Intuitions for cubical methods in nonabelian algebraic topology

Ronnie Brown

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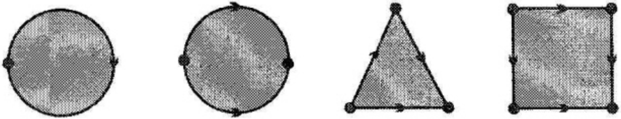
CONSTRUCTIVE MATHEMATICS AND MODELS OF TYPE THEORY

Motivation of talk

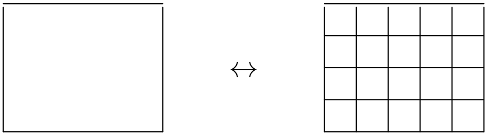
Modelling types by homotopy theory?
What do homotopy types look like?
How do they behave and interact?
Need to be eclectic, evaluate alternatives,
not assume standard models are "fixed" for all time!



disc	globe	simplex	square	geometry
crossed module	2-groupoid	??	double groupoid	algebra

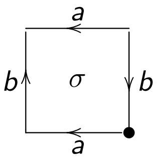
Four Anomalies in Algebraic Topology

1. Fundamental group: nonabelian,
   Homology and higher homotopy groups: abelian.
2. The traditional van Kampen Theorem does not compute the fundamental group of the circle,
   THE basic example in algebraic topology.
3. Traditional algebraic topology is fine with composing paths but does not allow for the algebraic expression of



From left to right gives subdivision.
From right to left should give composition.
What we need for higher dimensional, nonabelian, local-to-global problems is:
Algebraic inverses to subdivision.

4. For the Klein Bottle diagram



in traditional theory we have to write $\partial\sigma = 2b$, not

$$\partial(\sigma) = a + b - a + b.$$

All of 1–4 can be resolved by using
groupoids and their developments in some way.
Clue: while group objects in groups are just abelian groups,
group objects in groupoids are equivalent to
Henry Whitehead's crossed modules,

$$\pi_2(X, A, c) \to \pi_1(A, c),$$

a major example of
nonabelian structure in higher homotopy theory.

The origins of algebraic topology

The early workers wanted to define numerical invariants using cycles modulo boundaries but were not too clear about what these were!

Then Poincaré introduced

formal sums of oriented simplices

and so the possibility of the equation $\partial\partial = 0$.

The idea of formal sums of domains

came from integration theory,

$$\int_C f + \int_D f = \int_{C+D} f$$

with which many were concerned.

This automatically gives an abelian theory.

In our account we use actual compositions for homotopically defined functors.

We want to find and use algebraic structures

which better model the geometry, and the interaction of spaces.

Enter groupoids

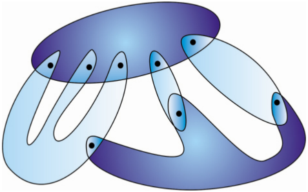
I was led into this area in the 1960s through writing a text on topology.
Fundamental group $\pi_1(X, c)$ of a space with base point.
Seifert-van Kampen Theorem: Calculate the fundamental group of a union of based spaces.

$$\begin{tikzcd}
\pi_1(U \cap V, c) \arrow[r] \arrow[d] & \pi_1(V, c) \arrow[d] \\
\pi_1(U, c) \arrow[r] & \pi_1(U \cup V, c)
\end{tikzcd}$$

pushout of groups if U, V are open and $U \cap V$ is path connected.

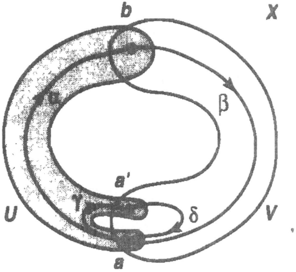
I'll sketch on the board key steps in the proof.

If $U \cap V$ is not connected, where to choose the basepoint?
Example: $X = S^1$.
Answer: hedge your bets, and use lots of base points!



Try dealing with that using covering spaces!

Actually Munkres' Topology book deals with the following example



by covering spaces!

RB 1967: The fundamental groupoid $\pi_1(X, C)$ on a set C of base points.

$$\begin{tikzcd} \pi_1(U \cap V, C) \arrow[r] \arrow[d] & \pi_1(V, C) \arrow[d] \\ \pi_1(U, C) \arrow[r] & \pi_1(U \cup V, C) \end{tikzcd}$$

pushout of groupoids if U, V are open and C meets each path component of $U, V, U \cap V$.

Get $\pi_1(S^1, 1)$: from $\pi_1(S^1, \{\pm 1\})$

"I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids."
Alexander Grothendieck

Proof of the pushout by verifying the universal property, so we don't need to know how to compute pushouts of groupoids to prove the theorem.

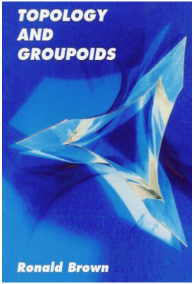
I've already sketched on the board key steps in the proof!

If $X = U \cup V$, and $U \cap V$ has n path components, then one can choose n, or more, base points.

$$(X, C) = (\text{union}) \xrightarrow{\text{SvKT}} \pi_1(X, C) \xrightarrow{\text{combinatorics}} \pi_1(X, c).$$

Strange. One can completely determine $\pi_1(X, C)$ and so any $\pi_1(X, c)$! A new anomaly!
Need to further develop
combinatorial and geometric groupoid theory,
including fibrations of groupoids, and orbit groupoids.

Groupoids have structure in dimensions 0 and 1,
and so can model homotopy 1-types.
All of 1-dimensional homotopy theory is
better modelled by groupoids than by groups.



2006 edition (previous differently titled editions 1968, 1988)

To model gluing you need to model spaces and maps.

Are there higher homotopical invariants with
structure in dimensions from 0 to n?
Colimit theorems in higher homotopy?
The proof of the groupoid theorem seemed to
generalise to dimension 2, at least, if one had the right algebra of double groupoids, and the right gadget, a
strict homotopy double groupoid of a space.
So this was an “idea of a proof in search of a theorem”.
First:
basic algebra of double categories/groupoids.
(due to Charles Ehresmann)

Compositions in a double groupoid:

$$\begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline \alpha & \gamma \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline \alpha & \gamma \\ \hline \beta & \delta \\ \hline \end{array} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha +_1 \beta \qquad \begin{bmatrix} \alpha & \gamma \end{bmatrix} = \alpha +_2 \gamma \qquad \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

That each is a morphism for the other gives the interchange law:

$$(\alpha \circ_2 \gamma) \circ_1 (\beta \circ_2 \delta) = (\alpha \circ_1 \beta) \circ_2 (\gamma \circ_1 \delta).$$

This illustrates that a 2-dimensional picture can be more comprehensible than a 1-dimensional equation.
Note: In these double groupoids the horizonal edges and vertical edges may come from different groupoids.

Need also "Commutative cubes"

In dimension 1, we still need the 2-dimensional notion of commutative square:

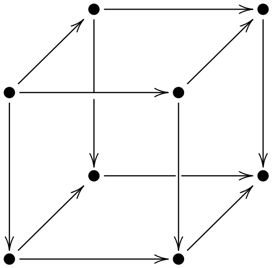
$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "c"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "d"'] & \cdot \end{tikzcd} \qquad ab = cd \quad a = cdb^{-1}$$

Easy result: any composition of commutative squares is commutative.
In ordinary equations:

$$ab = cd, ef = bg \text{ implies } aef = abg = cdg.$$

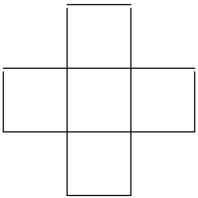
The commutative squares in a category form a double category!
Compare Stokes' theorem! Local Stokes implies global Stokes.

What is a commutative cube? in a double groupoid in which horizontal and vertical edges come from the same groupoid.

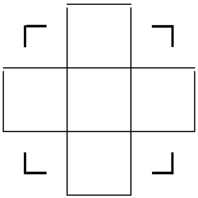


We want the faces to commute!

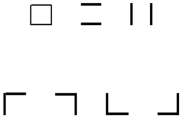
We might say the top face is the composite of the other faces: so fold them flat to give:



which makes no sense! Need fillers:



To resolve this, we need some special squares with commutative boundaries:



where a solid line indicates a constant edge.
The top line are just identities.
The bottom line are called connections.
Any well defined composition of these squares is called thin.

What are the laws on connections?

$$[\ulcorner \ \lrcorner] = | \ | \qquad \begin{bmatrix} \ulcorner \\ \lrcorner \end{bmatrix} = \overline{\underline{\quad}} \qquad \text{(cancellation)}$$

$$\begin{bmatrix} \ulcorner & \overline{\underline{\quad}} \\ | \ | & \ulcorner \end{bmatrix} = \ulcorner \qquad \begin{bmatrix} \lrcorner & | \ | \\ \overline{\underline{\quad}} & \lrcorner \end{bmatrix} = \lrcorner \qquad \text{(transport)}$$

The term transport law and the term connections came from laws on path connections in differential geometry.
It is a good exercise to prove that any composition of commutative cubes is commutative.
These are equations on turning left or right, and so
are a part of 2-dimensional algebra.

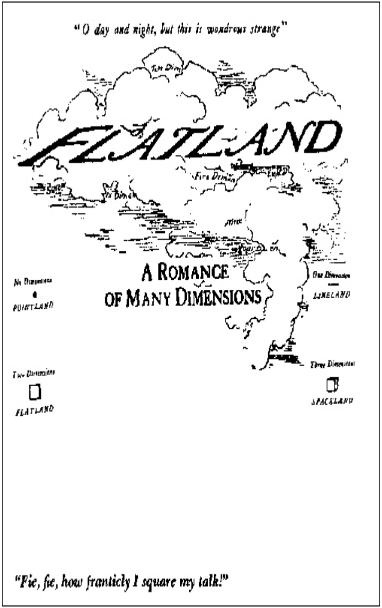
Double groupoids allow for:
multiple compositions,
2-dimensional formulae, and
2-dimensional rewriting.
As an example, we get a rotation

$$\sigma(\alpha) = \begin{bmatrix} \vert\,\vert & \ulcorner & = \\ \llcorner & \alpha & \urcorner \\ = & \lrcorner & \vert\,\vert \end{bmatrix}$$

Exercise: Prove $\sigma^4(\alpha) = \alpha$.

Hint: First prove $\sigma \begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} \sigma\alpha \\ \sigma\beta \end{bmatrix}$.

For more on this, see the Appendix.



Published in 1884, available on the internet.

The linelanders had limited interaction capabilities!
What is the logic for higher dimensional formulae?

Cubical sets in algebraic topology

Dan Kan's thesis and first paper (1955) were cubical, relying clearly on geometry and intuition.
It was then found that cubical groups, unlike simplicial groups were not Kan complexes.
There was also a problem on realisation of cartesian products.
The Princeton group assumed the cubical theory was quite unfixable.
So cubical methods were generally abandoned for the simplicial; although many workers found them useful.

The work with Chris Spencer on double groupoids in the early 1970s found it necessary to introduce an extra and new kind of "degeneracy" in cubical sets, using the monoid structures

$$\max, \min : [0,1]^2 \to [0,1].$$

We called these "connection operators".
Andy Tonks proved (1992) that cubical groups with connections are Kan complexes! G. Maltsiniotis (2009) showed that up to homotopy, connections correct the realisation problem.
Independently of these facts, we deal with
cubical sets with connections and compositions.
If all the compositions are groupoid structures, we get a
cubical ω-groupoid.

Strict homotopy double groupoids?

How to get a strict homotopy double groupoid of a space?
Group objects in groups are abelian groups, by the interchange law.
Chris Spencer and I found out in the early 1970s that group objects in groupoids are more complicated, in fact equivalent to Henry Whitehead's crossed modules! (This was known earlier to some.)
In the early 1970s Chris Spencer, Philip Higgins and I developed a lot of understanding of:
(i) relations between double groupoids and crossed modules; and
(ii) algebraic constructions on the latter, e.g. induced crossed modules, and colimit calculations.
In June, 1974, Phil and I did a strategic analysis as follows:

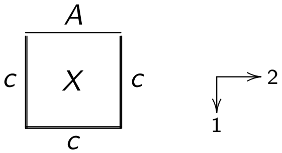
1. Whitehead had a subtle theorem (1949) that

$$\pi_2(A \cup \{e^2_\lambda\}, A, c) \to \pi_1(A, c)$$

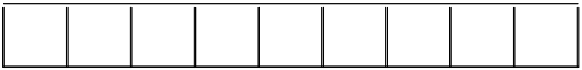
   is a free crossed module, and this was an example of a **universal property in 2-dimensional homotopy theory**.
2. If our conjectured but unformulated theorem was to be any good it should have Whitehead's theorem as a consequence.
3. But Whitehead's theorem was about second relative homotopy groups.
4. So we should look for a homotopy double groupoid in a relative situation, (X, A, c).
5. The simplest way to do this was to look at maps of the square I^2 to X which took the edges of the square to A and the vertices to c, and consider homotopy classes of these,
6. Because of all the preliminary work with Chris and Phil, this worked like a dream!
   (Submitted 1975, published 1978.)

Groupoids in higher homotopy theory?

Consider second relative homotopy groups $\pi_2(X, A, c)$.
(Traditionally, the structure has to be a group!)



Here thick lines show constant paths.
Note that the definition involves choices,
and is unsymmetrical w.r.t. directions. Unaesthetic!
All compositions are on a line:



Brown-Higgins 1974 $\rho_2(X, A, C)$: homotopy classes rel vertices of maps $[0,1]^2 \to X$ with edges to A and vertices to C

$$\begin{array}{ccc} C & \text{---}A\text{---} & C \\ | & & | \\ A & X & A \\ | & & | \\ C & \text{---}A\text{---} & C \end{array} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

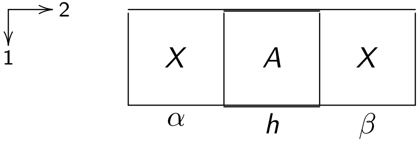
$$\begin{tikzcd} \rho_2(X, A, C) \arrow[r, shift left=2] \arrow[r, shift left=0.7] \arrow[r, shift right=0.7] \arrow[r, shift right=2] & \pi_1(A, C) \arrow[r, shift left] \arrow[r, shift right] & C \end{tikzcd}$$

Childish idea: glue two square if the right side of one is the same as the left side of the other. Geometric condition.

There is a horizontal composition

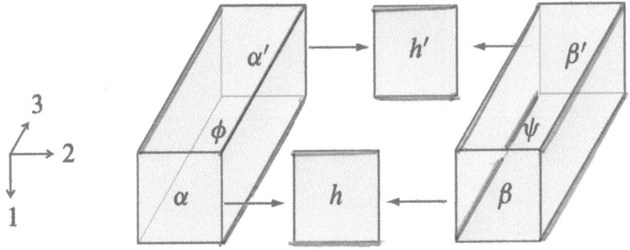
$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha +_2 h +_2 \beta \rangle\!\rangle$$

of classes in $\rho_2(X, A, C)$, where thick lines show constant paths.



Intuition: gluing squares exactly is a bit too rigid, while "varying edges in A" seems just about right!

To show $+_2$ well defined, let $\phi : \alpha \equiv \alpha'$ and $\psi : \beta \equiv \beta'$, and let $\alpha' +_2 h' +_2 \beta'$ be defined. We get a picture in which thick lines denote constant paths.
Can you see why the 'hole' can be filled appropriately?



Thus $\rho(X, A, C)$ has in dimension 2
compositions in directions 1 and 2 satisfying the interchange law
and is a double groupoid with connections,
containing as an equivalent substructure the classical

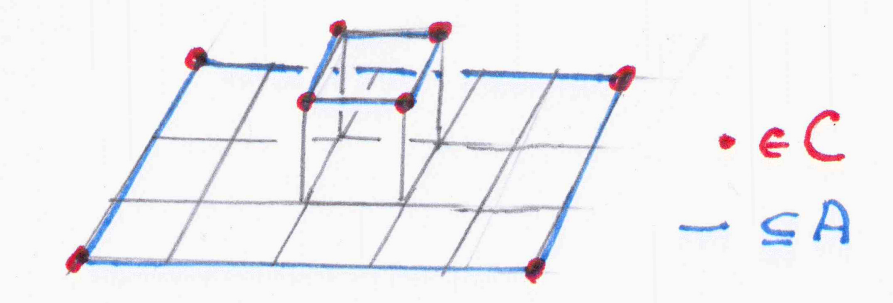
$$\Pi(X, A, C) = (\pi_2(X, A, C) \to \pi_1(A, C)),$$

a crossed module over a groupoid.
(All that needs proof, but in this dimension is not too hard.)

Now we can directly generalise the 1-dimensional proof:
since one has a homotopy double groupoid which is “equivalent”
to crossed modules but which can express:

- algebraic inverse to subdivision
- commmutative cubes such that any multiple composition of commutative cubes is commutative

A key deformation idea is shown in the picture:



We need to deform the bottom subdivided square into a subdivided square for which all the subsquares define an element of $\rho(X, A, C)$. This explains the connectivity assumptions for the theorem.
It is to express this diagram that $\rho(X, A, C)$ is designed.

We end up with a 2-d SvK theorem, namely a
pushout of crossed modules:

$$\begin{tikzcd}
\Pi((X,A,C)\cap U\cap V) \arrow[r] \arrow[d] & \Pi((X,A,C)\cap V) \arrow[d] \\
\Pi((X,A,C)\cap U) \arrow[r] & \Pi(X,A,C)
\end{tikzcd}$$

if $X = U \cup V$, U, V open, and some connectivity conditions hold.
Connectivity: (X, A, C) is connected if
(i) $\pi_0 C \to \pi_0 A, \pi_0 C \to \pi_0 X$ are surjective.
(ii) any map $(I, \partial I) \to (X, C)$ is deformable into A rel end points.
How do you glue homotopy 2-types?
Glue crossed modules!
This is a huge generalisation of Whitehead's theorem.
It enables some nonabelian computations of homotopy 2-types.

The proof of this works first in the category of double groupoids with connection, and then uses the equivalence with crossed modules. The notion of connection in a double groupoid allows for the proof the notion of commutative cube and for the equivalence with crossed modules.

This is a general pattern: need

"broad" algebraic structure for conjecturing and proving theorems

"narrow" algebraic structure for relating to classical theory and for calculations.

The algebraic equivalence between these is then quite powerful.

Filtered spaces

The success of the 2-d idea led to a look for the n-dimensional idea, and in view of other work of Whitehead it was natural to look at filtered spaces.

$$X_* := X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

(i) Skeletal filtration of a CW-complex: e.g. Δ^n_*, I^n_*

(ii) $* \subseteq A \subseteq \cdots \subseteq A \subseteq X \subseteq X \subseteq \cdots$

(iii) FM, the free monoid on a topological space with base point: filter by word length.

$$R_n X_* = \mathsf{FTop}(I^n_*, X_*).$$

$R(X_*)$ is a cubical set with compositions and connections.

Theorem (Brown-Higgins, JPAA, 1981)

Let the projection

$$p : RX_* \to \rho X_* = R(X_*)/\equiv$$

be given in dimension n by taking homotopies through filtered maps and rel vertices.
Define an element $\alpha \in \rho_n(X_*)$ to be thin if it has a representative a such that $a(I^n) \subseteq X_{n-1}$. Then
(i) $\rho(X_*)$ is a Kan complex in which every box has a unique thin filler;
(ii) compositions on RX_* are inherited by ρX_* to give it the structure of strict cubical ω-groupoid;
(iii) the projection p is a Kan fibration of cubical sets.

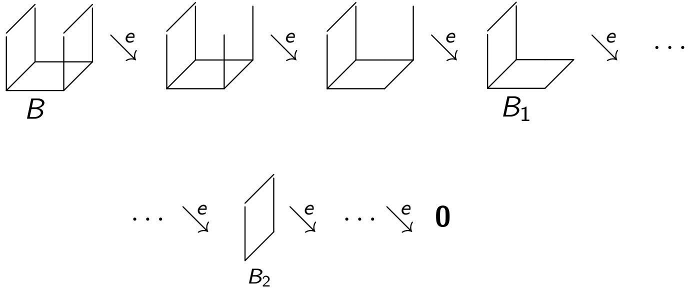
Look again at the fibration $p : RX_* \to \rho X_*$: a consequence of the fibration property is:

Corollary (Lifting composable arrays)

Let $(\alpha_{(i)})$ be a composable array of elements of $\rho_n(X_*)$.
Then there is a composable array $(a_{(i)})$ of elements of $R_n(X_*)$ such that for all (i), $p(a_{(i)}) = \alpha_{(i)}$.

Thus the weak cubical infinity groupoid structure of $R(X_*)$
has some kind of control
by the strict infinity groupoid structure of $\rho(X_*)$.
The theory is about compositions, as these are relevant to Higher Homotopy SvKT's, i.e. to gluing of homotopy types.

The proof of the Fibration Theorem, that $R(X_*) \to \rho(X_*)$ is a fibration, relies on a nice use of geometric cubical methods, and the Kan condition.
Apply the Kan condition by modelling in subcomplexes of real cubes and using expansions and collapsings of these.



Let C be an r-cell in the n-cube I^n. Two $(r-1)$-faces of C are called opposite if they do not meet.

A partial box in C is a subcomplex B of C generated by one $(r-1)$-face b of C (called a base of B) and a number, possibly zero, of other $(r-1)$-faces of C none of which is opposite to b.

The partial box is a box if its $(r-1)$-cells consist of all but one of the $(r-1)$-faces of C.

The proof of the fibration theorem uses a filter homotopy extension property and the following:

Theorem (Key Proposition)

Let B, B' be partial boxes in an r-cell C of I^n such that $B' \subset B$. Then there is a chain

$$B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1 = B'$$

such that

(i) *each B_i is a partial box in C;*

(ii) *$B_{i+1} = B_i \cup a_i$ where a_i is an $(r-1)$-cell of C not in B_i;*

(iii) *$a_i \cap B_i$ is a partial box in a_i.*

The proof is a kind of program.

Why filtered spaces?

In the proof of the above Theorems, particularly the proof that compositions are inherited, there is
exactly the right amount of "filtered room".
One then needs to evaluate the significance of that fact!
Grothendieck in "Esquisse d'un Programme" (1984) has attacked the dominance of the idea of topological space, which he says comes from the needs of analysis rather than geometry. He advocates some ideas of stratified spaces, and filtered spaces are a step in that direction.
A general argument is that to describe/specify a space you need some kind of data, and that data has some kind of structure. So it is reasonable for the invariants to be defined using that structure.

What kinds of algebraic model?

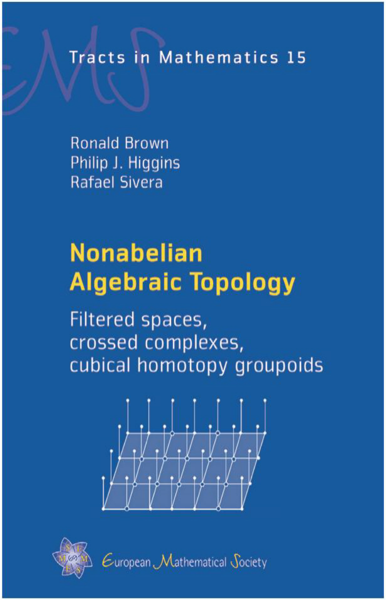
Our methodology is to use two types of categories of algebraic model; they are equivalent, but serve different purposes:
"broad" algebraic data:
geometric type of axioms, expressive,
useful for conjecturing and proving theorems, particularly colimit theorems, and for constructing classifying spaces;
"narrow" algebraic data:
complicated axioms, useful for explicit calculation and relating to classical theory,
colimit examples lead to new algebraic constructions.
The algebraic proof of equivalence, of "Dold-Kan type", is then a key to the power of the theory, and important in developing aspects of it.

Where occur? Obtained from certain
structured spaces (filtered spaces, n-cubes of spaces)
Aim is a gluing result (Seifert-van Kampen): these algebraic models are values of a homotopically defined functor from some topological data, and this functor preserves some colimits.
So ruled out, for this aim, are: simplicial groups, quadratic modules (Baues), 2-crossed modules, weak infinity groupoids, ...
Classical successful example in dimension 1:
spaces with base point; groups, fundamental group. (Seifert-van Kampen Theorem)
Two further successful examples:

- (1981, with P.J. Higgins) filtered spaces, strict cubical homotopy groupoids with connections, and crossed complexes;
- (1987, J.-L. Loday and RB, G. Ellis and R. Steiner) n-cubes of spaces, cat^n-groups, and crossed n-cubes of groups

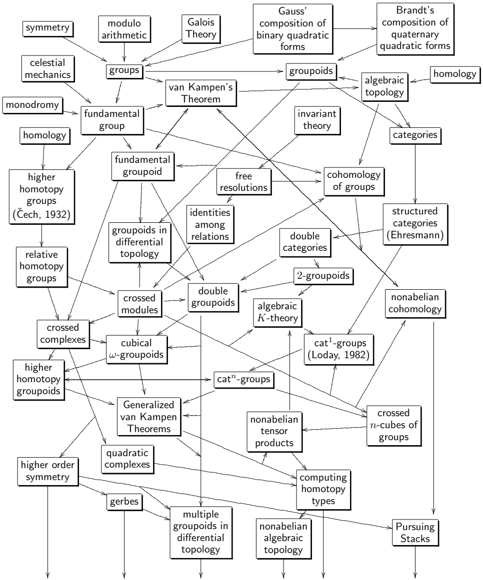
$$\begin{tikzcd}
 & & \left(\begin{array}{c}\text{“broad”}\\ \text{algebraic}\\ \text{model}\end{array}\right) \arrow[dll, shift left, "\mathbb{B}"] \arrow[d, shift left, "\simeq"] & \text{cubical omega-groupoids} \\
\left(\begin{array}{c}\text{topological}\\ \text{data}\end{array}\right) \arrow[urr, shift left, "\rho"] \arrow[drr, "\Pi"] \arrow[ddr, "U"'] & & & \text{filtered spaces} \\
 & & \left(\begin{array}{c}\text{“narrow”}\\ \text{algebraic}\\ \text{model}\end{array}\right) \arrow[u, shift left, "\simeq"] \arrow[dl, "B"] & \text{crossed complexes} \\
 & \left(\begin{array}{c}\text{topological}\\ \text{spaces}\end{array}\right) & &
\end{tikzcd}$$

- (HHSvKT): ρ, and hence also Π, preserves certain colimits, (hence some calculations);
- $\Pi \circ \mathbb{B}$ is naturally equivalent to 1 ;
- $B = U \circ \mathbb{B}$ is a kind of classifying space ;
- There is a natural transformation $1 \to \mathbb{B} \circ \Pi$ preserving some homotopical information.



Journey,
by John Robinson
(Macquarie University)

Some Context for Higher Dimensional Group Theory



Ronnie Brown, April, 2004.

Appendix: Rotations in a double groupoid with connections

To show some 2-dimensional rewriting, we consider the notion of rotations σ, τ of an element u in a double groupoid with connections:

$$\sigma(u) = \begin{bmatrix} \mid\mid & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & \mid\mid \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \urcorner & \mid\mid \\ \ulcorner & u & \lrcorner \\ \mid\mid & \llcorner & = \end{bmatrix}.$$

For any $u, v, w \in G_2$,

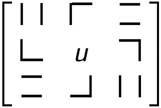
$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \text{and} \quad \sigma\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\sigma w, \sigma u]$$

$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \text{and} \quad \tau\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\tau u, \tau w]$$

whenever the compositions are defined.
Further $\sigma^2\alpha = -_1 -_2 \alpha$, and $\tau\sigma = 1$.

To prove the first of these one has to rewrite $\sigma(u +_2 v)$ until one ends up with an array, shown on the next slide, which can be reduced in a different way to $\sigma u +_1 \sigma v$. Can you identify σu, σv in this array? This gives some of the flavour of this 2-dimensional algebra of double groupoids.
When interpreted in $\rho(X, A, C)$ this algebra implies the existence, even construction, of certain homotopies which may be difficult to do otherwise.

$$\sigma(u) =$$



$$u +_1 v = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$u +_2 v = \begin{bmatrix} u & v \end{bmatrix}$$

$$\sigma(u +_2 v) = \sigma(u) +_1 \sigma(v)$$