

Future directions for crossed complexes in nonabelian algebraic topology?*

Ronald Brown
School of Computer Science,
Bangor University, LL57 1UT, Wales

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Yet all experience is an arch wherethro'
Gleams that untravelled world, whose margin fades
Forever and forever when I move.

from 'Ulysses', by Alfred Lord Tennyson.

We have in the previous Chapters developed a description of an intricate structure. We hope to have shown how it fits together and allows a new approach to algebraic topology, in which some nonabelian information in dimension 2 and the actions of the fundamental groupoid are successfully taken into account. We also wanted to convey how a key to the success of the theory has been the good modelling of the geometry by the algebra, and the way the algebra gives power and reality to some basic intuitions, revealing underlying processes.

We have presented the material in a way which we hope will convince you that the intricacy of the justification of the theory does not detract from the fact that crossed complexes are usable as a tool even without knowing exactly why they works. That is, we have given a pedagogical order rather than a logical and structural order. It should be emphasised that the order of discovery followed the logical order! The conjectures were made and verified in terms of ω -groupoids, and we were amazed that the theory of crossed complexes, which was in essence already available, fitted with this so nicely.

It is also surprising that this corpus of work followed from a simple aesthetic question posed in 1964-65, to find a determination of the fundamental group of the circle which avoided the detour of setting up covering space theory. This led to nonabelian cohomology, [Bro65a], and then to groupoids, [Bro67]. The latter suggested the programme of seeing if rewriting homotopy theory replacing the word 'group' by 'groupoid' could be done, and if so whether the result was an improvement! This naive question raised some new prospects.

There is much more to do, and we explain some potential areas of work in the next Section. It is not expected that these questions and problems are of equal interest or solvability!

*This is a slightly edited version of Chapter 16 of the book [BHS11], with a bibliography limited to the citations from this Chapter. References to pages, Sections, Chapters are to the earlier parts of the book.

Some of these matters discussed are speculative; it seems right to quote here from a letter of Alexander Grothendieck dated 14/06/83:

Of course, no creative mathematician can afford not to “speculate”, namely to do more or less daring guesswork as an indispensable source of inspiration. The trouble is that, in obedience to a stern tradition, almost nothing of this appears in writing, and preciously little even in oral communication. The point is that the disrepute of “speculation” or “dream” is such, that even as a strictly private (not to say secret!) activity, it has a tendency to vegetate - much like the desire and drive of love and sex, in too repressive an environment.

Any new idea has to be caught as it flashes across the mind, or it might vanish; Brown has found that talking about his ideas has helped to make them real, though it sometimes raised some funny looks!

16.1 Problems and questions

There are a number of standard methods and results in algebraic topology to which the techniques of crossed complexes given here have not been applied, or applied only partially. So we leave these open for work to be done, and to decide if the uses in these areas of crossed complexes and related structures can advance the subject of algebraic topology and homological algebra. We expect the reader to rely on the internet for additional references and sources for further details, with the usual cautions about not relying totally on all that is there. Also you must do your own assessment of the possible value of these questions.

Problem 16.1.1 There has been surprisingly little general use of the HHSvKT for crossed modules in low-dimensional topology and geometric group theory: this theorem is not even mentioned in [HAMS93], though some consequences are given. We mention again the important work of Papakyriakopoulos on relations between group theory and the Poincaré conjecture, [Pap63], which uses Whitehead’s theorem on free crossed modules, though of course the Poincaré Conjecture has been resolved by different means. Recent uses of the 2-dimensional van Kampen theorem are by [KFM08, Far08]. Perhaps even more surprising uses could be made of the triadic results in [BL87b, BL87a]? \square

Problem 16.1.2 Investigate applications of the enrichment of the category \mathbf{FTop} over the monoidal closed category \mathbf{Crs} in the spirit of the work on 2-groupoids in [KP02]. In fact, as an exercise, translate the work of the last paper into the language of crossed complexes and their internal homs. \square

Problem 16.1.3 Investigate and apply Mayer-Vietoris type exact sequences for a pullback of a fibration of crossed complexes, analogous to that given for a pullback of a covering morphism of groupoids in [Bro06, Section 10.7]. See also [HK81, BHK83]. \square

Problem 16.1.4 Can one use crossed complexes to give a finer form of Poincaré Duality? For an account of this duality, see for example [tD08, Chapter 16]. This might require developing cup and cap products, which should be no problem, and also coefficients in an object with an analogue of a ‘ring structure’. These could be the crossed differential algebras (i.e. monoid objects in the monoidal category \mathbf{Crs})

considered in [BT97], and the braided regular crossed modules of [BG89a], further developed in [AU07]. See also the preprint [Bro10]. □

Problem 16.1.5 Another standard area in algebraic topology is *fixed point theory*, which includes the Lefschetz theory, involving homology, and also Nielsen theory, involving the fundamental group. Can these be combined? Perhaps one needs abstract notions for the Lefschetz number analogous to those found for the Euler characteristic, and with values in some ring generalising the integers? Relevant papers on this are perhaps [Hea05, Pon09, PS09]. Note that the last two papers use symmetric monoidal categories, and all use groupoid techniques. □

Problem 16.1.6 Are there possible results on the fundamental crossed complex of an orbit space of a filtered space analogous to those for the fundamental groupoid of an orbit space given in [Bro06, Chapter 11]? Some related work is in [HT82]. But in [Bro06, Chapter 11] a key result is on path lifting. Can one get some homotopy lifting using subdivisions of a square and the retraction arguments used in the proof of Proposition 14.2.8? □

Problem 16.1.7 Are there applications of crossed complexes to the nonabelian cohomology of fibre spaces? Could the well developed acyclic model theory and fibre spaces of [GM57] be suitably modified and used? The spectral sequence of filtered crossed complexes has been developed by Baues in [Bau89], but surely more work needs to be done. Note also that while the theory of simplicial fibre bundles is well developed, the cubical theory has problems because the categorical product of cubical sets has poor homotopical properties. This might be solved by using cubical sets with connections. □

Problem 16.1.8 The category \mathbf{Gpds} of groupoids does not satisfy properties analogous to those of the category of groups, i.e. is not semi-abelian in the sense of [JMT02]. However it seems that each fibre of the functor $\text{Ob} : \mathbf{Gpds} \rightarrow \mathbf{Sets}$ is semi-abelian. Is it reasonable to investigate for purposes of homological algebra the general situation of fibrations of categories such that each fibre is semi-abelian, and can such a generalised theory be helpfully applied to crossed complexes? □

Problem 16.1.9 Can one apply to the cubical collapses of Section 11.3.i the methods of finite topological spaces as applied to collapses of simplicial complexes in [BM09]? □

Problem 16.1.10 Is there a nonabelian *homological perturbation theory* for constructing nonabelian twisted tensor products from fibrations? As a start in the literature, see [BL91]. Or for constructing small free crossed resolutions of groups? References for the standard theory, and the important relation to twisted tensor products, may be found by a web search. □

Problem 16.1.11 The standard theory of chain complexes makes much use of double chain complexes. Double crossed complexes have been defined in [Ton94] but presumably there is much more to be done here. □

Problem 16.1.12 The theory of equivariant crossed complexes has already been developed in [BGPT97, BGPT01]. However notions such as fibrations of crossed complexes have not been applied in that area.¹ □

Problem 16.1.13 Can one make progress with nonabelian cohomology operations? The tensor product of crossed complexes is symmetric, as proved in Section 15.4. So if K is a simplicial set, then we can consider the non-commutativity of the diagonal map $\Delta : \pi|K| \rightarrow \pi|K| \otimes |K|$. If T is the twisting map $A \otimes B \rightarrow B \otimes A$, then there is a natural homotopy $T\Delta \simeq \Delta$, by the usual acyclic models argument. This look like the beginnings of a theory of nonabelian Steenrod cohomology operations. Does such a theory exist and does it hold any surprises? By contrast, [Bau89] gives an obstruction to the existence of a Pontrjagin square with local coefficients. □

Problem 16.1.14 One use of chain complexes is in defining Kolmogorov-Steenrod homology. One takes the usual net of polyhedra defined as the nerves of open covers of a space X , with maps between them induced by choices of refinements. The result is a homotopy coherent diagram of polyhedra. This is also related to Čech homology theory. It is shown in [Cor87] that a strong homology theory results by taking the chain complexes of this net, and forming the chain complex which is the homotopy inverse limit. What sort of strong homology theory results from using the fundamental crossed complexes of the nerves instead of the chain complexes? Is there a kind of ‘strong fundamental groupoid’, and could this be related to defining universal covers of spaces which are not locally ‘nice’? □

Problem 16.1.15 There are a number of areas of algebraic topology where chain complexes with a group of operators are used, for example [RW90]. Is it helpful to reformulate this work in terms of crossed complexes? Note that Section 17 of [Whi50b] is given in terms of crossed complexes, but the exposition is sparse. We have earlier related this work to that of Baues in [Bau89, p.357]. A related work on simple homotopy theory is [Bro92], which is also related to generalisations of Tietze equivalences of presentations. Standard expositions of simple homotopy theory, for example [Coh73], are in terms of chain complexes with operators. It may be worth going back to the paper which introduced many of these ideas, namely [Whi41b]. Note that simple homotopy theory is applied to manifolds using filtrations defined by a Morse function in [Maz65]. □

Problem 16.1.16 Another example for the last problem of replacing chain complexes by crossed complexes is the work of Dyer and Vasquez in [DV73] on CW -models for one-relator groups. □

Problem 16.1.17 Can the use of crossed complexes in Morse theory explained by Sharko in [Sha93] be further developed? He writes at the beginning of Chapter VII:

The need to make use of homotopy systems [i.e. free crossed complexes] in order to study Morse functions on non-simply connected closed manifolds or on manifolds with one boundary component arises from the failure of the chain complexes constructed from the

¹Is there a notion of “freeness” relevant to G -CW-complexes, as there is for the crossed complex of an ordinary CW-complex?

Morse functions and gradient-like vector fields to capture completely the geometric aspects of the problem. This relates to application of the Whitney lemma to the reduction of the number of points of intersection of manifolds of complementary dimensions.

Problem 16.1.18 Baues and Tonks in [BT97] use crossed complexes to study the cobar construction. But the original work on the cobar construction in [AH56] was cubical. Can one do better by using many base points instead of just loop spaces, and also using ω -groupoids instead of crossed complexes? \square

Problem 16.1.19 Find applications of these non-abelian constructions to configuration space theory and mapping space theory, particularly the theory of spaces of rational maps. More generally, one can look at areas where the standard tools are simplicial abelian groups, classifying spaces, and some notion of freeness. \square

Problem 16.1.20 A further aim is to use these methods in the theory of stacks and gerbes, and more generally in differential topology and geometry. The ideas of Section 12.5.i are hopefully a start on this. The paper [FMP11] uses directly methods of our ω -groupoids, and for similar reasons to ours, but in the context of smooth manifolds rather than filtered spaces. \square

Problem 16.1.21 Investigate the relation between the cocycle approach to Postnikov invariants and that given using triple cohomology and crossed complexes in [BFGM05]. \square

Problem 16.1.22 One starting intuition for the proof of the HHSvKT was the wish to algebraicise the proof of the cellular approximation theorem due to Frank Adams, and given in [Bro68, Bro06]. Now a subtle proof of an excision connectivity theorem of Blakers and Massey is given in [tD08, Section 6.9]. Can one use methods of crossed squares or cat^n -groups to algebraicise this proof? \square

Problem 16.1.23 It would be good to have another proof of the main result of [BB93], using cubical ω -groupoids. Perhaps one needs also some of the methods of [tD08, Section 6.9]? \square

Problem 16.1.24 There are many problems associated with generalisation of the HHSvKT to n -cubes of spaces as given in [BL87b, BL87a]. For a survey, and references to related literature, see [Bro92]. A recent work in this area is [EM10]. It is not clear what should be the appropriate generalisation to a many base point approach of the work on the fundamental cat^n -group of an n -cube of spaces explained in [BL87b, Gil87]. Note the idea of a fundamental double groupoid of a map of spaces in [BJ04]. Can this be generalised to n -cubes of spaces? Grothendieck remarked in 1985 to Brown that the idea that (strict) n -fold groupoids model homotopy n -types was ‘absolutely beautiful!’. Some relation of cat^n -groups to other models is developed in [Pao09]. \square

Problem 16.1.25 The term ∞ -groupoid has been used for the simplicial singular complex $S^\Delta X$ of a space X and this has also been written ΠX . See for example [Ber02, Lur09, JT07]. However the axiomatic properties of the cubical singular complex $S^\square X$, with its multiple compositions which we use greatly in this book, have not been much investigated. We mention [Ste06] as an approach to use Kan fillers in a categorical situation. \square

Problem 16.1.26 The area of homological algebra has been invigorated by the notion of triangulated category and related areas, see for example [Nee01, K un07]. These are related to chain complexes, also called differential graded objects. However the work of Fr ohlich and of Lue, for which see references in [Lue71], shows the relevance of general notions of crossed modules. Crossed modules and triangulated categories are also used in [MTW10]. Again work of Tabuada [Tab09, Tab10] relates Postnikov invariants and monoidal closed categories. But this is done for dg-objects without the crossed module environment. \square

Problem 16.1.27 One intention of the work of Mosa, [Mos87], was to start on working out the homological algebra of algebroids (rings with several objects) by defining crossed resolutions of algebroids and obtaining a monoidal closed structure on crossed complexes of algebroids. However even the conjectured equivalence between crossed complexes of algebroids and higher dimensional cubical algebroids is unsolved. The difficulty is shown by the complexity of the arguments in [AABS02] compared with those of Chapter 13 of this book. \square

Problem 16.1.28 A programme set by Grothendieck in ‘Pursuing Stacks’ is related to the previous problem. We quoted on p. xiv his aim to understand non-commutative cohomology of topoi. Earlier in the same letter he writes:

For the last three weeks I haven’t gone on writing the notes, as what was going to follow next is presumably so smooth that I went out for some scratchwork on getting an idea about things more obscure still, particularly about understanding the basic structure of ‘(possibly non-commutative) “derived categories”, and the internal homotopy-flavoured properties of the “basic modelizer” (Cat), namely of functors between “small” categories, modelled largely on work done long time ago about  tale cohomology properties of maps of schemes. I am not quite through yet but hope to resume work on the notes next week.

For work of Grothendieck on ‘Modelizers’ and ‘Derivateurs’, see [Gro89, MalDer]. \square

Problem 16.1.29 The last problem is possibly related to the problem of relating the methods of this book to those of the modern theory of sheaves, as discussed in [Ive86], with applications to generalised Poincar e duality, known as Verdier duality. A related area is that of stratified spaces, on which a recent paper using higher order categories is [Woo10]. \square

Problem 16.1.30 A work on monoidal categories, Hopf algebras, species and related areas, and which strongly uses the Eilenberg-Zilber theorem for chain complexes, is [AM10]. There are possibilities of relating their work to that done here, or bringing in crossed complexes into the areas studied in that book. \square

Problem 16.1.31 Can this area of crossed complexes be helpfully related to that of complexes of groups, which generalises graphs of groups, as initiated by Haefliger in [Hae92]?

Problem 16.1.32 There is an extensive theory of quantum groups and of quantum groupoids. Can this be extended to ‘quantum crossed complexes’ using the methods of [Chi11], and related papers referenced there? □

Other problems in crossed complexes and related areas are given in [Bro90].

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