \in \mathcal{C}_n$ by $g^h$.

A *morphism of crossed complexes* $f : (\mathcal{C}, \chi) \to (\mathcal{D}, \delta)$ is a family of morphisms of groupoids $f_n : \mathcal{C}_n \to \mathcal{D}_n$ for $n \geqslant 0$, compatible with the $\mathcal{C}_1-$ and $\mathcal{D}_1-$ morphisms and the $\mathcal{C}_1$, $\mathcal{D}_1$ actions: $\delta_n f_n = f_{n-1}\chi_n$ and $f_n(g_n^h) = f_n(g_n)^{f_1(h)}$.

The category of crossed complexes, $\mathcal{C}rs$, has objects all crossed complexes and arrows all morphisms of crossed complexes.

An $m$-*truncated* crossed complex $(\mathcal{C}, \chi)$ consists of all the structure for a crossed complex but only for $n \leqslant m$. An $m$-truncated crossed complex for $m = 0,\ 1$ and 2 is a set, a groupoid, and a crossed module respectively. To turn an $m$-truncated crossed complex into a crossed complex we let $\mathcal{C}_n$ be a trivial $\mathcal{C}_1$-module for $n > m$.

As for groupoids we have an unit interval object in the category of crossed complexes which we will use to define a cylinder in the category of crossed complexes.

**Example 4.1.19** The unit interval crossed complex $\mathcal{I} := (\mathcal{I}, i)$ is determined by the groupoid $\mathcal{I}$ in dimensions 0 and 1 and $\mathcal{I}_n$ for $n \geqslant 2$ consists of two objects and their identity arrows. The morphisms $i_n$ for $n \geqslant 2$ are identity morphisms. $\Diamond$

The *fundamental groupoid* $\pi_1(\mathcal{C}, \chi)$ *of a crossed complex* $(\mathcal{C}, \chi)$ is the quotient of the groupoid $\mathcal{C}_1$ by the normal totally disconnected subgroupoid $\chi_2\mathcal{C}_2$. By definition of a crossed complex, $\mathcal{C}_n$ for $n \geqslant 3$ has the induced structure of a $\pi_1(\mathcal{C}, \chi)$-module.

A crossed complex is *free* if $\mathcal{C}_1$ is a free groupoid on some graph $\Gamma_1$, $\mathcal{C}_2$ is a free crossed $\mathcal{C}_1$-module for some $\gamma : \Gamma_2 \to \mathcal{C}_1$, and for $n \geqslant 3$, $\mathcal{C}_n$ is a free $\pi_1\mathcal{C}$-module on some graph $\Gamma_n$. A crossed complex $(\mathcal{C}, \chi)$ is *exact* if for $n \geqslant 2$, $\mathrm{Ker}(\chi_n) = \mathrm{Im}(\chi_{n+1})$.

If $(\mathcal{C}, \chi)$ is exact and $\mathcal{G}$ is a groupoid, then $(\mathcal{C}, \chi)$ together with an isomorphism $\pi_1(\mathcal{C}, \chi) \to \mathcal{G}$ is called a *crossed resolution* of $\mathcal{G}$. It is called a *free crossed resolution* of $\mathcal{G}$ if $(\mathcal{C}, \chi)$ is also free.

We now define the tensor product of crossed complexes constructed by Brown and Higgins [6].

**Definition 4.1.20** If $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are crossed complexes, then $(\mathcal{C}, \chi) = (\mathcal{A}, \alpha) \otimes (\mathcal{B}, \beta)$ is the crossed complex generated by elements $a \otimes b$ in dimension $m + n$, where $a \in \mathcal{A}_m$, $b \in \mathcal{B}_n$, with the following defining relations:

1. $ta \otimes tb = t(a \otimes b)$.

2. $(a \otimes b)^{ta \otimes b_1} = a \otimes b^{b_1}$ if $m \geqslant 0,\ n \geqslant 2,\ b_1 \in \mathcal{B}_1$.

3. $a^{a_1} \otimes b = (a \otimes b)^{a_1 \otimes tb}$ if $m \geqslant 2,\ n \geqslant 0,\ a_1 \in \mathcal{A}_1$.

4. If $b + b'$ is defined in $\mathcal{B}_n$, then
$$a \otimes (b + b') = \begin{cases} a \otimes b + a \otimes b' & \text{if } m = 0,\ n \geqslant 1 \text{ or if } m \geqslant 1,\ n \geqslant 2 \\ (a \otimes b)^{ta \otimes b'} + a \otimes b' & \text{if } m \geqslant 1,\ n = 1. \end{cases}$$

5. If $a + a'$ is defined in $\mathcal{A}_m$, then

$$(a + a') \otimes b = \begin{cases} a \otimes b + a' \otimes b & \text{if } m \geqslant 1,\ n = 0 \text{ or if } m \geqslant 2,\ n \geqslant 1 \\ a' \otimes b + (a \otimes b)^{a' \otimes tb} & \text{if } m = 1,\ n \geqslant 1. \end{cases}$$

(The reversal of addition is significant only when $m = n = 1$.)

6. $\chi(a \otimes b) =$

$$\begin{cases} \alpha a \otimes b + (-1)^m (a \otimes \beta b) & \text{if } m \geqslant 2,\ n \geqslant 2, \\ -(a \otimes \beta b) - (ta \otimes b) + (sa \otimes b)^{a \otimes tb} & \text{if } m = 1,\ n \geqslant 2, \\ (-1)^{m+1}(a \otimes tb) + (-1)^m (a \otimes sb)^{ta \otimes b} + \alpha a \otimes b & \text{if } m \geqslant 2,\ n = 1, \\ -ta \otimes b - a \otimes sb + sa \otimes b + a \otimes tb & \text{if } m = 1,\ n = 1, \\ a \otimes \beta b & \text{if } m = 0,\ n \geqslant 2, \\ \alpha a \otimes b & \text{if } m \geqslant 2,\ n = 0 \end{cases}$$

$$s(a \otimes b) = \begin{cases} a \otimes sb & \text{if } m = 0,\ n = 1, \\ sa \otimes b & \text{if } m = 1,\ n = 0 \end{cases}$$

$$t(a \otimes b) = \begin{cases} a \otimes tb & \text{if } m = 0,\ n = 1, \\ ta \otimes b & \text{if } m = 1,\ n = 0. \end{cases}$$

We now give an example of a tensor product of crossed complexes which we will use to define a cylinder and homotopy of crossed complexes.

**Example 4.1.21** The crossed complex $A \otimes \mathcal{I} := (\mathcal{A}, \alpha) \otimes (\mathcal{I}, i)$ is generated by elements $a_{n-1} \otimes \iota$, $a_n \otimes 0$ and $a_n \otimes 1$ in dimension $n$. The morphisms $\delta_n : (A \otimes \mathcal{I})_n \to (A \otimes \mathcal{I})_{n-1}$ are defined as follows:

$$\begin{aligned} \delta_n(a_n \otimes 0) &= \alpha_n a_n \otimes 0 \\ \delta_n(a_n \otimes 1) &= \alpha_n a_n \otimes 1 \\ \delta_n(a \otimes \iota) &= \begin{cases} a_{n-1} \otimes 1 - a_{n-1} \otimes 0 & \text{if } n = 2 \\ -a_{n-1} \otimes 1 + (a_{n-1} \otimes 0)^{*_A \otimes \iota} + \alpha_{n-1} a_{n-1} \otimes \iota & \text{if } n \text{ odd} \\ a_{n-1} \otimes 1 - (a_{n-1} \otimes 0)^{*_A \otimes \iota} + \alpha_{n-1} a_{n-1} \otimes \iota & \text{if } n > 2 \text{ and even} \end{cases} \end{aligned}$$

$\Diamond$

For work on graphs of crossed complexes we need a crossed complex cylinder $C \otimes \mathcal{I}$, and $\mathcal{I}$ in the category of crossed complexes is used to denote the crossed complex of the unit groupoid $\mathcal{I}$ see Example 4.1.19.

**Definition 4.1.22** A *cylinder* in the category of crossed complexes can be defined by the functor

$$M(\mathcal{C}) : \mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{I}$$

together with the natural transformations

$$\begin{aligned} i_0 : \mathcal{C} \to \mathcal{C} \otimes \mathcal{I} \qquad & i_0(c) = (c, 0) \\ i_1 : \mathcal{C} \to \mathcal{C} \otimes \mathcal{I} \qquad & i_1(c) = (c, 1) \end{aligned}$$

and $\sigma : \mathcal{C} \times \mathcal{I} \to \mathcal{C}$ is the projection onto $\mathcal{C}$.

We recall from Chapter 3 that homotopies of morphisms is defined using cylinders.

**Definition 4.1.23** A *homotopy* $h : f \simeq g$ of crossed complex morphisms $f, g : (\mathcal{C}, \chi) \to (\mathcal{D}, \delta)$ is given by a crossed complex morphism $h : (\mathcal{C}, \chi) \otimes (\mathcal{I}, i) \to (\mathcal{D}, \delta)$ such that the following diagram commutes.

$$\begin{tikzcd} & \mathcal{C}, \chi) \arrow[dl, "i_0"'] \arrow[dr, "f"] & \\ (C, \chi) \otimes (\mathcal{I}, i) \arrow[rr, "h"] & & (\mathcal{D}, \delta) \\ & (\mathcal{C}, \chi) \arrow[ul, "g"] \arrow[ur, "i_1"'] & \end{tikzcd}$$

**Proposition 4.1.24** *Specifying a homotopy $h : f \sim g$ is equivalent to specifying the morphism $g$ together with a map $\phi_n : \mathcal{C}_n \to \mathcal{D}_{n+1}$ which satisfies the following.*

*(i)* $t(\phi_0 c_0) = g c_0$

*(ii)* $t(\phi_n c_n) = t(g c_n)$ *for* $n \geqslant 1$

*(iii)* $\phi_n(c_n \cdot c_1) = (\phi_n c_n) \cdot g c_1$ *for* $n \geqslant 2$

*(iv)* $\phi_1(c_1 + c_1') = (\phi_1 c_1) \cdot g c_1 + \phi_1 c_1'$

*(v)* $\phi_n(c_1 + c_n') = \phi_n c_n) + \phi_n c_n'$ *for* $n \geqslant 2$.

*The morphism* $f$ *is then completely determined by*

$$\begin{aligned} s(\phi_0 c_0) &= f c_0 \\ \delta_2(\phi_1 c_1) &= (g c_1)^{-1} + (\phi_0 s c_1)^{-1} + f c_1 + \phi_0 t c_1 \\ \delta_{n+1}(\phi_n c_n) &= (g c_n)^{-1} + (f c_n) \cdot \phi_0 t c_n + (\phi_{n-1} \delta_n c_n)^{-1} \text{ for } n \geqslant 2 \end{aligned}$$

**Definition 4.1.25** Two crossed complexes are *homotopy equivalent* if there exists crossed complex morphisms $f : (\mathcal{C}, \chi) \to (\mathcal{D}, \delta)$ and $g : (\mathcal{D}, \delta) \to (\mathcal{C}, \chi)$ together with homotopies $h : fg \simeq Id_{\mathcal{C}}$ and $k : gh \simeq Id_{\mathcal{D}}$.

**Proposition 4.1.26** *Given free crossed resolutions* $(\mathcal{C}, \chi)$ *and* $(\mathcal{D}, \delta)$ *of two groups* $G$ *and* $H$*, the tensor product* $(\mathcal{C}, \chi) \otimes (\mathcal{D}, \delta)$ *gives a free crossed resolution of their product* $G \times H$.

We refer the reader to the proof of Theorem 3.1.5.*Theory and Application of Crossed Complexes* [24] for the proof.

### 4.1.3 Free Crossed Resolutions

In this subsection we will give a small free crossed resolution $\mathcal{F}_{C_r}$ of the cyclic group $C_r$ of order $r$. We will then consider a corresponding free crossed resolution $\mathcal{F}_{C_{rl}}$ of the cyclic group of order $rl$ and construct a morphism of crossed complexes from $\mathcal{F}_{C_r}$ to $\mathcal{F}_{C_{rl}}$. The aim of this subsection is to move away from the abstract definitions and to exploit the computational features of crossed complexes.

We give a free crossed resolution $(\mathcal{F}, \phi)$ over a group $G = \langle X \mid R \rangle$ by considering the following diagram of a morphism of a free crossed resolution to a crossed complex. We omit the object set $\mathcal{C}_0$ of a crossed complex when working with groups.

$$\begin{tikzcd} \cdots \arrow[r, "\delta_6"] & \mathcal{F}_5 \arrow[r, "\phi_5"] \arrow[d, "0"] & \mathcal{F}_4 \arrow[r, "\phi_4"] \arrow[d, "0"] & \mathcal{F}_3 \arrow[r, "\phi_3"] \arrow[d, "0"] & \mathcal{F}_2 \arrow[r, "\phi_2"] \arrow[d, "0"] & \mathcal{F}_1 \arrow[d, "\phi^*"] \\ \cdots \arrow[r, "1"] & 1 \arrow[r, "1"] & 1 \arrow[r, "1"] & 1 \arrow[r, "1"] & 1 \arrow[r, "1"] & G \end{tikzcd}$$

We let $\mathcal{F}_1$ be the free group on the set of generators of $G$ and $\phi^*$ is the identity on generators. We choose $\phi_2 : \mathcal{F}_2 \to \mathcal{F}_1$ to be the free crossed module of the presentation and we can recover the group $G$ from the top resolution by the quotient $\mathcal{F}_1/\phi_2(\mathcal{F}_2)$ which is isomorphic to $G$.

Given free crossed resolutions $\mathcal{F}_G$ and $\mathcal{F}_H$ of groups $G$ and $H$ and a group homomorphism $\mathrm{f} : G \to H$, then f can be lifted to a morphism $f : \mathcal{F}_G \to \mathcal{F}_H$ which is unique up to homotopy. The following result is well known but given for completeness.

**Theorem 4.1.27** *Let $(\mathcal{C}, \chi)$ be a free crossed complex of the group $G$, let $(\mathcal{D}, \delta)$ be an exact crossed complex of the group $H$, and let $\alpha : G \to H$ be a group homomorphism. Then there exists a morphism of crossed complexes $k : (\mathcal{C}, \chi) \to (\mathcal{D}, \delta)$ such that $\delta^* k_1 = \alpha\chi^*$ where $\delta^* : \mathcal{D}_1 \to H$ and $\chi^* : \mathcal{C}_1 \to G$ and any two such morphisms of crossed complexes is unique up to homotopy.*

**Proof** We first define $k_1 : \mathcal{C}_1 \to \mathcal{D}_1$. The group $\mathcal{C}_1$ is free on $X_1$ say and $\chi^*$ is surjective, so for each $x \in X_1$ choose $k_1(x)$ such that $\delta^* k_1(x) = \alpha\chi^*(x)$. Then extend to a morphism.

The group $\mathcal{C}_2$ is a free crossed $\mathcal{C}_1$-module on ${\chi_2}' : X_2 \to \mathcal{C}_1$. If $x \in X_2$ then $\delta^* k_1 \chi_2(x) = \alpha\chi^*\chi_2(x) = 1$. By exactness $k_1\chi_2(x) \in \mathrm{Im}\delta_2$.

We choose $k_2(x) \in \mathcal{D}_2$ such that $\delta_2 k_2(x) = k_1\chi_2(x)$. Since $\delta_2 : \mathcal{D}_2 \to \mathcal{D}_1$ is a crossed module and $\mathcal{C}_2 \to \mathcal{C}_1$ is free on $X_2$ then we extend $k_2$ on $X_2$ uniquely to a morphism $k_2 : \mathcal{C}_2 \to \mathcal{D}_2$ such that $\delta_2 k_2 = k_1\chi_2$.

For $n \geqslant 2$ we consider the following diagram.

$$\begin{array}{ccccc}
\mathcal{C}_{n+1} & \xrightarrow{\chi_{n+1}} & \mathcal{C}_n & \xrightarrow{\chi_n} & \mathcal{C}_{n-1} \\
\downarrow{\scriptstyle k_{n+1}} & & \downarrow{\scriptstyle k_n} & & \downarrow{\scriptstyle k_{n-1}} \\
\mathcal{D}_{n+1} & \xrightarrow{\delta_{n+1}} & \mathcal{D}_n & \xrightarrow{\delta_n} & \mathcal{D}_{n-1}
\end{array}$$

We know that $\delta_{n-1}k_{n-1}\chi_n(x_n) = 0$. By exactness $k_{n-1}\chi_n(x_n) = \delta_n(y_n)$ for some $y_n$. Let $k_n x_n = y_n$. By freeness this defines $k_n$. As $x_n$ generates $\mathcal{C}_n$, $\delta_n k_n = k_{n-1}\chi_n$.

So suppose $k, l : (\mathcal{C}, \chi) \to (\mathcal{D}, \delta)$ such that $\delta^* k_1 = \alpha\chi^*$ and $\delta^* l_1 = \alpha\chi^*$. Then there exists a homotopy $h : k \simeq l$.

Recall $X_1$ is the generating set for $\mathcal{C}_1$. By properties of morphisms of crossed complexes, for all $x \in X_1$, $\psi f_1 x = f_0\phi x = \psi g_1 x$. Therefore $\psi(f_1 x - g_1 x) = 0$ and there exists $h_1x \in \mathcal{D}_2$ such that $\delta_2 h_1 x = f_1x - g_1x$. Since $\mathcal{C}_1$ is a free group there exists a unique $g_1$-derivation $h_1 : \mathcal{C}_1 \to \mathcal{D}_2$ such that $\delta_2 h_1 c_1 = f_1 c_1 - g_1 c_1$ for all $c_1 \in \mathcal{C}_1$.

Let $x \in X_2$ where $X_2$ is a generating set for $\mathcal{C}_2$ and $\delta_3 h_2 x = f_2x - g_2x - h_1\chi_2 x$. Then there exists an $h_2x$ such that $\delta_2(f_2x - g_2x - h_1\chi_2x) = 0$.

$$\begin{aligned}
\delta_2\delta_3 h_2 x &= \delta_2 f_2 x - \delta_2 g_2 x - \delta_2 h_1 \chi_2 x \\
&= f_1\chi_2 x - g_1\chi_2 x - \delta_2 h_1\chi_2 x \\
&= f_1\chi_2 x - g_1\chi_2 x + g_1\chi_2 x - f_1\chi_2 x \\
&= 0.
\end{aligned}$$

For $x \in X_n$ $(n > 2)$ where $X_n$ is a generating set for $\mathcal{C}_n$ and $\delta_{n+1} h_n x = f_n x - g_n x - h_{n-1}\chi_n x$. Then there exists an $h_n x$ such that $\delta_n(f_n x - g_n x - h_{n-1}\chi_n x) = 0$.

$$\begin{aligned}
\delta_n\delta_{n+1} h_n x &= \delta_n f_n x - \delta_n g_n x - \delta_n h_{n-1}\chi_n x \\
&= f_{n_1}\chi_n x - g_{n-1}\chi_n x + h_{n-2}\chi_{n-1}\chi_n x + g_{n-1}\chi_n x - f_{n-1}\chi_n x \\
&= 0 \text{ since } h_{n-2}\chi_{n-1}\chi_n x = 0.
\end{aligned}$$

$\square$

We will illustrate the above process by lifting the injective morphism $\mathrm{f} : C_r \to C_{rm}$ of cyclic groups to a morphism of small free crossed resolutions of finite cyclic groups. This is needed for homotopy pushout calculations.

A free crossed resolution for finite cyclic groups is given by Brown and Wensley [10] which we now describe.

A free crossed resolution $\mathcal{F}_{C_r} := (\mathcal{A}, \alpha)$ of the group $C_r$ is given by a sequence

$$\cdots \xrightarrow{\alpha_{n+1}} \mathcal{A}_n \xrightarrow{\alpha_n} \mathcal{A}_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_4} \mathcal{A}_3 \xrightarrow{\alpha_3} \mathcal{A}_2 \xrightarrow{\alpha_2} \mathcal{A}_1$$

where each $\mathcal{A}_n$ is free on one generator $a_n$, say.

We choose $C_r$ to have group presentation $\langle a \mid a^r = 1\rangle$, $\mathcal{A}_1 := \langle a_1 \rangle$ to be the free group on one generator $a_1$, and $\alpha^* : \mathcal{A}_1 \to C_r$ to be defined by $\alpha^*(a_1) = a$.

The crossed $\mathcal{A}_1$-module $\mathcal{A}_2$ is free on one generator $a_2$ with $\alpha_2(a_2) = a_1^r$. By Lemma 4.1.17 $\mathcal{A}_2$ is also a free $C_r$-module on the generator $a_2$. Thus in dimensions $n \geqslant 2$, we have an isomorphism of $C_r$-modules $\mathcal{A}_n \simeq \mathbb{Z}[C_r]$, but it is convenient to keep the separate notation. We denote the $\mathcal{A}_1$-action on an element $a_n \in \mathcal{A}_n$ by $a_n \cdot a_1$ An element of $\mathcal{A}_n$ is of the form $a_n \cdot z$ where $z \in \mathbb{Z}[C_r]$ and so $\mathcal{A}_n$ is additively generated by $a_n, a_n \cdot a, \ldots, a_n \cdot a^{r-1}$. The group $\mathcal{A}_1$ acts on $\mathcal{A}_n$ by its image under $\alpha^*$ so an element $a^j$ of $C_r$ acts on the generators $a_n \cdot a^i$ of $\mathcal{A}_n$ by cyclically permuting them, $(a_n \cdot a^i) \cdot a^j = a_n \cdot a^{(i+j)}$. We set $\alpha_2(a_2) = a_1^r$ and $\alpha_n : \mathcal{A}_n \to \mathcal{A}_{n-1}$ for $n > 2$ is given by

$$\alpha_n(a_n) = \begin{cases} a_{n-1} \cdot (a-1) & \text{for } n \text{ odd} \\ a_{n-1} \cdot N_r(a) & \text{for } n \text{ even and } n \geqslant 4. \end{cases}$$

where $N_r(a) := 1 + a + \cdots + a^{r-1}$ and $\alpha_n(a_n \cdot a^i) = \alpha_n(a_n) \cdot a^i$. We note that $N_r(a) \cdot (a-1) = 0$ which we use to check the crossed complex $\mathcal{F}_{C_r}$ is exact.

An element of $\mathcal{A}_i$ can be represented by $\sum\limits_{i=0}^{r-1} x_i a^i$. To check $\mathrm{Im}\alpha_{n+1} = \mathrm{Ker}\alpha_n$ we have two cases where $n$ is odd and $n$ is even.

For $n$ odd, the kernel of $\alpha_n$ is determined by,

$$\begin{aligned} \alpha_n(\sum_{i=0}^{r-1} x_i a^i) &= \sum_{i=0}^{r-1} x_i a^i \cdot (a-1) \\ &= (x_0 + x_1 a + \cdots + x_{r-1} a^{r-1}) \cdot (a-1) \\ &= (x_{r-1} - x_0) + (x_0 + x_1)a + \cdots + (x_{r-2} - x_{r-1})a^{r-1}. \end{aligned}$$

For the last equation to equal 0, all the $x_i$ equal the same value. Hence the kernel of $\alpha_n = lN_r(a)$ for all $l \in \mathbb{Z}$.

The image of $\alpha_{n+1}$ is given by,

$$\begin{aligned} \alpha_{n+1}(\sum_{i=0}^{r-1} x_i a^i) &= \sum_{i=0}^{r-1} x_i a^i \cdot N_r(a) \\ &= x_0 + x_1 a + \cdots + x_{r-1} a^{r-1} \\ &\quad + x_{r-1} + x_0 a + \cdots + x_{r-2} a^{r-1} + \cdots \\ &\quad + x_1 + x_2 a + \cdots + x_0 a^{r-1} \\ &= k(1 + a + a^2 + \cdots + a^{r-1}) \text{ where } k = (x_0 + \cdots + x_{r-1}) \\ &= kN_r(a) \end{aligned}$$

which gives the result $\mathrm{Im}(\alpha_{n+1}) = \mathrm{Ker}(\alpha_n)$ where $n$ is odd. Similarly for $n$ even.

We call this free crossed resolution of the cyclic group $C_r$ the *small free crossed resolution* of $C_r$.

Similarly the small free crossed resolution $\mathcal{F}_{C_{rl}} := (\mathcal{B}, \beta)$ of the group $C_{rl}$ is given by the sequence

$$\cdots \xrightarrow{\beta_{n+1}} \mathcal{B}_n \xrightarrow{\beta_n} \mathcal{B}_{n-1} \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_4} \mathcal{B}_3 \xrightarrow{\beta_3} \mathcal{B}_2 \xrightarrow{\beta_2} \mathcal{B}_1,$$

where $C_{rl} := \langle b \mid b^{rl} = 1 \rangle$, where $\mathcal{B}_n$ for $n \geqslant 2$ is generated by $b_n$ as a free $\mathbb{Z}[C_{rl}]$-module, and morphisms $\beta_n$ are defined on generators as follows:

$$\beta_n(b_n) = \begin{cases} b_1^{rl} & \text{for } n = 2 \\ b_{n-1} \cdot (b-1) & \text{for } n \text{ odd} \\ b_{n-1} \cdot N_{rl}(b) & \text{for } n \text{ even and } n \geqslant 4. \end{cases}$$

Given small free crossed resolutions $\mathcal{F}_{C_r}$ and $\mathcal{F}_{C_{rl}}$ for $C_r$ and $C_{rl}$ and the inclusion $\mathrm{f} : C_r \to C_{rl}$ defined by $\mathrm{f}(a) = b^l$ we can construct a morphism $f : (\mathcal{A}, \alpha) \to (\mathcal{B}, \beta)$ of free crossed resolutions and label the family of morphisms $f_i$ mapping an element in dimension $i$ of $\mathcal{F}_{C_r}$ to an element of dimension $i$ in $\mathcal{F}_{C_{rl}}$. We consider the following two diagrams where we use the notation $\mathbb{Z}[C_r]$ and $\mathbb{Z}[C_{rl}]$ in the resolutions as this is more suggestive.

$$\begin{tikzcd} \cdots \arrow[r, "\alpha_6"] & \mathbb{Z}[C_r] \arrow[r, "\alpha_5"] \arrow[d, "f_5"] & \mathbb{Z}[C_r] \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & \mathbb{Z}[C_r] \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & \mathbb{Z}[C_r] \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & \mathcal{A}_1 \arrow[d, "f_1"] \\ \cdots \arrow[r, "\beta_6"] & \mathbb{Z}[C_{rl}] \arrow[r, "\beta_5"] & \mathbb{Z}[C_{rl}] \arrow[r, "\beta_4"] & \mathbb{Z}[C_{rl}] \arrow[r, "\beta_3"] & \mathbb{Z}[C_{rl}] \arrow[r, "\beta_2"] & \mathcal{B}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \arrow[r, "\alpha^*"] \arrow[d, "f_1"] & C_r \arrow[d, "\mathrm{f}"] \\ \mathcal{B}_1 \arrow[r, "\beta^*"] & C_{rl} \end{tikzcd}$$

The morphism $f_1$ on the free groups should make the second diagram commute. Since $\mathrm{f}\alpha^*(a_1) = \mathrm{f}(a) = b^l$ and $\beta^*(b_1) = b$ we choose $f_1(a_1) = b_1^l$.

To construct the morphisms $f_i$ we use the conditions of a crossed complex morphism that squares commute, $f_{n-1}\alpha_n = \beta_n f_n$, and the action conditions are preserved.

**Theorem 4.1.28** *The inclusion* $\mathrm{f} : C_r \to C_{rl}$*, given by* $\mathrm{f}(a) = b^l$ *lifts to a morphism of small free crossed resolutions of groups* $C_r$ *and* $C_{rl}$ *defined on generators by*

$$f_n(a_n) \quad = \quad \begin{cases} b_1^l & \text{for } n = 1 \\ b_n & \text{for } n \text{ even} \\ b_n \cdot N_l(b) & \text{for } n \geqslant 3 \text{ and } n \text{ odd.} \end{cases}$$

**Proof** We first check the action conditions and note that $f_n(a_n \cdot a^i) = f_n(a_n) \cdot \mathrm{f}(a^i)$. In calculating $a^p$ and $b^q$ we regard $p$ as in $\mathbb{Z}_r$ and $q$ as in $\mathbb{Z}_{rl}$.

$$\begin{aligned} f_n(a_n \cdot a^i \cdot a^j) &= f_n(a_n \cdot a^{i+j}) \\ &= f_n(a_n) \cdot \mathrm{f}(a^{i+j}) \\ &= f_n(a_n) \cdot b^{l(i+j)} \end{aligned}$$

$$\begin{aligned} f_n(a_n \cdot a^i) \cdot \mathrm{f}(a^j) &= f_n(a_n) \cdot \mathrm{f}(a^i) \cdot \mathrm{f}(a^j) \\ &= f_n(a_n) \cdot b^{li} \cdot b^{lj} \\ &= f_n(a_n) \cdot b^{l(i+j)} \end{aligned}$$

We now check the squares commute. We begin by checking the first square $f_1\alpha_2 = \beta_2 f_2$.

$$\begin{aligned} f_1\alpha_2(a_2 \cdot a^i) &= f_1(\alpha_2(a_2) \cdot a^i) \\ &= f_1(a_1^r \cdot a^i) \\ &= f_1(a_1^r) \cdot \mathrm{f}(a^i) \\ &= b_1^{rl} \cdot b^{il} \end{aligned} \qquad \begin{aligned} \beta_2 f_2(a_2 \cdot a^i) &= \beta_2(f_2(a_2) \cdot \mathrm{f}(a^i)) \\ &= \beta_2(b_2 \cdot b^{il}) \\ &= \beta_2(b_2) \cdot b^{il} \\ &= b_1^{rl} \cdot b^{il} \end{aligned}$$

We now consider the following two squares for the general cases $f_{n-1}\alpha_n = \beta_n f_n$ where $n > 2$.

$$\begin{tikzcd} \mathbb{Z}[C_r] \arrow[r, "\cdot(a-1)"] \arrow[d, "\cdot N_l(b)"] & \mathbb{Z}[C_r] \arrow[r, "\cdot N_r(a)"] \arrow[d, "1"] & \mathbb{Z}[C_r] \arrow[d, "\cdot N_l(b)"] \\ \mathbb{Z}[C_{rl}] \arrow[r, "\cdot(b-1)"] & \mathbb{Z}[C_{rl}] \arrow[r, "\cdot N_{rl}(b)"] & \mathbb{Z}[C_{rl}] \end{tikzcd}$$

For $n$ odd we have the following checks.

$$\begin{aligned}
f_{n-1}\alpha_n(a_n \cdot a^i) &= f_{n-1}(\alpha_n(a_n) \cdot a^i) \\
&= f_{n-1}(a_{n-1} \cdot (a-1) \cdot a^i) \\
&= f_{n-1}(a_{n-1}) \cdot \mathrm{f}(a-1) \cdot \mathrm{f}(a^i) \\
&= b_{n-1} \cdot (b^l - 1) \cdot b^{il}
\end{aligned}$$

$$\begin{aligned}
\beta_n f_n(a_n \cdot a^i) &= \beta_n(f_n(a_n) \cdot \mathrm{f}(a^i)) \\
&= \beta_n(b_n \cdot N_l(b) \cdot b^{il}) \\
&= \beta_n(b_n) \cdot N_l(b) \cdot b^{il} \\
&= b_{n-1} \cdot (b-1) \cdot N_l(b) \cdot b^{il} \\
&= b_{n-1} \cdot (b^l - 1) \cdot b^{il}
\end{aligned}$$

Similarly for $n$ even we have the following checks.

$$\begin{aligned}
f_{n-1}\alpha_n(a_n \cdot a^i) &= f_{n-1}(\alpha_n(a_n) \cdot a^i) \\
&= f_{n-1}(a_{n-1} \cdot N_r(a) \cdot a^i) \\
&= f_{n-1}(a_{n-1}) \cdot \mathrm{f}(N_r(a)) \cdot \mathrm{f}(a^i) \\
&= b_{n-1} \cdot N_l(b) \cdot N_r(b^l) \cdot b^{il} \\
&= b_{n-1} \cdot N_{rl}(b) \cdot b^{il}
\end{aligned}$$

$$\begin{aligned}
\beta_n f_n(a_n \cdot a^i) &= \beta_n(f_n(a_n) \cdot \mathrm{f}(a^i)) \\
&= \beta_n(b_n \cdot b^{il}) \\
&= \beta_n(b_n) \cdot b^{il} \\
&= b_{n-1} \cdot N_{rl}(b) \cdot b^{il}
\end{aligned}$$

□

### 4.1.4 Fundamental Crossed Complexes

In this subsection we define the fundamental crossed complex functor which provides the link between CW and crossed complexes. To obtain relationships between CW and crossed complexes we will consider the following categories and functors.

$$\begin{tikzcd}
\mathcal{C}rs \arrow[rr, bend left, "B"] \arrow[dr, bend left, "N"'] & & \mathcal{T}op \arrow[ll, bend left, "\pi"] \\
& \mathcal{S}impSet \arrow[ur, "|\ |"'] &
\end{tikzcd}$$

The motivating example for crossed complexes is given by the fundamental crossed complex functor $\pi$. The *fundamental crossed complex* $\pi(X_*)$ *of a filtered space*

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X,$$

where $\pi_1(X_*)$ is the fundamental groupoid $\pi_1(X_1, X_0)$ and for $n \geqslant 2$, $\pi_n(X_*)$ are the relative homotopy groups $\pi_n(X_n, X_{n-1}, x)$ for all $x$ in $X_0$. For $n \geqslant 2$ there is an action of $\pi_1(X)$ on $\pi_n(X_*)$, a boundary map $\delta : \pi_n(X_*) \to \pi_{n-1}(X_*)$ and source and target maps $s,\ t : \pi_1(X_*) \to X_0$. This defines a crossed complex. Hence given a CW-complex $X$ we can obtain a crossed complex $C$ such that $\pi(X) \simeq C$. To show that given a crossed complex $C$ we can obtain a CW-complex $X$ such that $\pi(X) \simeq C$ we need to define the functors $N$, $B$ and $|\ |$.

The category $\mathcal{S}impset$ has objects simplicial set and arrows simplicial maps. The functor $|\ |$ is the *geometric realisation* of a simplicial set. If $K$ is a simplicial set, then

$$|K| = \big(\bigsqcup_n (K_n \times \Delta^n)\big)/\sim$$

where $\Delta^n$ is the standard $n$-simplex and $\sim$ is the equivalence relation generated by $(d_i x, t) \sim (x, \delta^i t)$ if $x \in K_n$ and $t \in \Delta^{n-1}$ and $(s_i x, t) \sim (x, \sigma^i t)$ if $x \in K_n$ and $t \in \Delta^{n+1}$.

We have already defined the category of crossed complexes. The *nerve functor* $N$ applied to $C$ a crossed complex gives a simplicial set defined in dimension $n$ by $(NC)_n = \mathcal{C}rs(\pi\Delta^n, C)$.

The functor $B$ is called the *classifying space functor*. Given a crossed complex $C$, $B(C)$ is the geometric realisation of the nerve of $C$, $B(C) = |NC|$.

We can now show that given a crossed complex $C$ we can obtain a space $X$ such that $\pi(X) \simeq C$. Let $C$ be a crossed complex and $C_m$ be the $m$-truncation of $C$. We then have a crossed complex which we call a filtered crossed complex

$$C_* : C_0 \subseteq C_1 \subseteq \cdots \subseteq C_m \subseteq \cdots$$

This gives rise to a filtered space $B(C_*)$. It is proved in Ashley, *T-complexes and Crossed Complexes* [1] that there is a natural isomorphism $\pi B(C^m) \simeq C$.

We now state results on crossed complexes and their relationship with CW-complexes to prove theorem 4.2.3.

If $X_s$ is a filtered space defined by the skeletons of a CW-complex $X$ then $\pi(X_s)$ is a free crossed complex. Further, if $X$ is aspherical then $\pi(X_s)$ is exact and so a free crossed resolution of $\pi_1(X_2, X_0) = \pi_1(X_n, X_0)$ for all $n \geqslant 2$. We refer the reader to Whitehead, *Combinatorial Homotopy 2* [25] for details and proof.

If $X$ and $Y$ are CW-complexes with skeletal filtration $X_s$, $Y_s$ then we have the isomorphisms of crossed complexes

$$\pi(X_s) \otimes \pi(Y_s) \simeq \pi(X_s \otimes Y_s)$$

where $X_s \otimes Y_s = (X \otimes Y)_s$ is the skeletal filtration of the product $X \times Y$ and $\pi(X_s) \otimes \pi(Y_s)$ denotes the tensor product of crossed complexes. We refer the reader to Brown and Higgins [8] for details.

**Theorem 4.1.29** *If the CW-complex $X$ is the union of a family of sub-complexes $X_\lambda$, $\lambda \in \Lambda$, closed under finite intersection, then the natural map*

$$\mathrm{colim}_\lambda \pi(X_*)\lambda \to \pi(X_*)$$

*is an isomorphism.*

If $C$ is a free crossed resolution of a group $G$, with basis elements $z_n$ in dimensions $n$, then there is a CW-complex $X$ with $n$-cells in one-to-one correspondence with the elements of $z_n$, $n \geqslant 0$, and an isomorphism of based CW-complexes $\pi X^* \simeq C$. Further a morphism $\phi : C \to D$ of such based crossed complexes is realised by a map $f : X \to Y$ of based CW-complexes such that $\pi f = \phi$. For further details refer to Whitehead, *Combinatorial Homotopy 2* [25] and *Simple Homotopy Types* [26].

## 4.2 Graphs of Crossed Complexes

In this section we adapt the methods used in subsection 3.2.2 on graphs of CW-complexes by defining graphs of crossed complexes and the total crossed complex of a graph of crossed complexes. The total crossed complex is used to obtain a free crossed resolution. We give details of the relationship between the total crossed complex and the fundamental crossed complex of a total space. The second section gives details of computations and gives concrete presentations for free crossed resolutions built from small free crossed resolutions.

### 4.2.1 Total Crossed Complex

In this subsection we define a graph of crossed complexes and a graph of free crossed resolutions induced from a graph of groups. We then give an example of a graph of small free crossed resolutions which will be used in subsection 4.2.2 to obtain information about fundamental groups of graphs of groups.

We then define the total crossed complex of a graph of crossed complexes by double mapping cylinders of crossed complexes in an analogous way to homotopy colimits of graphs of groups.

The main result of this chapter is that the total crossed complex of a graph of reduced free crossed complexes gives a free crossed resolution of the fundamental groupoid of the graph of fundamental groups.

We adapt definition 3.2.15 of a graph of CW-complexes to define a graph of crossed complexes $\Gamma_{\mathrm{C}}$ as we will be exploiting the relationship between CW-complexes and crossed complexes in theorem 4.2.3.

**Definition 4.2.1** A graph of crossed complexes $\Gamma_{\mathrm{C}}$ is given by a graph $\Gamma$ with involution, crossed complexes $C_y$ and $C_u$ associated to each edge and vertex respectively with $C_y = C_{\overline{y}}$ and a morphism of crossed complexes $\mu_y : C_y \to C_{t(y)}$ for each edge $y$..

We form a graph of crossed complexes $\Gamma_{\mathrm{C}}$ based on a graph of groups $\Gamma_{\mathrm{G}}$ by choosing a free crossed resolution for each group, and morphisms of free crossed resolutions for each morphism of groups.

We now define the total crossed complex of a graph of crossed complexes. We model this definition on Definition 3.2.16 of the total space of a graph of spaces. This will enable the modelling of Proposition 3.2.19 by crossed complexes.

**Definition 4.2.2** Given a graph of crossed complexes $\Gamma_{\mathrm{C}}$ the total crossed complex $\mathrm{Tot}(\Gamma_{\mathrm{C}})$ is defined as the quotient of

$$\{\cup\, C_u : u \in V(\Gamma)\} \cup \{\cup\, C_y \otimes \mathcal{I} : y \in E(\Gamma)\}$$

by the identifications

$$\begin{array}{ll} C_y \otimes \mathcal{I} \to C_{\overline{y}} \otimes \mathcal{I} & (c_n^y \otimes \iota) \to (c_n^{\overline{y}} \otimes \iota^{-1}) \\ C_y \otimes 0 \to C_{t(y)} & (c_n^y \otimes 1_0) \to \mu_y(c_n^y). \end{array}$$

Given a graph of free crossed resolutions we model the free crossed resolution by $K(G,1)$ spaces and hence obtain an associated graph of aspherical spaces. By Proposition 3.2.19 the total space of a graph of aspherical spaces is aspherical; and by the fundamental crossed complex of an aspherical space is a free crossed resolution we have the result that the total crossed complex of a graph of free crossed resolutions is a also a free crossed resolution.

The results of subsection 4.1.4 are used to realise the graph of crossed complexes by a graph of CW-complexes $\Gamma_{\mathrm{X}}$. By Scott and Wall, proposition 3.2.19 $\mathrm{Tot}(\Gamma_{\mathrm{X}})$ is aspherical. The above results show $\pi(\mathrm{Tot}(\Gamma_{\mathrm{X}})) \simeq \mathrm{Tot}(\Gamma_{\mathrm{C}})$. Hence $\mathrm{Tot}(\Gamma_{\mathrm{C}})$ is a free crossed resolution and we obtain the following theorem.

**Theorem 4.2.3** *The crossed complex* $\mathrm{Tot}(\Gamma_{\mathrm{C}})$ *is a free crossed resolution of* $\pi_1(\Gamma_{\mathrm{G}})$.

### 4.2.2 Computations

Given the graph of groups with graph $\Gamma := u \underset{\overline{y}}{\overset{y}{\rightleftarrows}} v$, groups $G_y = G_{\overline{y}} := C_r$, $G_u := C_{rl}$, $G_v := C_{rm}$ and morphisms $\mu_y(a) = b^l$ and $\mu_{\overline{y}}(a) = c^m$ we can form a graph of crossed complexes. We choose small free crossed resolutions $\mathcal{F}_{C_r} = (\mathcal{A}, \alpha)$ $F_{rl} = (\mathcal{B}, \beta)$ and $\mathcal{F}_{C_{rm}} = (\mathcal{C}, \chi)$ of the groups $C_r$, $C_{rl}$ and $C_{rm}$ respectively and lifted morphisms $\mathcal{F}_{C_r} \to \mathcal{F}_{C_{rl}}$ and $\mathcal{F}_{C_r} \to F_{rm}$ which we label $f$ and $g$ of the group morphisms $C_r \to C_{rl}$ and $C_r \to C_{rm}$.

The total crossed complex has generators $b_n \in F_{C_{rl}}$, $c_n \in F_{C_{rl}}$ and $a_{n-1} \otimes \iota$, $a_n \otimes 0$, $a_n \otimes 1 \in F_{C_r} \otimes \mathcal{I}$ in dimension $n$. The morphisms $\alpha$, $\beta$ and $\chi$ are the small free crossed resolution morphisms. The morphisms $\delta_n(a_{n-1} \otimes \iota)$ are given by example 4.1.21.

We also have the following identifications, given by the definition of the total crossed complex. We identify $(a_n \otimes 0)$ with $f_n(a_n)$ and $(a_n \otimes 1)$ with $g_n(a_n)$.

Combining the definitions of $\delta_n(a_{n-1} \otimes \iota)$, $\beta_n(b_n)$ and $\chi_n(c_n)$ and the identifications we can define the morphisms of the total crossed complex of the given graph of free crossed resolutions.

$$\begin{aligned}
\delta_2(a_1 \otimes \iota) &= -(*_A \otimes \iota) - (a_1 \otimes 0) + (* \otimes \iota) + (a_1 \otimes 1) \\
&= (a_1 \otimes 1) - (a_1 \otimes 0) \\
&= g_1(a_1) - f_1(a_1) \\
&= c_1^m - b_1^l
\end{aligned}$$

The formulae for $\delta_n$ where $n > 2$ can be given in general for $n$ odd and $n$ even. We begin with $n$ odd.

$$\begin{aligned}
\delta_n(a_{n-1} \otimes \iota) &= -(a_{n-1} \otimes 1) + (a_{n-1} \otimes 0)^{(*_A \otimes \iota)} + (\alpha_{n-1} a_{n-1} \otimes \iota) \\
&= -g_{n-1}(a_{n-1}) + (f_{n-1}(a_{n-1}))^{(*_A \otimes \iota)} + a_{n-1} \cdot N_r(a) \otimes \iota \\
&= -c_{n-1} + (b_{n-1})^{(*_A \otimes \iota)} + a_{n-2} \cdot N_r(a) \otimes \iota
\end{aligned}$$

Similarly with $n$ even.

$$\begin{aligned}
\delta_n(a_{n-1} \otimes \iota) &= (a_{n-1} \otimes 1) - (a_{n-1} \otimes 0)^{(*_A \otimes \iota)} + (\alpha_{n-1} a_{n-1} \otimes \iota) \\
&= g_{n-1}(a_{n-1}) - (f_{n-1}(a_{n-1}))^{(*_A \otimes \iota)} + a_{n-1} \cdot N_r(a) \otimes \iota \\
&= c_{n-1} \cdot N_m(c) + (b_{n-1} \cdot N_l(b))^{(*_A \otimes \iota)} + a_{n-2} \cdot (a-1) \otimes \iota
\end{aligned}$$

Hence the total crossed complex is generated by $b_n$, $c_n$, $a_n \otimes 0$, $a_n \otimes 1$ and $a_{n-1} \otimes \iota$ in dimension $n$ and the morphisms are defined as follows:

$$\delta_n(a_n \otimes \iota) \\
= \begin{cases} c_1^m - b_1^l & \text{for } n = 2 \\ -c_{n-1} + (b_{n-1})^{(*_A \otimes \iota)} + a_{n-2} \cdot N_r(a) \otimes \iota & \text{for } n \text{ odd} \\ c_{n-1} \cdot N_m(c) + (b_{n-1} \cdot N_l(b))^{(*_A \otimes \iota)} + a_{n-2} \cdot (a-1) \otimes \iota & \text{for } n \geqslant 3 \text{ and } n \text{ even.} \end{cases}$$

# Conclusion

# Notation

| | | |
|---|---|---|
| **Categories** | | |
| $\mathcal{C}$ | category | 7 |
| $\mathrm{Ob}(\mathcal{C})$ | objects of $\mathcal{C}$ | 7 |
| $\mathrm{Arr}(\mathcal{C})$ | arrows of $\mathcal{C}$ | 7 |
| $\Gamma\mathcal{C}$ | underlying graph of $\mathcal{C}$ | 7 |
| $1_v$ | identity element at $v$ | 7 |
| $\mathcal{C}(u,v)$ | set of arrows from $u$ to $v$ in $\mathcal{C}$ | 7 |
| $\mathcal{C}^{op}$ | opposite category of $\mathcal{C}$ | 7 |
| $\mathcal{C} \times \mathcal{D}$ | direct product of categories $\mathcal{C}$ and $\mathcal{D}$ | 7 |
| $\mathcal{G}ps$ | category of groups | 8 |
| $\mathcal{G}phs$ | category of graphs | 8 |
| $\mathcal{S}ets$ | category of sets | 8 |
| $\mathcal{T}op$ | category of topological spaces | 8 |
| $\mathcal{G}pds$ | category of groupoids | 8 |
| $\mathcal{C}rs$ | category of crossed complexes | 8 |
| $P\Gamma$ | category of directed paths in $\Gamma$ | 12 |
| | | |
| **Crossed Complexes** | | |
| $\mathbb{Z}[G]$ | free abelian group | ?? |
| $(\mathcal{F}, C_r)$ | free $C_r$-module | ?? |
| $\mathfrak{X}$ | crossed module | ?? |
| $\mathcal{C} = (\mathcal{C}, \chi)$ | crossed complex | ?? |
| $\mathcal{I} = (\mathcal{I}, i)$ | unit interval crossed complex | ?? |
| $\pi_1(\mathcal{C}, \chi)$ | fundamental groupoid of a crossed complex | ?? |
| $(\mathcal{A}, \alpha) \otimes (\mathcal{B}, \beta)$ | tensor product of crossed complexes | ?? |
| $\mathcal{F}_{C_r}$ | small free crossed resolution of $C_r$ | ?? |
| $\mathcal{C}_*$ | filtered crossed complex | ?? |
| | | |
| **Graphs** | | |
| $\Gamma$ | directed graph | 5 |

| | | |
|---|---|---|
| $V(\Gamma)$ | set of vertices of $\Gamma$ | 5 |
| $E(\Gamma)$ | set of edges of $\Gamma$ | 5 |
| $s(y)$ | source vertex of edge $e$ | 5 |
| $t(y)$ | target vertex of edge $e$ | 5 |
| $\Gamma(u,v)$ | set of edges from $u$ to $v$ | 5 |
| $()_v$ | empty path at $v$ | 5 |
| $\overrightarrow{\Gamma}(u,v)$ | set of directed paths from $u$ to $v$ | 5 |
| $\mathbb{D}$ | graph map $\mathbb{D} : \Gamma \to \Gamma\mathcal{C}$ | 10 |
| $\mathcal{G}^\sigma$ | graph | 21 |
| $\Gamma_{\mathrm{G}}$ | graph of groups | |
| $\Gamma_{\mathcal{G}}$ | graph of groupoids | |
| $\Gamma_{\mathcal{X}}$ | graph of spaces | |
| $\Gamma_{\mathcal{K}}$ | graph of aspherical CW-complexes | |

**Groupoids**

| | | |
|---|---|---|
| $\mathcal{G}$ | groupoid | 7 |
| $\mathcal{F}(\Gamma)$ | free and fundamental groupoid of $\Gamma$ | 16 |
| $\mathcal{I}$ | unit groupoid | 13 |
| $\mathcal{I}_n$ | tree groupoid | 13 |
| $G \times \mathcal{I}_n$ | direct product of a group and tree groupoid | 13 |
| $\mathcal{X}(\ )$ | generating set | 13 |
| $\mathcal{N}$ | normal subgroupoid | 17 |
| $\mathcal{G}/\mathcal{N}$ | quotient groupoid | 17 |
| $g\mathcal{H}$ | left cosets | 18 |
| $U_\sigma(\mathcal{G})$ | universal groupoid | 21 |
| $\overline{G}$ | disjoint union of vertex groups | **??** |
| $\mathcal{F}(\Gamma_{\mathrm{G}})$ | fundamental groupoid of a graph of groups | **??** |

**Functors**

| | | |
|---|---|---|
| $F : \mathcal{C} \to \mathcal{D}$ | functor | 7 |
| $\|\ \|$ | geometric realisation functor | **??** |

| | | |
|---|---|---|
| $N$ | nerve functor | **??** |
| $B$ | classifying space functor | **??** |
| $\pi(X^*)$ | fundamental crossed complex functor | **??** |

**Misc**

| | | |
|---|---|---|
| $M_f$ | mapping cylinder of $f$ | **??** |
| $M_{f,g}$ | double mapping cylinder of $f$ and $g$ | **??** |
| $\mathbb{Z}$ | integers | **??** |
| $\mathbb{R}$ | real numbers | **??** |
| $I$ | unit interval | **??** |
| $S^n$ | $n$-sphere | **??** |
| $D^n$ | $n$-disk | **??** |
| $X_*$ | filtered space | **??** |

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