A non abelian tensor product of groups

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My involvement: Papers with
Jean-Louis Loday:
1984 (CRASP), 1987 (Topology, Proc LMS)
D.L. Johnson-E.F. Robertson,
1987 (J. Algebra)

Earlier work:
Claire Miller 1952
Abe Lue 1976
R K Dennis 1976 (preprint)

Now see bibliography of 74 papers
http://www.bangor.ac.uk/~mas010/nonabtens.html

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PHILOSOPHY: programme since 1967 was to investigate for description and computation in topology the extensions:

groups $\subseteq$ groupoids $\subseteq$ multiple groupoids.

First observation (1967):
the fundamental groupoid $\pi_1(X, A)$ on a **set** A of base points gives better results than the fundamental group.
So why not rewrite higher homotopy theory replacing **groups** by **groupoids**?
Motivation: Curiosity!

Second observation:
group objects in groups = abelian groups
(so higher homotopy groups are abelian).

What about **groupoid objects in groups**?
These are more complicated than groups!
Non-commutative structures
Higher dimensional algebra

Expect to use such structures for:

Higher order Van Kampen Theorems
Local-to-global problems
Algebraic inverse to subdivision

Implications of Higher Order Van Kampen Theorems:

Interested in colimit constructions for these new algebraic structures. It was one such construction which led RB and J-LL to the non abelian tensor product in 1982 - though it had precursors - and also gave a ready made application to homotopy theory.

Part of the interest is possibly that:

(i) it suggests a range of analogous algebraic constructions,

tensor products mod q
tensor products of Lie algebras
non abelian homology

(ii) this construction impinges on but a fragment of the potential theory (it involves only triple groupoids).

Groupoid objects in groups are equivalent to:
Crossed modules
Morphism $\mu : M \to P$ of groups
action of P on the left of M by $(p, m) \mapsto {}^p m$, so that

CM1) $\mu({}^p m) = p(\mu m)p^{-1}$;

CM2) $mnm^{-1} = {}^{\mu m} n$

for all $m, n \in M, p \in P$.

We can conveniently regard crossed modules as

2-dimensional groups.

Examples:

1. A normal subgroup $M \lhd P$.

2. The inner automorphism map $\chi : M \to Aut(M)$.

3. The trivial map $M \to P$ when M is a P-module.

4. Any surjection $M \to P$ with central kernel.

5. The *free crossed P-module* $C(w) \to P$ on a function $w : R \to P$ on a set R.

6. The induced morphism $\pi_1(F) \to \pi_1(E)$ when $F \to E \to B$ is a based fibration.

If M, N are normal subgroups of the group P then we are interested in the *commutator map*

$$[-,-] : M \times N \to P, [m,n] = mnm^{-1}n^{-1}.$$

This function is **not** bimultiplicative. Instead it is what we might call a **biderivation**. That is

$$[mm', n] = [{}^m m', {}^m n][m, n]$$

$$[m, nn'] = [m, n][{}^n m, {}^n n']$$

for all $m, m' \in M, n, n' \in N$. Notice that M, N operate on each other via conjugation in P and of course on themselves by conjugation.

So we form the universal object for biderivations, i.e. define the *non abelian tensor product* $M \otimes N$ as the group generated by elements $m \otimes n$ subject to the relations

$$mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n)$$

$$m \otimes nn' = (m \otimes n)({}^n m \otimes {}^n n')$$

for all $m, m' \in M, n, n' \in N$.

One of the consequences is that P operates on $M \otimes N$ so that

$${}^p(m \otimes n) = {}^p m \otimes {}^p n.$$

Homotopical consequence:

Consider the homotopy pushout

$$\begin{array}{ccc} BP & \longrightarrow & B(P/M) \\ \downarrow & & \downarrow \\ B(P/N) & \longrightarrow & X \end{array}$$

Then (Brown-Loday, 1987)

$$\pi_n(X) \cong \begin{cases} P/(MN) & \text{if } n = 1, \\ (M \cap N)/[M,N] & \text{if } n = 2, \\ \mathrm{Ker}\,(M \otimes N \to P) & \text{if } n = 3. \end{cases}$$

Consequence of a Van Kampen Theorem for the fundamental crossed square of a square of spaces.

This was the first time this 3rd homotopy group was calculated!

The above result leads to explicit calculations, as expected by a theorem of Ellis:

M, N finite implies $M \otimes N$ finite.

(Brown/Loday) Exact sequence

$$H_3(M) \to \Gamma(M^{\mathrm{ab}}) \to M \otimes M.$$

So if M is free, then $M \otimes M$ is isomorphic to $[M, M] \times \Gamma(M^{\mathrm{ab}})$.

Here Γ is Whitehead's universal quadratic functor on abelian groups.

The general case when M is infinite and non commutative is quite hard (L-C. Kappe and students).

Example: $M = N = D_n$ the dihedral group of order $2n$ with presentation $\langle x, y : x^n, y^2, xyxy \rangle$. Then $M \otimes M$ is isomorphic to:

if n is odd

$\mathbb{Z}_2 \times \mathbb{Z}_n$ generated by $y \otimes y, x \otimes y$;

if n is even

$\mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by
$y \otimes y, x \otimes y, x \otimes x, (x \otimes y)(y \otimes x)$

This calculation is by hand. Lots more calculations available, some by computer (see bibliography).

Back to the definition: What happens now if you expand $mm' \otimes nn'$ in two ways? After some reduction and manipulation you get

$$({}^{mn}m' \otimes {}^{mn}n')(m \otimes n) = (m \otimes n)({}^{nm}m' \otimes {}^{nm}n').$$

This can be rewritten as

$$[m \otimes n, m' \otimes n'] = [m, m'] \otimes [n, n'].$$

We then get the beautiful diagram

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow[\mu]{} & P \end{array},$$

where

$$\lambda(m \otimes n) = m^n(m^{-1}), \lambda'(m \otimes n) = {}^m n n^{-1},$$

together with a map $h : M \times N \to M \otimes N$ making a so called *crossed square*.

crossed square (crossed module in the category of crossed modules)

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow[\mu]{} & P \end{array} \qquad \text{and } h : M \times N \to L$$

actions of P on L, M, N, so that M and N act on L, M, N via P
the morphisms $\lambda, \lambda', \mu, \nu$ and $\mu\lambda = \nu\lambda'$ are crossed modules
λ, λ' are P-equivariant;
and for all $l \in L, m, m' \in M, n, n' \in N, p \in P$:

$$\begin{aligned} h(mm', n) &= h({}^m m', {}^m n)h(m, n) \\ h(m, nn') &= h(m, n)h({}^n m, {}^n n') \\ \lambda h(m, n) &= m({}^n m^{-1}), \\ \lambda' h(m, n) &= ({}^m n)n^{-1} \\ h(\lambda l, n) &= l({}^n l)^{-1}, \\ h(m, \lambda' l) &= ({}^m l)l^{-1} \\ h({}^p m, {}^p n) &= {}^p h(m, n) \end{aligned}$$

Notice this is a mouthful, and cumbersome to verify in many cases.

But as crossed modules are just groupoid objects in groups, so crossed squares are **double groupoid objects in groups**. This gives a relation to geometry, suggest higher dimensional versions, and also versions for other algebraic objects (Lie algebras, DGA algebras, . . .). There is a lot to explore!

Philosophy: The aim of this work is **not** to solve other people's problems in homotopy theory, but to use homotopy theory as a test bed for the tools. It is a useful test bed because:

1. Calculations in homotopy theory are notoriously hard.

2. The notion of homotopy, of deformation, as a mode of classification is important across a range of areas, so we are interested in new methods, which might generalise.

3. We are trying to understand the underlying structures, which seem to be non abelian.

Relation with homology:

Let $M, N \triangleleft P$. Define

$$M \wedge N = (M \otimes N)/\{m \otimes m\} \quad m \in M \cap N$$

Claire Miller:

$$H_2(M) \cong \mathrm{Ker}(\kappa : M \wedge M \to M)$$

Def: (Graham Ellis)

$$H_2(P, M) = \mathrm{Ker}(\kappa : P \wedge M \to P)$$

Exact sequence:

$$H_3(P) \xrightarrow{\eta} H_3(P/M) \longrightarrow H_2(P, M) \longrightarrow H_2(P) \longrightarrow$$

$$H_2(P/M) \longrightarrow P/[P, M] \longrightarrow P^{\mathrm{ab}} \longrightarrow (P/M)^{\mathrm{ab}}$$

This is the start of work of Ellis to obtain many new results on classical Schur Multiplier.

Nice exact sequence

$$\Gamma((M \cap N)^{\mathrm{ab}}) \to M \otimes N \to M \wedge N \to 0$$

Other nice tricks:

Define variants of the centre of a group P as:

1. $\{p \in P : p \otimes q = 1, \forall q \in P\}$

2. $\{p \in P : p \wedge q = 1, \forall q \in P\}$

3. $\{p \in P : p \widetilde{\wedge} q = 1, \forall q \in P\}$

In the last, $P \widetilde{\wedge} P = (P \otimes P)/\{(p \otimes q)(q \otimes p)\}$.

Ellis uses these in relation to capability.

Theorem (Loday and Guin-Waléry)

$$\text{(crossed squares)} \simeq \text{(cat}^2\text{-groups)}$$

where cat^2-groups are double groupoids internal to the category of groups.

If $X_* = (X : A, B)$ is a based triad, let Φ be the maps of cubes $I^3 \to X$ which take faces in direction:
1 to $*$
2 to A
3 to B
edges to C .
Let G be homotopy classes of such maps.
Then G gets a group structure in direction 1.
Claim: the other partial gluings of cubes inherit to make G a cat^2-group. (Loday)

Conclusion: The origin of this work was a simple question:

what rôle could and should groupoids play in higher homotopy theory?

It now seems that n-fold groupoids model all homotopy information up to level n (Loday, Porter).

Out of this approach we obtained a new construction in group theory, which refines commutator theory, motivated by the properties of and constructions with some special kinds of triple groupoids.

Surely we have just picked some nice nuggets from the surface, in one locality! How deep and wide does this vein run?

Other points:

1. M perfect implies $M \otimes M \to M$ is the universal covering group.

2. The definition of $M \otimes N$ requires only that M, N act on each other, and on themselves by conjugation. However the main properties (e.g. the actions of M, N on $M \otimes N$) require *compatability* which is that

$$ {}^{{}^n m} n' = {}^n({}^m({}^{n^{-1}} n')) \qquad {}^{{}^m n} m' = {}^m({}^n({}^{m^{-1}} m')) $$

for $m, m' \in M, n, n' \in N$. This holds for crossed P-modules.

3. In order to define derived functors of tensor products, Niko Inassaridze has relaxed compatability but imposed extra relations to recover the desired properties.

For lots more topics, see the bibliography on the web.

Non abelian derived functors of $G \otimes -$.

Tensor products mod q

Other algebraic structures

Conclusion

The theme of higher dimensional group theory might be seen as a useful research strategy, and source of problems and questions for research students! Surely we have so far only scratched the surface.