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The fundamental groupoid of the quotient of a Hausdorff space by a discontinuous action of a discrete group is the orbit groupoid of the induced action

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Abstract

The main result is that the fundamental groupoid of the orbit space of a discontinuous action of a discrete group on a Hausdorff space which admits a universal cover is the orbit groupoid of the fundamental groupoid of the space. We also describe work of Higgins and of Taylor which makes this result usable for calculations. As an example, we compute the fundamental group of the symmetric square of a space.

The main result, which is related to work of Armstrong, is due to Brown and Higgins in 1985 and was published in sections 9 and 10 of Chapter 9 of the first author's book on Topology [3]. This is a somewhat edited, and in one point (on normal closures) corrected, version of those sections. Since the book is out of print, and the result seems not well known, we now advertise it here.

It is hoped that this account will also allow wider views of these results, for example in topos theory and descent theory.

Because of its provenance, this should be read as a graduate text rather than an article. The Exercises should be regarded as further propositions for which we leave the proofs to the reader. It is expected that this material will be part of a new edition of the book.

MATH CLASSIFICATION: 20F34, 20L13, 20L15, 57S30

1 Groups acting on spaces

In this section we show some of the theory of a group G acting on a topological space X, and describe the *orbit topological space*, which is written X/G.

There arises the problem of relating topological invariants of the orbit space X/G to those of X and the group action. In particular, it is a complicated and interesting question to find, if at all possible, relations between the fundamental groups and groupoids of X and X/G. This we shall do for a particular family of actions which arise commonly, namely the *discontinuous actions*. The resulting theory generalises that of regular covering spaces, and has a number of important applications. A useful special case of a discontinuous

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action is the action of a finite group on a Hausdorff space (see below); there are in the literature many interesting cases of discontinuous actions of infinite groups (see [2]).

We now come to formal definitions.

Let G be a group, with its group operation written as multiplication, and let X be a set. An *action* of G on X is a function $G \times X \to X$, written $(g, x) \mapsto g \cdot x$, satisfying the following properties for all x in X and g, h in G:

1.1 (i) $1 \cdot x = x$,
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$.

Thus the first rule says that the identity of G acts as identity, and the second rule says that two elements of G, acting successively, act as the product of the two elements.

There are some standard notions associated with such an action. First, an equivalence relation is defined on X by $x \sim y$ if and only if there is an element g of G such that $y = g \cdot x$. This is an equivalence relation. Reflexivity follows since G has an identity. Symmetry follows from the existence of inverses in G, using 1.1 (i), (ii). Transitivity follows from the product of two elements in G being in G. The equivalence classes under this relation are the *orbits* of the action. The set of these orbits is written X/G.

Suppose given an action of the group G on the set X. If $x \in X$, then the *group of stability of* x is the subgroup of G

$$G_x = \{g \in G : g \cdot x = x\}$$

The elements of G_x are said to *stabilise* x, that is, they leave x fixed by their action. If G_x is the whole of G, then x is said to be a *fixed point* of the action. The set of fixed points of the action is often written X^G. The action is said to be *free* if all groups of stability are trivial.

Another useful condition is the notion of *effective* action of a group. This requires that if $g, h \in G$ and for all x in X, $g \cdot x = h \cdot x$, then $g = h$. In this case the elements of G are entirely determined by their action on X.

We now turn to the topological situation. Let X be a topological space, and let G be a group. An *action* of G on X is again a function $G \times X \to X$ with the same properties as given in 1.1, but with the additional condition that when G is given the discrete topology then the function $(g, x) \mapsto g \cdot x$ is continuous. This amounts to the same as saying that for all $g \in G$, the function $g_\sharp : x \mapsto g \cdot x$ is continuous. Note that $g_\sharp$ is a bijection with inverse $(g^{-1})_\sharp$, and since these two functions are continuous, each is a homeomorphism.

Let X/G be the set of orbits of the action and let $p : X \to X/G$ be the *quotient map*, which assigns to each x in X its orbit. For convenience we will write the orbit of x under the action as $\bar{x}$. So the defining property is that $\bar{x} = \bar{y}$ if and only if there is a g in G such that $y = g \cdot x$. Now a topology has been given for X. We therefore give the orbit space X/G the identification topology with respect to the map p. This topology will always be assumed in what follows. The first result on this topology, and one which is used a lot, is as follows.

Proposition 1.2 *The quotient map $p : X \to X/G$ is an open map.*

Proof Let U be an open set of X. For each $g \in G$ the set $g \cdot U$, by which is meant the set of $g \cdot x$ for all x in U, is also an open set of X, since $g_\sharp$ is a homeomorphism, and $g \cdot U = g_\sharp[U]$. But

$$p^{-1}p[U] = \bigcup_{g \in G} g \cdot U.$$

Since the union of open sets is open, it follows that $p^{-1}p[U]$ is open, and hence $p[U]$ is open. □

Definition 1.3 An action of the group G on the space X is called *discontinuous* if the stabiliser of each point of X is finite, and each point x in X has a neighbourhood V_x such that any element g of G not in the stabiliser of x satisfies $V_x \cap g \cdot V_x = \emptyset$.

Suppose G acts discontinuously on the space X. For each x in X choose such an open neighbourhood V_x of x. Since the stabiliser G_x of x is finite, the set

$$U_x = \bigcap \{g \cdot V_x : g \in G_x\}$$

is open; it contains x since the elements of G_x stabilise x. Also if $g \in G_x$ then $g \cdot U_x = U_x$. We say U_x is *invariant* under the action of the group G_x. On the other hand, if $h \notin G_x$ then

$$(h \cdot U_x) \cap U_x \subseteq (h \cdot V_x) \cap V_x = \emptyset.$$

An open neighbourhood U of x which satisfies $(h \cdot U) \cap U = \emptyset$ for $h \notin G_x$ and is invariant under the action of G_x is called a *canonical neighbourhood* of x. Note that any neighbourhood N of x contains a canonical neighbourhood: the proof is obtained by replacing V_x in the above by $N \cap V_x$. The image in X/G of a canonical neighbourhood U of x is written $\overline{U}$ and called a *canonical neighbourhood* of $\bar{x}$.

In order to have available our main example of a discontinuous action, we prove:

Proposition 1.4 *An action of a finite group on a Hausdorff space is discontinuous.*

Proof Let G be a finite group acting on the Hausdorff space X. Then the stabiliser of each point of X is a subgroup of G and so is finite.

Let $x \in X$. Let $x_0, x_1, \ldots, x_n$ be the distinct points of the orbit of x, with $x_0 = x$. Suppose $x_i = g_i \cdot x, g_i \in G, i = 1, \ldots, n$, and set $g_0 = 1$. Since X is Hausdorff, we can find pairwise disjoint open neighbourhoods N_i of $x_i, i = 0, \ldots, n$. Let

$$N = \bigcap \{g_i^{-1} \cdot N_i : i = 0, 1, \ldots n\}$$

Then N is an open neighbourhood of x. Also, if $g \in G$ does not belong to the stabiliser G_x of x, then for some $j = 1, \ldots, n, g \cdot x = x_j$, whence $g \cdot N \subseteq N_j$. Hence $N \cap g \cdot N = \emptyset$, and the action is discontinuous. $\Box$

Our main result in general topology on discontinuous actions is the following.

Proposition 1.5 *If the group G acts discontinuously on the Hausdorff space X, then the quotient map $p : X \to X/G$ has the path lifting property: that is, if $\bar{a} : \mathbf{I} \to X/G$ is a path in X/G and x_0 is a point of X such that $p(x_0) = \bar{a}(0)$, then there is a path $a : \mathbf{I} \to X$ such that $pa = \bar{a}$ and $a(0) = x_0$.*

Proof If there is a lift a of $\bar{a}$ then there is an element g of G such that $g \cdot a(0) = x_0$ and so $g \cdot a$ is a lift of $\bar{a}$ starting at x_0. So we may ignore x_0 in what follows.

Since the action is discontinuous, each point $\bar{x}$ of X/G has a canonical neighbourhood. By the Lebesgue covering lemma, there is a subdivision

$$\bar{a} = \bar{a}_n + \cdots + \bar{a}_1$$

of $\bar{a}$ such that the image of each $\bar{a}_i$ is contained in a canonical neighbourhood. So if the path lifting property holds for each canonical neighbourhood in X/G, then it holds for X/G.

Now $p : X \to X/G$ is an open map. Hence for all x in X the restriction $p_x : U_x \to \overline{U}_x$ of p to a canonical neighbourhood U_x is also open, and hence is an identification map. So we can identify $\overline{U}_x$ with the orbit space $(U_x)/G_x$. The key point in this case is that the group G_x is finite.

Thus it is sufficient to prove the path lifting property for the case of the action of a *finite* group G, and this we do by induction on the order of G. That is, we assume that path lifting holds for any action of any proper subgroup of G on a Hausdorff space, and we prove that path lifting holds for the action of G. The case $|G| = 1$ is trivial.

Again let $\bar{a}$ be as in the proposition, and we are assuming G is finite. Let F be the set of fixed points of the action. Then F is the intersection for all $g \in G$ of the sets $X^g = \{x \in X : g \cdot x = x\}$. Since X is Hausdorff, the set X^g is closed in X, and hence F is closed in X. So $p[F]$ is closed, since $p^{-1}p[F] = F$. Let A be the subspace of $\mathbf{I}$ of points t such that $\bar{a}(t)$ belongs to $p[F]$, that is, $A = \bar{a}^{-1}p[F]$. Then A is closed.

The restriction of the quotient map p to $p' : F \to p[F]$ is a homeomorphism. So $\bar{a} \mid_A$ has a unique lift to a map $a \mid_A$:A$\to X$. So we have to show how to lift $\bar{a} \mid_{(\mathbf{I}\backslash A)}$ to give a map $a \mid_{\mathbf{I}\backslash A}$ and then show that the function $a : \mathbf{I} \to X$ defined by these two parts is continuous.

In order to construct $a \mid_{\mathbf{I}\backslash A}$, we first assume $A = \{1\}$.

Let S be the set of $s \in \mathbf{I}$ such that $\bar{a} \mid_{[0,s]}$ has a lift to a map a_s starting at x. Then S is non-empty, since $0 \in S$. Also S is an interval. Let $u = \sup S$. Suppose $u < 1$. Then there is a $y \in X\backslash F$ such that $p(y) = \bar{a}(u)$. Choose a canonical neighbourhood U of y. If $u > 0$, there is a $\delta > 0$ such that $\bar{a}[u - \delta, u + \delta] \subseteq \bar{U}$. Then $u - \delta \in S$ and so there is a lift $a_{u-\delta}$ on $[0, u - \delta]$. Also the stabiliser of y is a proper subgroup of G, since $y \notin F$, and so by the inductive assumption there is a lift of $\bar{a} \mid_{[u-\delta,u+\delta]}$ to a path starting at $a_{u-\delta}(u-\delta)$. Hence we obtain a lift $a_{u+\delta}$, contradicting the definition of u. We get a similar contradiction to the case $u = 0$ by replacing in the above $u - \delta$ by 0. It follows that $u = 1$. We are not quite finished because all we have thus ensured is that there is a lift on $[0, s]$ for each $0 \leqslant s < 1$, but this is not the same as saying that there is a lift on $[0, 1)$. We now prove that such a lift exists.

By the definition of u, and since $u = 1$, for each integer $n \geqslant 1$ there is a lift a^n of $\bar{a} \mid_{[0,1-n^{-1}]}$. Also there is an element g_n of G such that

$$g_n \cdot a^{n+1}(1 - n^{-1}) = a^n(1 - n^{-1})$$

Hence a^n and $g_n \cdot (a^{n+1} \mid_{[1-n^{-1},1-(n+1)^{-1}]})$ define a lift of $\bar{a} \mid_{[0,1-(n+1)^{-1}]}$. Starting with $n = 1$, and continuing in this way, gives a lift of $\bar{a} \mid_{[0,1)}$. This completes the construction of the lift on $\mathbf{I}\backslash A$ in the case $A = \{1\}$.

We now construct a lift of $\bar{a} \mid_{\mathbf{I}\backslash A}$ in the general case. Since A is closed, $\mathbf{I}\backslash A$ is a union of disjoint open intervals each with end points in $\{0, 1\} \cup A$. So the construction of the lift is obtained by starting at the mid point of any such interval and working backwards and forwards, using the case $A = \{1\}$, which we have already proved.

The given lift of $\bar{a} \mid_A$ and the choice of lift of $\bar{a} \mid_{\mathbf{I}\backslash A}$ together define a lift $a : \mathbf{I} \to X$ of $\bar{a}$ and it remains to prove that a is continuous.

Let $t \in \mathbf{I}$. If $t \notin A$, then a is continuous at t by construction. Suppose then $t \in A$ so that $y = a(t) \in F$. Let N be any neighbourhood of y. Then N contains a canonical neighbourhood U of y. If $g \in G$ then $g \cdot y = y$, and so U is invariant under the action of G. Hence $p^{-1}p[U] = U$. Since $\bar{a}$ is continuous, there is a neighbourhood M of t such that $\bar{a}[M] \subseteq \bar{U}$. Since $pa = \bar{a}$ it follows that $a[M] \subseteq p^{-1}[\bar{U}] = U$. This proves continuity of A, and the proof of the proposition is complete. □

In our subsequent results we shall use the path lifting property rather than the condition of the action being discontinuous.

Our aim now is to determine the fundamental groupoid of the orbit space X/G. In general it is difficult to say much. However we can give reasonable and useful conditions for which the question can be completely answered. From our point of view the result is also of interest in that our statement and proof use groupoids in a crucial way. This use could be overcome, but at the cost of complicating both the statement of the theorem and its proof.

In order to make the transition from the topology to the algebra, it is necessary to introduce the notion of a group acting on a groupoid.

Let G be a group and let Γ be a groupoid. We will write the group structure on G as multiplication,

and the groupoid structure on Γ as addition. An *action* of G on Γ assigns to each $g \in G$ a morphism of groupoids $g_\sharp : \Gamma \to \Gamma$ with the properties that $1_\sharp = 1 : \Gamma \to \Gamma$, and if $g, h \in G$ then $(hg)_\sharp = h_\sharp g_\sharp$. If $g \in G, x \in \mathrm{Ob}(\Gamma), a \in \Gamma$, then we write $g \cdot x$ for $g_\sharp(x), g \cdot a$ for $g_\sharp(a)$. Thus the rules 1.1 apply also to this situation, as well as the laws **1.1** (iii) $g.(a+b) = g.a + g.b$, and (iv) $g.0_x = 0_{g.x}$
for all $g \in G, x \in \mathrm{Ob}(\Gamma), a, b \in Gamma$ such that $a + b$ is defined.

The action of G on Γ is *trivial* if $g_\sharp = 1$ for all g in G.

Definition 1.6 Let G be a group acting on a groupoid Γ. An *orbit groupoid* of the action is a groupoid $\Gamma//G$ together with a morphism $p : \Gamma \to \Gamma//G$ such that:

(i) If $g \in G, \gamma \in \Gamma$, then $p(g \cdot \gamma) = p(\gamma)$.

(ii) The morphism p is universal for (i), *i.e.* if $\phi : \Gamma \to \Phi$ is a morphism of groupoids such that $\phi(g \cdot \gamma) = \phi(\gamma)$ for all $g \in G, \gamma \in \Gamma$, then there is a unique morphism $\phi^* : \Gamma//G \to \Phi$ of groupoids such that $\phi^* p = \phi$.

The morphism $p : \Gamma \to \Gamma//G$ is then called an *orbit morphism*. □

The universal property (ii) implies that $\Gamma//G$, if it exists, is unique up to a canonical isomorphism. At the moment we are not greatly concerned with proving any general statement about the *existence* of the orbit groupoid. One can argue that $\Gamma//G$ is obtained from Γ by imposing the relations $g \cdot \gamma = \gamma$ for all $g \in G$ and all $\gamma \in \Gamma$; however we have not yet explained quotients in this generality. We will later prove existence by giving a *construction* of $\Gamma//G$ which will be useful in interpreting our main theorem. But our next result will give conditions which ensure that the induced morphism $\pi X \to \pi(X/G)$ is an orbit morphism, and our proof will not *assume* general results on the existence of the orbit groupoid. The reason we can do this is that our proof directly verifies a universal property.

First we must point out that if the group G acts on the space X, then G acts on the fundamental groupoid πX, since each g in G acts as a homeomorphism of X and $g_\sharp : \pi X \to \pi X$ may be defined to be the induced morphism. This is one important advantage of groupoids over groups: by contrast, the group G acts on the fundamental group $\pi(X, x)$ only if x is a fixed point of the action.

Suppose now that G acts on the space X. Our purpose is to give conditions on the action which enable us to prove that

$$p_* : \pi X \to \pi(X/G)$$

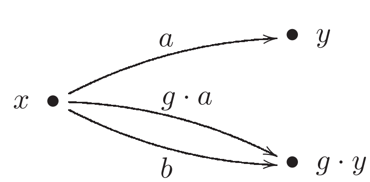
determines an isomorphism $(\pi X)//G \to \pi(X/G)$, by verifying the universal property for p_*. We require the following conditions:

Conditions 1.7 (i) The projection $p : X \to X/G$ has the path lifting property: *i.e.* if $\bar{a} : I \to X/G$ is a path, then there is a path $a : I \to X$ such that $pa = \bar{a}$.

(ii) If $x \in X$, then x has an open neighbourhood U_x such that

(a) if $g \in G$ does not belong to the stabiliser G_x of X, then $U_x \cap (g \cdot U_x) = \emptyset$;

(b) if a and b are paths in U_x beginning at x and such that pa and pb are homotopic rel end points in X/G, then there is an element $g \in G_x$ such that $g \cdot a$ and b are homotopic in X rel end points.



For a discontinuous action, 1.7(ii:a) trivially holds, while 1.7(i) holds by virtue of 1.5. However, 1.7(ii:b) is an extra condition. It does hold if X is semi-locally simply-connected, since then for sufficiently small U and

$x, g \cdot y \in U$, any two paths in U from x to $g \cdot y$ are homotopic in X rel end points; so 1.7(ii:b) is a reasonable condition to use in connection with covering space theory.

A neighbourhood U_x of $x \in X$ satisfying 1.7(i) and 1.7(ii) will be called a *strong canonical neighbourhood* of X. The image $p[U_x]$ of U_x in X/G will be called a *strong canonical neighbourhood* of px.

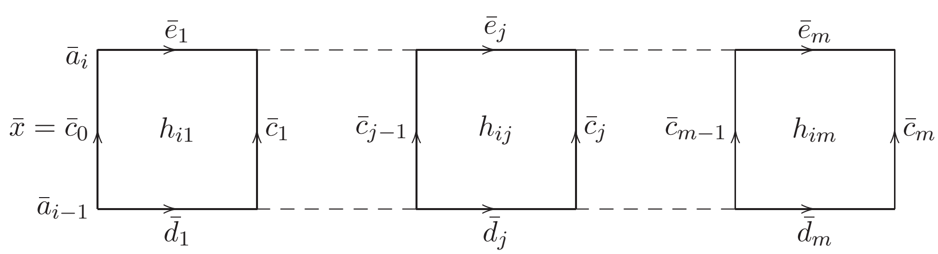
Proposition 1.8 *If the action of G on X satisfies 1.7(i) and 1.7(ii) above, then the induced morphism $p_* : \pi X \to \pi(X/G)$ makes $\pi(X/G)$ the orbit groupoid of πX by the action of G.*

Proof Let $\phi : \pi X \to \Phi$ be a morphism to a groupoid Φ such that $\phi(g \cdot \gamma) = \phi(\gamma)$ for all $\gamma \in \pi X$ and $g \in G$. We wish to construct a morphism $\phi^* : \pi(X/G) \to \Phi$ such that $\phi^* p = \phi$.

Let $\bar{a}$ be a path in X/G. Then $\bar{a}$ lifts to a path a in X. Let $[b]$ denote the homotopy class rel end points of a path b. We prove that $\phi[a]$ in Φ is independent of the choice of $\bar{a}$ in its homotopy class and of the choice of lift a; hence we can define $\phi^*[\bar{a}]$ to be $\phi[a]$.

Suppose given two homotopic paths $\bar{a}$ and $\bar{b}$ in X/G, with lifts a and b which without loss of generality we may assume start at the same point x in X. (If they do not start at the same point, then one of them may be translated by the action of G to start at the same point as the other.) Let $h : \mathbf{I} \times \mathbf{I} \to X/G$ be a homotopy rel end points $\bar{a} \simeq \bar{b}$. The method now is not to lift the homotopy h itself, but to lift pieces of a subdivision of h; it is here that the method differs from that used in the theory of covering spaces given in Section 9.1 of [3].

Subdivide $\mathbf{I} \times \mathbf{I}$, by lines parallel to the axes, into small squares each of which is mapped by h into a strong canonical neighbourhood in X/G. This subdivision determines a sequence of homotopies $h_i : \bar{a}_{i-1} \simeq \bar{a}_i, i = 1, 2, \ldots, n$, say, where $\bar{a}_0 = \bar{a}, \bar{a}_n = \bar{b}$. Keep i fixed for the present. Each h_i is further expressed by the subdivision as a composite of homotopies $h_{ij} (j = 1, 2, \ldots, m)$ as shown in the following picture in which for convenience the boundaries of the h_{ij} are labelled:



Choose lifts a_{i-1}, a_i of $\bar{a}_{i-1}, \bar{a}_i$ respectively; express a_{i-1} as a sum $a_{i-1} = d_m + \cdots + d_1$ and a_i as a sum $a_i = e_m + \cdots + e_1$ where d_j lifts $\bar{d}_j$ and e_j lifts $\bar{e}_j$. Choose for each j a lift c_j of $\bar{c}_j$ (with c_0 the constant path at x). For fixed j choose $f, g, h \in G$ such that $g \cdot d_j$ has the same initial point as c_{j-1} and the sums

$$f \cdot c_j + g \cdot d_j, \quad h \cdot e_j + c_{j-1}$$

are defined. This is possible because of the boundary relations between the projections in X/G of the various paths.

Now our assumption (ii:b) of Conditions 1.7 implies that there is an element $k \in G$ such that the following paths in X

$$k \cdot (f \cdot c_j + g \cdot d_j), \quad h \cdot e_j + c_{j-1}$$

are homotopic rel end points in X. On applying ϕ to homotopy classes of paths in X and using equations such as $\phi(g \cdot \gamma) = \phi(\gamma)$ we find that

$$\begin{aligned} \phi[e_j] + \phi[c_{j-1}] &= \phi[h \cdot e_j] + \phi[c_{j-1}] \\ &= \phi[h \cdot e_j + c_{j-1}] \\ &= \phi[k \cdot (f \cdot c_j + g \cdot d_j)] \\ &= \phi[k \cdot f \cdot c_j] + \phi[g \cdot d_j] \\ &= \phi[c_j] + \phi[d_j]. \end{aligned}$$

This proves that
$$\phi[e_j] = \phi[c_j] + \phi[d_j] - \phi[c_{j-1}].$$
It follows easily that
$$\phi[a_{i-1}] = \phi[a_i].$$
and hence by induction on i that $\phi[a] = \phi[b]$. From this it follows that $\phi^* : \pi(X/G) \to \Phi$ is a well defined function such that $\phi^* p = \phi$. The uniqueness of ϕ^* is clear since p_* is surjective on elements, by the path lifting property of Conditions 1.7. The proof that ϕ^* is a morphism is simple. This completes the proof of Proposition 1.8. □

In the next section, we introduce some further constructions in the theory of groupoids and groups acting on groupoids, in order to interpret Proposition 1.8 in a manner suitable for calculations. Once again, we will find that an apparently abstract result involving a universal property can, when appropriately interpreted, lead to specific calculations.

EXERCISES 1

1. Let $\lambda \in \mathbb{R}$ and let the additive group $\mathbb{R}$ of real numbers act on the torus $T = \mathbf{S}^1 \times \mathbf{S}^1$ by
$$t \cdot (e^{2\pi i\theta}, e^{2\pi i\phi}) = (e^{2\pi i(\theta+t)}, e^{2\pi i(\phi+\lambda t)})$$
for $t, \theta, \phi \in \mathbb{R}$. Prove that the orbit space has the indiscrete topology if and only if λ is irrational. [You may assume that the group generated by 1 and λ is dense in $\mathbb{R}$ if and only if λ is irrational.]

2. Let G be a group and let X be a G-space. Prove that the quotient map $p : X \to X/G$ has the following universal property: if Y is a space and $f : X \to Y$ is a map such that $f(g \cdot x) = f(x)$ for all $x \in X$ and $g \in G$, then there is a unique map $f^* : X/G \to Y$ such that $f^* p = f$.

3. Let X, Y, Z be G-spaces and let $f : X \to Z, h : Y \to Z$ be G-maps (*i.e.* $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$, and similarly for h). Let $W = X \times_Z Y$ be the pullback. Prove that W becomes a G-space by the action $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Prove also that if $Z = X/G$ and f is the quotient map, then W/G is homeomorphic to Y.

4. Let G and Γ be groupoids and let $w : \Gamma \to \mathrm{Ob}(G)$ be a morphism where $\mathrm{Ob}(G)$ is considered as a groupoid with identities only. An *action of* G *on* Γ *via* w is an assignment to each $g \in G(x, y)$ and $\gamma \in w^{-1}[x]$ an element $g \cdot \gamma \in w^{-1}[y]$ and with the usual rules: $h \cdot (g \cdot \gamma) = (hg) \cdot \gamma; 1 \cdot \gamma = \gamma; g \cdot (\gamma + \delta) = g \cdot \gamma + g \cdot \delta$. In this case Γ is called a G-groupoid. Show how to define a category of G-groupoids so that this category is equivalent to the functor category $\mathsf{FUN}(G, \mathsf{Set})$.

5. Prove that if Γ is a G-groupoid via w, then $\pi_0\Gamma$ becomes a G-set via $\pi_0(w)$.

6. If Γ is a G-groupoid via w, then the action is *trivial* if for all $x, y \in \mathrm{Ob}(G), g, h \in G(x, y)$ and $\gamma \in w^{-1}[x]$, we have $g \cdot \gamma = h \cdot \gamma$. Prove that the action is trivial if for all $x \in \mathrm{Ob}(G)$, the action of the group $G(x)$ on the groupoid $w^{-1}[x]$ is trivial. Prove also that Γ contains a unique maximal subgroupoid Γ^G on which G acts trivially. Give examples to show that Γ^G may be empty.

7. Continuing the previous exercise, define a G-*section* of w to be a morphism $s : \mathrm{Ob}(G) \to \Gamma$ of groupoids such that $ws = 1$ and s commutes with the action of G, where G acts on $\mathrm{Ob}(G)$ via the source map by $g \cdot x = y$ for $g \in G(x, y)$. Prove that Γ^G is non-empty if w has a G-section, and that the converse holds if G is connected. Given a G-section s, let $\Gamma^G(s)$ be the set of functions $a : \mathrm{Ob}(G) \to \Gamma$ such that $wa = 1$ and a commutes with the action of G (but we do not assume a is a morphism). Show that $\Gamma^G(s)$ forms a group under addition of values, and that if G is connected and $x \in \mathrm{Ob}(G)$, then $\Gamma^G(s)$ is isomorphic to the group $\Gamma(sx)^{G(x)}$ of fixed points of $\Gamma(sx)$ under the action of $G(x)$.

2 The computation of orbit groupoids: quotients and semidirect products

The theory of quotient groupoids is modelled on that of quotient groups, but differs from it in important respects. In particular, the First Isomorphism Theorem of group theory (that every surjective morphism of groups is obtained essentially by factoring out its kernel) is no longer true for groupoids, so we need to characterise those groupoid morphisms (called quotient morphisms) for which this isomorphism theorem holds. The next two propositions achieve this; they were first proved in [5].

Let $f : K \to H$ be a morphism of groupoids. Then f is said to be a *quotient morphism* if $\mathrm{Ob}(f) : \mathrm{Ob}(K) \to \mathrm{Ob}(H)$ is surjective and for all x, y in $\mathrm{Ob}(K)$, $f : K(x, y) \to H(fx, fy)$ is also surjective. Briefly, we say f is *object surjective* and *full*.

Proposition 2.1 *Let $f : K \to H$ be a quotient morphism of groupoids. Let $N = \mathrm{Ker}\, f$. The following hold:*

(i) If $k, k' \in K$, then $f(k) = f(k')$ if and only if there are elements $m, n \in N$ such that $k' = m + k + n$.

(ii) If x is an object of K, then $H(fx)$ is isomorphic to the quotient group $K(x)/N(x)$.

Proof (i) If k, k' satisfy $k' = m + k + n$ where $m, n \in N$, then clearly $f(k) = f(k')$.

Suppose conversely that $f(k) = f(k')$, where $k \in K(x, y), k' \in K(x', y')$. Then

$$f(x) = f(x'), f(y) = f(y').$$

$$\begin{tikzcd} x \arrow[r, "k"] & y \arrow[d, dashed, "m"] \\ x' \arrow[u, dashed, "n"] \arrow[r, "k'"'] & y' \end{tikzcd}$$

Since $f : K(y, y') \to H(fy, fy')$ is surjective, there is an element $m \in K(y, y')$ such that $f(m) = 0_{f(y)}$. Similarly, there is an element $n \in K(x', x)$ such that $f(n) = 0_{f(x)}$. It follows that if

$$n' = -k' + m + k + n \in K(x'),$$

then $f(n') = 0_{f(x')}$, and so $n' \in N$. Hence $k' = m + k + n - n'$, where $m, n - n' \in N$. This proves (i).

(ii) By definition of quotient morphism, the restriction $f' : K(x) \to H(fx)$ is surjective. Also by 2.1(i), if $f'(k) = f'(k')$ for $k, k' \in K(x)$, then there are $m, n \in N(x)$ such that $k' = m + k + n$. Since $N(x)$ is normal in $K(x)$, there is an m' in $N(x)$ such that $m + k = k + m'$. Hence $k' + N(x) = k + N(x)$. Conversely, if $k' + N(x) = k + N(x)$ then $f'(k') = f'(k)$. So f' determines an isomorphism $K(x)/N(x) \to H(fx)$. □

We recall the definition of normal subgroupoid.

Let G be a groupoid. A subgroupoid N of G is called *normal* if N is wide in G (i.e. $\mathrm{Ob}(N) = \mathrm{Ob}(G)$) and, for any objects x, y of G and $a \in G(x, y)$, $aN(x)a^{-1} \subseteq N(y)$, from which it easily follows that

$$aN(x)a^{-1} = N(y).$$

We now prove a converse of the previous result. That is, we suppose given a normal subgroupoid N of a groupoid K and use (i) of Proposition 2.1 as a model for constructing a quotient morphism $p : K \to K/N$.

The object set of K/N is to be $\pi_0 N$, the set of components of N. Recall that a normal subgroupoid is, by definition, wide in K, so that $\pi_0 N$ is also a quotient set of $X = \mathrm{Ob}(K)$. Define a relation on the elements of

K by $k' \sim k$ if and only if there are elements m, n in N such that $m + k + n$ is defined and equal to k'. It is easily checked, using the fact that N is a subgroupoid of K, that $\sim$ is an equivalence relation on the elements of K. The set of equivalence classes is written K/N. If $\mathsf{cls}\, k$ is such an equivalence class, and $k \in K(x, y)$, then the elements $\mathsf{cls}\, x$, $\mathsf{cls}\, y$ in $\pi_0 N$ are independent of the choice of k in its equivalence class. So we can write $\mathsf{cls}\, k \in K/N(\mathsf{cls}\, x, \mathsf{cls}\, y)$. Let $p : K \to K/N$ be the quotient function. So far, we have not used normality of N. Not surprisingly, normality is used to give K/N an addition which makes it into a groupoid.

Suppose
$$\mathsf{cls}\, k_1 \in (K/N)(\mathsf{cls}\, x, \mathsf{cls}\, y), \quad \mathsf{cls}\, k_2 \in (K/N)(\mathsf{cls}\, y, \mathsf{cls}\, z).$$

Then we may assume $k_1 \in K(x, y), k_2 \in K(y', z)$, where $y \sim y'$ in $\pi_0 N$. So there is an element $l \in N(y, y')$, and we define
$$\mathsf{cls}\, k_2 + \mathsf{cls}\, k_1 = \mathsf{cls}\,(k_2 + l + k_1).$$

We have to show that this addition is well defined. Suppose then
$$k_1' = m_1 + k_1 + n_1,$$
$$k_2' = m_2 + k_2 + n_2,$$
where $m_1, n_1, m_2, n_2 \in N$. Choose any l' such that $k_2' + l' + k_1'$ is defined. Then we have the following diagram, in which a, a' are to be defined:

$$\begin{tikzcd}
\bullet \arrow[r, "k_1"] & \bullet \arrow[r, "l"] \arrow[d, "m_1"] & \bullet \arrow[r, "k_2"] \arrow[loop above, "a"] & \bullet \arrow[loop above, "a'"] \arrow[d, "m_2"] \\
\bullet \arrow[u, "n_1"] \arrow[r, "k_1'"'] & \bullet \arrow[r, "l'"'] & \bullet \arrow[u, "n_2"'] \arrow[r, "k_2'"'] & \bullet
\end{tikzcd}$$

Let $a = n_2 + l' + m_1 - l$. Then $a \in N$, and $l = -a + n_2 + l' + m_1$. Since N is normal there is an element $a' \in N$ such that $a' + k_2 = k_2 - a$. Hence
$$\begin{aligned} k_2 + l + k_1 &= k_2 - a + n_2 + l' + m_1 + k_1 \\ &= a' + k_2 + n_2 + l' + m_1 + k_1 \\ &= a' - m_2 + k_2' + l' + k_1' - n_1. \end{aligned}$$

Since $a', m_2, n_1 \in N$, we obtain $\mathsf{cls}\,(k_2 + l + k_1) = \mathsf{cls}\,(k_2' + l' + k_1')$ as was required.

Now we know that the addition on K/N is well defined, it is easy to prove that the addition is associative, has identities, and has inverses. We leave the details to the reader. So we know that K/N becomes a groupoid.

Proposition 2.2 *Let N be a normal subgroupoid of the groupoid K, and let K/N be the groupoid just defined. Then*

(i) the quotient function $p : k \mapsto \mathsf{cls}\, k$ is a quotient morphism $K \to K/N$ of groupoids;

(ii) if $f : K \to H$ is any morphism of groupoids such that Ker f *contains N, then there is a unique morphism $f^* : K/N \to H$ such that $f^* p = f$.*

Proof The proof of (i) is clear. Suppose f is given as in (ii). If $m + k + n$ is defined in K and $m, n \in N$, then $f(m + k + n) = f(k)$. Hence f^* is well defined on K/N by $f^*(\mathsf{cls}\, k) = f(k)$. Clearly $f^* p = f$. Since p is surjective on objects and elements, f^* is the only such morphism. $\Box$

In order to apply these results, we need generalisations of some facts on normal closures which were given in Section 3 of Chapter 8 for the case of a family $R(x)$ of subsets of the object groups $K(x), x \in \mathrm{Ob}(K)$, of a groupoid K. The argument here is based on [6, Exercise 4, p.95].

Suppose that R is *any* set of elements of the groupoid K. The *normal closure* of R in K is the smallest normal subgroupoid $N(R)$ of K containing R. Clearly $N(R)$ is the intersection of all normal subgroupoids of K containing R, but it is also convenient to have an explicit description of $N(R)$.

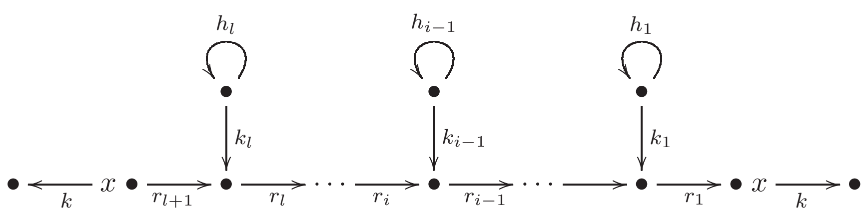
Proposition 2.3 *Let $\langle R\rangle$ be the wide subgroupoid of K generated by R. Then the normal closure $N(R)$ of R is the subgroupoid of K generated by $\langle R\rangle$ and all conjugates khk^{-1} for $k \in K, h \in \langle R\rangle$.*

Proof Let $\widehat{R}$ be the subgroupoid of K generated by $\langle R\rangle$ and all conjugates khk^{-1} for $k \in K, h \in \langle R\rangle$. Clearly any normal subgroupoid of K containing R contains $\widehat{R}$, so it is sufficient to prove that $\widehat{R}$ is normal.

Suppose then that $k + a - k$ is defined where $k \in K$ and $a \in \widehat{R}$ so that

$$a = r_1 + c_1 + r_2 + c_2 + \cdots + r_l + c_l + r_{l+1}$$

where each $r_i \in \langle R\rangle$ and each $c_i = k_i + h_i - k_i$ is a conjugate of a loop h_i in $\langle R\rangle$ by an element $k_i \in K$.



Then a is a loop, since $k + a - k$ is defined, and so also is

$$b = r_1 + r_2 + \cdots + r_{l+1}.$$

Let

$$d_i = k + r_1 + \cdots + r_i + c_i - r_i - \cdots - r_1 - k$$

so that d_i is a conjugate of a loop in $\langle R\rangle$ for $i = 1, \ldots, l$. Then it is easily checked that

$$k + a - k = d_1 + \cdots + d_l + k + b - k$$

and hence $k + a - k \in \widehat{R}$. □

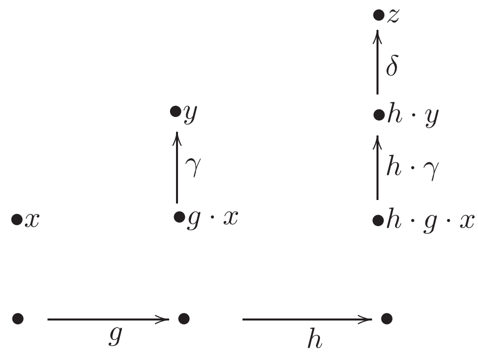
Notice that the loop b in the proof belongs to $\langle R\rangle$ rather than to R, and this shows why it is not enough just to take $N(R)$ to be the subgroupoid generated by R and conjugates of loops in R.

The elements of $N(R)$ as constructed above may be called the *consequences* of R.

We next give the definition of the *semidirect product* of a group with a groupoid on which it acts. Let G be a group and let Γ be a groupoid with G acting on the left. The *semidirect product groupoid* $\Gamma \rtimes G$ has object set $\mathrm{Ob}(\Gamma)$ and arrows $x \to y$ the set of pairs (γ, g) such that $g \in G$ and $\gamma \in \Gamma(g \cdot x, y)$. The sum of $(\gamma, g) : x \to y$ and $(\delta, h) : y \to z$ in $\Gamma \rtimes G$ is defined to be

$$(\delta, h) + (\gamma, g) = (\delta + h \cdot \gamma, hg).$$

This is easily remembered from the following picture.



Proposition 2.4 *The above addition makes* $\Gamma \rtimes G$ *into a groupoid and the projection*

$$q : \Gamma \rtimes G \to G, \quad (\gamma, g) \mapsto g,$$

is a fibration of groupoids. Further:

(i) q *is a quotient morphism if and only if* Γ *is connected;*

(ii) q *is a covering morphism if and only if* Γ *is discrete;*

(iii) q *maps* $(\Gamma \rtimes G)(x)$ *isomorphically to* G *for all* $x \in \mathrm{Ob}(\Gamma)$ *if and only if* Γ *has trivial object groups and* G *acts trivially on* $\pi_0\Gamma$.

Proof The proof of the axioms for a groupoid is easy, the negative of (γ, g) being $(g^{-1} \cdot (-\gamma), g^{-1})$. We leave the reader to check associativity.

To prove that q is a fibration, let $g \in G$ and $x \in \mathrm{Ob}(\Gamma)$. Then $(0_{g \cdot x}, g)$ has source x and maps by q to g.

We now prove (i). Let x, y be objects of Γ. Suppose q is a quotient morphism. Then q maps $(\Gamma \rtimes G)(x, y)$ surjectively to G and so there is an element (γ, g) such that $q(\gamma, g) = 1$. So $g = 1$ and $\gamma \in \Gamma(x, y)$. This proves Γ is connected.

Suppose Γ is connected. Let $g \in G$. Then there is a $\gamma \in \Gamma(g \cdot x, y)$, and so $q(\gamma, g) = g$. Hence q is a quotient morphism.

We now prove (ii). Suppose Γ is discrete, so that Γ may be thought of as a set on which G acts. Then $\Gamma \rtimes G$ is simply the covering groupoid of the action as constructed in a previous section. So q is a covering morphism.

Let x be an object of Γ. If γ is an element of Γ with source x then $(\gamma, 1)$ is an element of $\Gamma \rtimes G$ with source x and which lifts 1. So if q is a covering morphism then the star of Γ at any x is a singleton, and so Γ is discrete.

The proof of (iii) is best handled by considering the exact sequence based at $x \in \mathrm{Ob}(\Gamma)$ of the fibration q. This exact sequence is by 7.2.10 of [3]

$$1 \to \Gamma(x) \to (\Gamma \rtimes G)(x) \xrightarrow{q'} G \to \pi_0\Gamma \to \pi_0(\Gamma \rtimes G) \to 1.$$

It follows that q' is injective if and only if $\Gamma(x)$ is trivial. Exactness also shows that q' is surjective for all x if and only if the action of G on $\pi_0\Gamma$ is trivial. $\square$

Here is a simple application of the definition of semidirect product which will be used later.

Proposition 2.5 *Let G be a group and let Γ be a G-groupoid. Then the formula*

$$(\gamma, g) \cdot \delta = \gamma + g \cdot \delta$$

for $\gamma, \delta \in \Gamma, g \in G$, defines an action of $\Gamma \rtimes G$ on the set Γ via the target map $\tau : \Gamma \to \mathrm{Ob}(\Gamma)$.

Proof This says in the first place that if $(\gamma, g) \in (\Gamma \rtimes G)(y, z)$ and δ has target y, then $\gamma + g \cdot \delta$ has target z, as is easily verified. The axioms for an action are easily verified. The formula for the action also makes sense if one notes that

$$(\gamma, g)(\delta, 1) = (\gamma + g \cdot \delta, g).$$

□

If X is a G-space, and $x \in X$, let $\sigma(X, x, G)$ be the object group of the semidirect product groupoid $\pi X \rtimes G$ at the object x. This group is called by Rhodes in [9] and [10] the *fundamental group of the transformation group* (although he defines it directly in terms of paths). The following result from [10] gives one of the reasons for its introduction.

Corollary 2.6 *If X is a G-space, $x \in X$, and the universal cover $\tilde{X}_x$ exists, then the group $\sigma(X, x, G)$ has a canonical action on $\tilde{X}_x$.*

Proof By 9.5.8 of [3], we may identify the universal cover $\tilde{X}_x$ of X at x with $\mathrm{St}_{\pi X} x$. The function $\pi X \to \pi X, \delta \mapsto -\delta$, transports the action of $\pi X \rtimes G$ on πX via the target map τ to an action of the same groupoid on πX via the source map σ. Hence the object group $(\pi X \rtimes G)(x)$ acts on $\mathrm{St}_{\pi X} x$ by

$$(\gamma, g) * \delta = -((\gamma, g) \cdot (-\delta)) = g \cdot \gamma - \gamma.$$

The continuity of the action follows easily from the detailed description of the lifted topology (see also the remarks on topological groupoids after 9.5.8 of [3]). □

Now we start using the semidirect product to compute orbit groupoids. The next two results may be found in [7], [11] and [12].

Proposition 2.7 *Let N be the normal closure in $\Gamma \rtimes G$ of the set of elements of the form $(0_x, g)$ for all $x \in \mathrm{Ob}(\Gamma)$ and $g \in G$. Let p be the composite*

$$\Gamma \xrightarrow{i} \Gamma \rtimes G \xrightarrow{\nu} (\Gamma \rtimes G)/N,$$

in which the first morphism is $\gamma \mapsto (\gamma, 1)$ and the second morphism is the quotient morphism. Then

(i) p is a surjective fibration;

(ii) p is an orbit morphism and so determines an isomorphism

$$\Gamma /\!/ G \cong (\Gamma \rtimes G)/N;$$

(iii) the function $\mathrm{Ob}(\Gamma) \to \mathrm{Ob}(\Gamma /\!/ G)$ is an orbit map, so that $\mathrm{Ob}(\Gamma /\!/ G)$ may be identified with the orbit set $\mathrm{Ob}(\Gamma)/G$.

Proof Let $\Delta = (\Gamma \rtimes G)/N$. We first derive some simple consequences of the definition of Δ. Let $\gamma \in \Gamma(x, y), g, h \in G$. Then

$$(0_{g \cdot y}, g) + (\gamma, 1) = (g \cdot \gamma, g), \tag{1}$$

$$(\gamma, 1) + (0_x, h) = (\gamma, h). \tag{2}$$

It follows that in Δ we have
$$\nu(h\cdot\gamma,1)=\nu(\gamma,g). \tag{3}$$
Note also that the set R of elements of $\Gamma\rtimes G$ of the form $(0_{g\cdot x},g)$ is a subgroupoid of G, since
$$(0_{hg\cdot x},h)+(0_{g\cdot x},g)=(0_{hg\cdot x},hg),$$
and $-(0_{g\cdot x},g)=(0_x,g^{-1})$. It follows that $\pi_0 N=\pi_0 R=\mathrm{Ob}(\Gamma)/G$, the set of orbits of the action of G on $\mathrm{Ob}(\Gamma)$. Hence $\mathrm{Ob}(p)$ is surjective. This proves (iii), once we have proved (ii).

We now prove easily that $p:\Gamma\to\Delta$ is a fibration. Let $(\gamma,g):x\to y$ in $\Gamma\rtimes G$ be a representative of an element of Δ, and suppose $\nu z=\nu x$, where $z\in\mathrm{Ob}(\Gamma)$. Then z and x belong to the same orbit and so there is an element h in G such that $h\cdot x=z$. Clearly $h\cdot\gamma$ has source z and by (3), $p(h\cdot\gamma)=\nu(\gamma,g)$.

Suppose now $g\in G$ and $\gamma:x\to y$ in Γ. Then by (3) $p(g\cdot\gamma)=p(\gamma)$. This verifies (i).

To prove the other condition for an orbit morphism, namely Definition 1.6(ii), suppose $\phi:\Gamma\to\Phi$ is a morphism of groupoids such that Φ has a trivial action of the group G and $\phi(g\cdot\gamma)=\phi(\gamma)$ for all $\gamma\in\Gamma$ and $g\in G$. Define $\phi':\Gamma\rtimes G\to\Phi$ on objects by $\mathrm{Ob}(\phi)$ and on elements by $(\gamma,g)\mapsto\phi(\gamma)$. That ϕ is a morphism follows from the trivial action of G on Φ, since
$$\begin{aligned}\phi'((\delta,h)+(\gamma,g))&=\phi(\gamma+h.\delta)\\&=\phi(\gamma)+\phi(h.\delta)\\&=\phi(\gamma)+\phi(\delta)\\&=\phi'(\delta,h)+\phi'(\gamma,g).\end{aligned}$$
Also $\phi'(0_x,g)=\phi(0_x)=0_{\phi x}$, and so $N\subseteq\operatorname{Ker}\phi'$. By (ii), there is a unique morphism $\phi^*:(\Gamma\rtimes G)/N\to\Phi$ such that $\phi^*\nu=\phi'$. It follows that $\phi^*p=\phi^*\nu i=\phi' i=\phi$. The uniqueness of ϕ^* follows from the fact that p is surjective on objects and on elements.

Finally, the isomorphism $\Gamma/\!/G\cong\Delta$ follows from the universal property. □

In order to use the last result we analyse the morphism $p:\Gamma\to\Gamma/\!/G$ in some special cases. The construction of the orbit groupoid T in Proposition 2.7 is what makes this possible.

Proposition 2.8 *The orbit morphism $p:\Gamma\to\Gamma/\!/G$ is a fibration whose kernel is generated as a subgroupoid of Γ by all elements of the form $\gamma-g\cdot\gamma$ where g stabilises the source of γ. Furthermore,*

(i) if G acts freely on Γ, by which we mean no non-identity element of G fixes an object of Γ, then p is a covering morphism;

(ii) if Γ is connected and G is generated by those of its elements which fix some object of Γ, then p is a quotient morphism; in particular, p is a quotient morphism if the action of G on $\mathrm{Ob}(\Gamma)$ has a fixed point;

(iii) if Γ is a tree groupoid, then each object group of $\Gamma/\!/G$ is isomorphic to the factor group of G by the (normal) subgroup of G generated by elements which have fixed points.

Proof We use the description of p given in the previous Proposition 2.7, which already implies that p is a fibration.

Let R be the subgroupoid of $\Gamma\rtimes G$ consisting of elements $(0_{g\cdot x},g), g\in G$. Let N be the normal closure of R. By the construction of the normal closure in Proposition 2.3, the elements of N are sums of elements of R and conjugates of loops in R by elements of $\Gamma\rtimes G$. So let $(0_{g\cdot x},g)$ be a loop in R. Then $g\cdot x=x$. Let $(\gamma,h):x\to y$ in $\Gamma\rtimes G$, so that $\gamma:h\cdot x\to y$. Then we check that
$$(\gamma,h)+(0_x,g)-(\gamma,h)=(\gamma-hgh^{-1}\cdot\gamma,hgh^{-1}).$$

Writing $k = hgh^{-1}$, we see that $(\gamma - k \cdot \gamma, k) \in N$ if k stabilises the initial point of γ.

Now $\gamma \in \operatorname{Ker} p$ if and only if $(\gamma, 1) \in N$. Further, if $(\gamma, 1) \in N$ then $(\gamma, 1)$ is a consequence of R and so $(\gamma, 1)$ is equal to

$$(\gamma_1 - k_1 \cdot \gamma_1, k_1) + (\gamma_2 - k_2 \cdot \gamma_2, k_2) + \cdots + (\gamma_r - k_r \cdot \gamma_r, k_r)$$

for some γ_i, k_i where k_i stabilises the initial point of $\gamma_i, i = 1, \ldots, r$. Let $h_1 = 1, h_i = k_1 \ldots k_{i-1} (i \geqslant 2), \delta_i = h_i.\gamma, g_i = h_i k_i h_i^{-1}, i \geqslant 1$. Then

$$(\gamma, 1) = (\delta_1 - g_1 \cdot \delta_1 + \delta_2 g_2 \cdot \delta_2 + \cdots + \delta_r - g_r \cdot \delta_r, 1)$$

and so γ is a sum of elements of the form $\delta - g \cdot \delta$ where g stabilises the initial point of δ. This proves our first assertion.

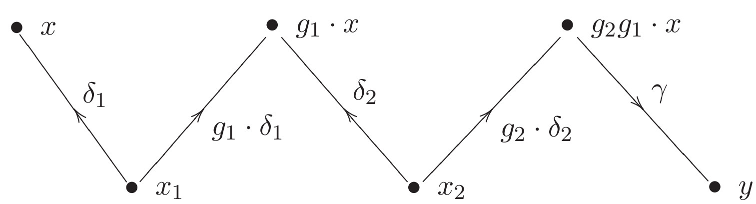
The proof of (i) is simple. We know already that $p : \Gamma \to \Gamma /\!/ G$ is a fibration. If G acts freely, then by the result just proved, p has discrete kernel. It follows that if $x \in \operatorname{Ob}(\Gamma)$, then $p : \operatorname{St}_\Gamma x \to \operatorname{St}_{\Gamma/\!/G} px$ is injective. Hence p is a covering morphism.

Now suppose Γ is connected and G is generated by those of its elements which fix some object of Γ. To prove p a quotient morphism we have to show that for $x, y \in \operatorname{Ob}(\Gamma)$, the restriction $p' : \Gamma(x, y) \to (\Gamma/\!/G)(px, py)$ is surjective.

Let (γ, g) be an element $x \to y$ in $\Gamma \rtimes G$, so that $\gamma : g \cdot x \to y$ in Γ. Using the notation of 2.7, we have to find $\delta \in \Gamma(x, y)$ such that $p\delta = \nu(\gamma, g)$. As shown in 2.7, $\nu(\gamma, g) = \nu(\gamma, 1)$. By assumption, $g = g_n g_{n-1} \ldots g_1$ where g_i stabilises an object x_i, say. Since Γ is connected, there are elements

$$\delta_1 \in \Gamma(x_1, x), \quad \delta_i \in \Gamma(x_i, (g_{i-1} \ldots g_1) \cdot x), i \geqslant 1.$$

The situation is illustrated below for $n = 2$.



Let

$$\delta = \gamma + (g_n \cdot \delta_n - \delta_n) + \cdots + (g_1 \cdot \delta_1 - \delta_1) : x \to y.$$

Then $p(\delta) = \nu(\gamma, 1)$. This proves (ii).

For the proof of (iii), let x be an object of Γ. Since Γ is a tree groupoid, the projection $(\Gamma \rtimes G)(x) \to G$ is an isomorphism which sends the element $(0_x, g)$, where $g \cdot x = x$, to the element g. Also if g fixes x and $h \in G$ then hgh^{-1} fixes $h \cdot x$. Thus the image of $N(x)$ is the subgroup K of G generated by elements of G with a fixed point, and K is normal in G. Let $\bar{x}$ denote the orbit of x. By (ii), and Proposition 2.7, the group $(\Gamma/\!/G)(\bar{x})$ is isomorphic to the quotient of $(\Gamma \rtimes G)(x)$ by $N(x)$. Hence $(\Gamma/\!/G)(\bar{x})$ is also isomorphic to G/K. $\square$

In Proposition 2.8, the result (i) relates the work on orbit groupoids to work on covering morphisms. The result (ii) will be used below. The result (iii) is particularly useful in work on discontinuous actions on Euclidean or hyperbolic space. In the case of a discontinuous action of a group G on a space X which satisfies the additional condition (ii) of 1.7, and which has a fixed point x, we obtain from Proposition 2.8 a convenient description of $\pi(X/G, \bar{x})$ as a quotient of $\pi(X, x)$. In the more general case, $\pi(X/G, \bar{x})$ has to be computed as a quotient of $\sigma(X, x, G) = (\pi X \rtimes G)(x)$, as given in Proposition 2.7. Actually the case corresponding to (iii) was the first to be discovered, for the case of simplicial actions [1]. The general case followed from the fact that if X has a universal cover $\tilde{X}_x$ at X, then $\sigma(X, x, G)$ acts on $\tilde{X}_x$ with orbit space homeomorphic to X/G.

We give next two computations for actions with fixed points.

Example 2.9 Let the group $\mathbb{Z}_2 = \{1, g\}$ act on the circle $\mathbf{S}^1$ in which g acts by reflection in the x-axis. The orbit space of the action can be identified with E^1_+, which is contractible, and so has trivial fundamental group.

To see how this agrees with the previous results, let $\Gamma = \pi\mathbf{S}^1, G = \mathbb{Z}_2$. It follows from Proposition 1.8 that the induced morphism $\pi\mathbf{S}^1 \to \pi(\mathbf{S}^1/G)$ is an orbit morphism, and so $\pi(\mathbf{S}^1/G) \cong \Gamma//G$. By Proposition 2.7, $p : \Gamma \to \Gamma//G$ is a quotient morphism, since the action has a fixed point 1 (and also -1). Let the two elements $a_\pm \in \Gamma(1, -1)$ be represented by the paths $[0, 1] \to \mathbf{S}^1, t \mapsto e^{\pm i\pi t}$ respectively. The non trivial element g of G satisfies $g \cdot a_+ = a_-$. Hence the kernel of the quotient morphism $p : \Gamma(1) \to (\Gamma//G)(p1)$ contains the element $-a_- + a_+$. But this element generates $\Gamma(1) \cong \mathbb{Z}$. So we confirm the fact that $(\Gamma//G)(p1)$ is the trivial group. $\Box$

Before our next result we state and prove a simple group theoretic result. First let H be a group. It is convenient to write the group structure on H as multiplication. The *abelianisation* H^{ab} of H is formed from H by imposing the relations $hk = kh$ for all $h, k \in H$. Equivalently, it is the quotient of H by the (normal) subgroup generated by all commutators $hkh^{-1}k^{-1}$, for all $h, k \in H$.

Proposition 2.10 *The quotient $(H \times H)/K$ of $H \times H$ by the normal subgroup K of $H \times H$ generated by the elements (h, h^{-1}) is isomorphic to H^{ab}.*

Proof We can regard $H \times H$ as the group with generators $[h], \langle k\rangle$ for all $h, k \in H$ and relations $[hk] = [h][k], \langle hk\rangle = \langle h\rangle\langle k\rangle, [h]\langle k\rangle = \langle k\rangle[h]$ for all $h, k \in H$, where we may identify $[h] = (h, 1), \langle k\rangle = (1, k), [h]\langle k\rangle = (h, k)$. Factoring out by K imposes the additional relations $[h]\langle h^{-1}\rangle = 1$, or equivalently $[h] = \langle h\rangle$, for all $h \in H$. It follows that $(H \times H)/K$ is obtained from H by imposing the additional relations $hk = kh$ for all $h, k \in H$. $\Box$

Definition 2.11 (The symmetric square of a space) Let $G = \mathbb{Z}_2$ be the cyclic group of order 2, with non-trivial element g. For a space X, let G act on the product space $X \times X$ by interchanging the factors, so that $g \cdot (x, y) = (y, x)$. The fixed point set of the action is the diagonal of $X \times X$. The orbit space is called the *symmetric square* of X, and is written Q^2X.

Proposition 2.12 *Let X be a connected, Hausdorff, semilocally 1-connected space, and let $x \in X$. Let $\langle x\rangle$ denote the class in Q^2X of (x, x). Then the fundamental group $\pi(Q^2X, \langle x\rangle)$ is isomorphic to $\pi(X, x)^{\mathrm{ab}}$, the fundamental group of X at x made abelian.*

Proof Since $G = \mathbb{Z}_2$ is finite, the action is discontinuous. Because of the assumptions on X, we can apply Proposition 1.8, and hence also the results of this section. We deduce that $p_* : \pi(X) \times \pi(X) \to \pi(Q^2X)$ is a quotient morphism and that if $x \in X, z = p(x, x)$, then the kernel of the quotient morphism

$$p' : \pi(X, x) \times \pi(X, x) \to \pi(Q^2X, z)$$

is the normal subgroup K generated by elements $(a, b) - g \cdot (a, b) = (a - b, b - a), a, b \in \pi X(y, x)$, for some $y \in X$. Equivalently, K is the normal closure of the elements $(c, -c), c \in \pi(X, x)$. The result follows. $\Box$

Taylor [12] has given extensions of previous results which we give without proof.

Proposition 2.13 *Let G be a group and let Γ be a G-groupoid. Let A be a subset of $\mathrm{Ob}(\Gamma)$ such that A is G-invariant and for each $g \in G$, A meets each component of the subgroupoid of G left fixed by the action of G. Let Ξ be the full subgroupoid of Γ on A. Then the orbit groupoid $\Xi//G$ is embedded in $\Gamma//G$ as the full subgroupoid on the set of objects A/G and the orbit morphism $\Xi \to \Xi//G$ is the restriction of the orbit morphism $\Gamma \to \Gamma//G$.*

Corollary 2.14 *If the action of G on X satisfies Conditions 1.7, and A is a G-stable subset of X meeting each path component of the fixed point set of each element of G, then $\pi(X/G)(A/G)$ is canonically isomorphic to $(\pi XA)//G$.*

The results of this section answer some cases of the following question:

Question 2.15 *Suppose $p : K \to H$ is a connected covering morphism, and $x \in \mathrm{Ob}(K)$. Then p maps $K(x)$ isomorphically to a subgroup of the group $H(px)$. What information in addition to the value of $K(x)$ is needed to reconstruct the group $H(px)$?*

There is an exact sequence

$$0 \to K(x) \to H(px) \to H(px)/p[K(x)] \to 0$$

in which $H(px)$ and $K(x)$ are groups while $H(px)/p[K(x)]$ is a pointed set with base point the coset $p[K(x)]$. Suppose now that p is a regular covering morphism. Then the group G of covering transformations of p is anti-isomorphic to $H(px)/p[K(x)]$, by [3, Cor. 9.6.4]. Also G acts freely on K.

Proposition 2.16 *If $p : K \to H$ is a regular covering morphism, then p is an orbit morphism with respect to the action on K of the group G of covering transformations of p. Hence if $x \in \mathrm{Ob}(K)$, then $H(px)$ is isomorphic to the object group $(K \rtimes G)(x)$.*

Proof Let G be the group of covering transformations of p. Then G acts on K. Let $q : K \to K//G$ be the orbit morphism. If $g \in G$ then $pg = p$, and so G may be considered as acting trivially on H. By Definition 1.6(i), there is a unique morphism $\phi : K//G \to H$ of groupoids such that $\phi q = p$. By Proposition 2.8(i), q is a covering morphism. Hence ϕ is a covering morphism [3, 9.2.3]. But ϕ is bijective on objects, because p is regular. Hence ϕ is an isomorphism.

Since G acts freely on K, the group $N(x)$ of Proposition 2.7 is trivial. So the description of $H(px)$ follows from Proposition 2.7(ii). □

The interest of the above results extends beyond the case where G is finite, since general discontinuous actions occur in important applications in complex function theory, concerned with Fuchsian groups and Kleinian groups.We refer the reader to [2] and [8].

EXERCISES

1. Let $f : G \to H$ be a groupoid morphism with kernel N. Prove that the following are equivalent:
   (a) f is a quotient morphism;
   (b) f is surjective and any two vertices of G having the same image in H lie in the same component of G.
2. Prove that a composite of quotient morphisms is a quotient morphism.
3. Let H be a subgroupoid of the groupoid G with inclusion morphism $i : H \to G$. Let $f : G \to H$ be a morphism with kernel N. Prove that the following are equivalent:
   (a) f is a deformation retraction;
   (b) f is piecewise bijective and $fi = 1_H$;
   (c) f is a quotient morphism, N is simply connected, and $fi = 1_H$.

4. Suppose the following diagram of groupoid morphisms is a pushout

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

   and f is a quotient morphism. Prove that g is a quotient morphism.

5. Let $f, g : H \to G$ be two groupoid morphisms. Show how to construct the coequaliser $c : G \to C$ of f, g as defined in Exercise 6.6.4 of [3]. Show how this gives a construction of the orbit groupoid. [Hint: First construct the coequaliser $\sigma : \mathrm{Ob}(G) \to Y$ of the functions $\mathrm{Ob}(f), \mathrm{Ob}(g)$, then construct the groupoid $U_\sigma(G)$, and finally construct C as a quotient of $U_\sigma(G)$.]

6. Suppose the groupoid G acts on the groupoid Γ via $w : \Gamma \to \mathrm{Ob}(G)$ as in Exercise 4 of Section 1. Define the *semidirect product groupoid* $\Gamma \rtimes G$ to have object set $\mathrm{Ob}(\Gamma)$ and elements the pairs $(\gamma, g) : x \to y$ where $g \in G(wx, wy)$ and $\gamma \in \Gamma(g \cdot x, y)$. The sum in $\Gamma \rtimes G$ is given by $(\delta, h) + (\gamma, g) = (\delta + h \cdot \gamma, hg)$. Prove that this does define a groupoid, and that the projection $p : \Gamma \rtimes G \to G, (\gamma, g) \mapsto G$, is a fibration of groupoids. Prove that the quotient groupoid $(\Gamma \rtimes G)/\operatorname{Ker} p$ is isomorphic to $(\pi_0 \Gamma G)$.

7. Let G and Γ be as in Exercise 6, and let the groupoid H act on the groupoid Δ via $v : \Delta \to \mathrm{Ob}(H)$. Let $f : G \to H$ and $\theta : \Gamma \to \Delta$ be morphisms of groupoids such that $v\theta = \mathrm{Ob}(f)w$ and $\theta(g \cdot \gamma) = (fg) \cdot (\theta\gamma)$ whenever the left hand side is defined. Prove that a morphism of groupoids (θ, f) is defined by $(\gamma, g) \mapsto (\theta\gamma, fg)$. Investigate conditions on f and θ for (θ, f) to have the following properties:

   (a) injective,

   (b) connected fibres,

   (c) quotient morphism,

   (d) discrete kernel,

   (e) covering morphism.

   In the case that (θ, f) is a fibration, investigate the exact sequences of the fibration.

8. Generalise the Corollary 2.6 from the case of the universal cover to the case of a regular covering space of X determined by a subgroup N of $\pi(X, x)$.

9. Let $1 \to N \to E \to G \to 1$ be an exact sequence of groups. Prove that there is an action of G on a connected groupoid Γ and an object x of Γ such that the above exact sequence is isomorphic to the exact sequence of the fibration $\Gamma \rtimes G \to G$ at the object x. [4]

10. Let X^n be the n-fold product of X with itself, and let the symmetric group S_n act on X^n by permuting the factors. The orbit space is called the *n-fold symmetric product of* X and is written $Q^n X$. Prove that for $n \geqslant 2$ the fundamental group of $Q^n X$ at an image of a diagonal point $(x, \ldots, x)$ is isomorphic to the fundamental group of X at x made abelian.

11. Investigate the fundamental groups of quotients of X^n by the actions of various proper subgroups of the symmetric group for various n and various subgroups. [Try out first the simplest cases which have not already been done in order to build up your confidence. Try and decide whether or not it is reasonable to expect a general formula.]

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