Applications of a non abelian tensor product of groups

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My involvement: Definitions and applications in papers with Jean-Louis Loday:
1984 (CRASP), 1987 (Topology, Proc LMS)
Computations with D.L. Johnson and E.F. Robertson, 1987 (J. Algebra)
Precursors:
Clair Miller 1952
Abe Lue 1976
R K Dennis 1976 (preprint)
Now see bibliography of 114 items (including Lie algebras)
http://www.bangor.ac.uk/r.brown/nonabtens.html
This story is a kind of nugget from the mine of the extensions

$$(\text{groups}) \subseteq (\text{groupoids}) \subseteq (\text{multiple groupoids}).$$

In the late 1960s I wondered:
given the known importance of groups
what might be the significance of these extensions for mathematics and science? How to probe?

Over the next 20 years it became clear that one could in homotopy theory use this extension

groups $\to$ multiple groupoids

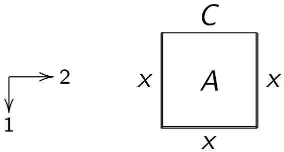
provided one generalised as follows:
pointed spaces $\to$ pointed pairs of spaces $\to$ filtered spaces $\to$ pointed n-cubes of spaces
and one could obtain categories Alg of nonabelian algebraic objects with structure in a range of dimensions $0 \leqslant i \leqslant n$ and homotopically defined functors

$$\Pi : (\text{topological data}) \to \mathsf{Alg}$$

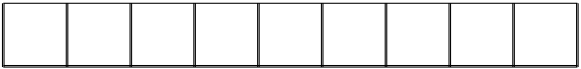
which preserved certain colimits of 'connected' data.
This replacement of the traditional exact sequences by precise colimits of algebraic models of homotopy types is one of the chief features of this method. Where this method works it gives precise algebraic and computable information on certain homotopy types from which information on homotopy groups needs to be extracted.

Excision in homotopy theory

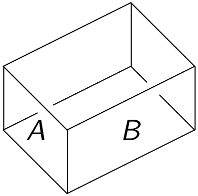
Illustrate the above by applications to homotopical excision.
Homotopical problem: $x \in C \subseteq A$.
There are relative homotopy groups $\pi_i(A, C)$ $i \geqslant 1$
(based sets if $i = 1$, commutative groups if $i \geqslant 3$).



where thick lines show constant paths.
Compositions are on a line:



Now suppose $x \in C = A \cap B \subseteq A \cup B \subseteq X$.
Triad homotopy groups $\pi_j(X; A, B), j \geqslant 2$
(based sets if $j = 2$, commutative groups if $j \geqslant 4$).



For $j = 3$ they are given by homotopy classes of maps of a cube $f : I^3 \to X$ such that f maps one front face into A, another front face into B and the other faces to x. Compose vertically. This leads to an exact sequence

$$\pi_3(X; A, B) \to \pi_2(A, A \cap B) \xrightarrow{\varepsilon_*} \pi_2(X, B) \to \pi_2(X; A, B)$$

where the last object is just a set with base point. Thus the triad homotopy group measures the failure of excision.

The geometry of the boundary of

$$I^p \times I^q \cong I^{p+q}$$

and homological methods lead to:

Theorem (Blakers-Massey, 1953)

There is a natural map

$$\pi_p(A, C) \otimes_{\mathbb{Z}} \pi_q(B, C) \to \pi_{p+q-1}(X; A, B)$$

which is an isomorphism if $(A, C), (B, C)$ are respectively $(p-1), (q-1)$-connected, all the spaces are connected, C is simply connected, $X = A \cup B$, and A, B are open. Further the triad $(X : B, C)$ is $(p+q-2)$-connected.

This determines the first non vanishing triad homotopy group.
Proof uses homological, so abelian, methods.
Currently, homotopy theory tries to move away from the fundamental group and nonabelian group methods.

Problem: What happens if C is not simply connected, and p or q is 2? For example,
$\pi_2(A, C), \pi_2(B, C)$ might be nonabelian groups.
Usual tensor product not suitable: If M, N are groups, and $M \otimes N$ is defined as the group with generators $m \otimes n$ and relations

$$mm' \otimes n = (m \otimes n)(m' \otimes n)$$

$$m \otimes nn' = (m \otimes n)(m \otimes n')$$

then by expanding

$$mm' \otimes nn'$$

in two ways we deduce

$$(m \otimes n')(m' \otimes n) = (m' \otimes n)(m \otimes n')$$

and from this that

$$M \otimes N \cong M^{\mathrm{ab}} \otimes_{\mathbb{Z}} N^{\mathrm{ab}}.$$

I.e., bimultiplicative maps are boring!

What has gone wrong?
$\pi_1(C)$ operates on the relative homotopy group $\pi_n(A, C)$.
For $n = 2$ the boundary

$$\pi_2(A, C) \to \pi_1(C)$$

is a crossed module (Whitehead, 1946): i.e.
a morphism $\mu : M \to P$ of groups and an action of P on the left of M by $(p, m) \mapsto {}^p m$ such that:

CM1) $\mu({}^p m) = p(\mu m)p^{-1}$;

CM2) $mnm^{-1} = {}^{\mu m} n$

for all $m, n \in M, p \in P$.
The second rule says that the operation of M on itself via P is just conjugation.
Crossed modules are thought of as 2-dimensional groups!
They model homotopy 2-types (pointed, connected, weak).

Examples of crossed modules

1. A normal subgroup inclusion $M \lhd P$.
2. The inner automorphism map $\chi : M \to Aut(M)$.
3. The trivial map $M \xrightarrow{0} P$ when M is a P-module.
4. Any epimorphism $M \to P$ with central kernel.
5. The *free crossed P-module* $C(w) \to P$ on a function $w : R \to P$ on a set R. Relevant to identities among relations (nonabelian syzygies) (Reidemeister, Whitehead).
6. The induced morphism $\pi_1(F) \to \pi_1(E)$ when $F \to E \to B$ is a based fibration. (Quillen).

Quotienting, and symmetry, lead, for many algebraic structures, to nonabelian 2-dimensional algebraic structures!

Mystic statement: Here be groupoids!

You all know here the transition
Betti numbers and torsion coefficients $\longrightarrow$Homology groups
Analogously,
Postnikov invariants of 2-type $\longrightarrow$ algebraic models of 2-type.
Algebraic models allow limits and colimits;
colimits give some information on gluing spaces by
Higher Homotopy Seifert-van Kampen Theorems. 2-d version:
Brown and Higgins, 1978.
Grothendieck liked to call this integration of homotopy types.

Biderivations

Consider the normal subgroup example.
If $M, N \lhd P$ then we are interested in the *commutator map*

$$[-,-] : M \times N \to P, \quad [m,n] = mnm^{-1}n^{-1}.$$

This function is **not** bimultiplicative. Instead it is what we might call a **biderivation**.

$$[mm', n] = [{}^m m', {}^m n][m, n]$$

$$[m, nn'] = [m, n][{}^n m, {}^n n']$$

for all $m, m' \in M, n, n' \in N$. Notice that M, N operate on each other and on themselves via P.

Assume that $\mu : M \to P, \nu : N \to P$ are crossed P-modules. Then M, N operate on each other via P. So we define the non abelian tensor product $M \otimes N$ as the group generated by elements $m \otimes n$ subject to the relations

$$mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n)$$

$$m \otimes nn' = (m \otimes n)({}^n m \otimes {}^n n')$$

for all $m, m' \in M, n, n' \in N$.

Compare

$$[mm', n] = [{}^m m', {}^m n][m, n]$$

$$[m, nn'] = [m, n][{}^n m, {}^n n']$$

$$u : (m, n) \mapsto m \otimes n$$

$$\begin{array}{ccc} M \times N & & \\ {\scriptstyle u}\downarrow & \searrow^{[\ ,\ ]} & \\ M \otimes N & \xrightarrow[\kappa]{} & P. \end{array}$$

The commutator map factors through a homomorphism on the universal object for biderivations and in particular for commutators!

The following is one consequence of a Higher Homotopy Seifert-van Kampen Theorem for n-cubes of spaces.

Theorem (Brown-Loday, 1984)

Under the same assumptions as Blakers-Massey, but without assuming C simply connected, the natural map

$$\pi_p(A, C) \otimes \pi_q(B, C) \to \pi_{p+q-1}(X : A, B)$$

is an isomorphism, and in particular

$$\pi_2(A, C) \otimes \pi_2(B, C) \xrightarrow{\cong} \pi_3(X : A, B).$$

Corollary If M is a group, then

$$\pi_3 SK(M, 1) \cong \mathrm{Ker}(M \otimes M \to M).$$

This was the first time this homotopy group was calculated!
Proof of Corollary: Exact sequences of triads and pairs.

Example Calculation:

$M = N = D_{2n}$ the dihedral group of order $2n$ with presentation $\langle x, y : x^n, y^2, xyxy \rangle$. Then $M \otimes M$ is isomorphic to:

$$\begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_n \quad \text{generated by} \\ y \otimes y, x \otimes y, & \text{if } n \text{ odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2 \quad \text{generated by} \\ y \otimes y, x \otimes y, x \otimes x, (x \otimes y)(y \otimes x) & \text{if } n \text{ even} \end{cases}$$

For n even the elements

$$x \otimes x,\ (x \otimes y)(y \otimes x) \in \mathrm{Ker}(D_{2n} \otimes D_{2n} \to D_{2n}) \cong \pi_3(SK(D_{2n}, 1))$$

have homotopical interpretations.

This calculation by hand. Lots more calculations available, some by computer (see bibliography). The case M is infinite and non commutative is quite hard (L-C. Kappe and students).

For more computer calculations see Graham Ellis: http://hamilton.nuigalway.ie/Hap/www/SideLinks/About/aboutTensorSquare.html

Back to formalities: Expand $mm' \otimes nn'$ in two ways? After some reduction and manipulation you get

$$({}^{mn}m' \otimes {}^{mn}n')(m \otimes n) = (m \otimes n)({}^{nm}m' \otimes {}^{nm}n').$$

This can be rewritten as

$$[m \otimes n, m' \otimes n'] = [m, m'] \otimes [n, n'].$$

We then get the beautiful diagram

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow[\mu]{} & P \end{array} \qquad \begin{array}{c} h : M \times N \to M \otimes N \\ (m, n) \mapsto m \otimes n \end{array}$$

making a *crossed square*.

Crossed squares may be thought of as 3-dimensional groups!

They give algebraic models of homotopy 3-types! (weak, pointed, connected)

crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow[\mu]{} & P \end{array} \qquad \text{and} \qquad \begin{array}{c} h : M \times N \to L \\ P \text{ acts on } L, M, N \\ \text{so } M, N \text{ act on } L, M, N \end{array}$$

$\lambda, \lambda', \mu, \nu$ and $\mu\lambda = \nu\lambda'$ are crossed modules,
λ, λ' are P-equivariant; and

$$\begin{aligned} h(mm', n) &= h({}^m m', {}^m n)h(m, n), \\ h(m, nn') &= h(m, n)h({}^n m, {}^n n'), \\ \lambda h(m, n) &= m({}^n m^{-1}), \\ \lambda' h(m, n) &= ({}^m n)n^{-1}, \\ h(\lambda l, n) &= l({}^n l)^{-1}, \\ h(m, \lambda' l) &= ({}^m l)l^{-1}, \\ h({}^p m, {}^p n) &= {}^p h(m, n), \end{aligned}$$

for all $l \in L, m, m' \in M, n, n' \in N, p \in P$.

One can consider colimits of crossed squares. The nonabelian tensor product arises as a pushout of crossed squares

$$\begin{array}{ccc} \begin{matrix} 1 & 1 \\ 1 & P \end{matrix} & \longrightarrow & \begin{matrix} 1 & N \\ 1 & P \end{matrix} \\ \downarrow & & \downarrow \\ \begin{matrix} 1 & 1 \\ M & P \end{matrix} & \longrightarrow & \begin{matrix} M \otimes N & N \\ M & P \end{matrix} \end{array}$$

then $L \cong M \otimes N$.
This is related to the pushout of squares of spaces when $X = A \cup B, C = A \cap B$:

$$\begin{array}{ccc} \begin{matrix} C & C \\ C & C \end{matrix} & \longrightarrow & \begin{matrix} C & B \\ C & B \end{matrix} \\ \downarrow & & \downarrow \\ \begin{matrix} C & C \\ A & A \end{matrix} & \longrightarrow & \begin{matrix} C & B \\ A & A \cup B \end{matrix} \end{array}$$

Hard to prove directly a colimit of crossed squares, because of the many axioms.

Theorem (Loday and Guin-Waléry, 1980)

$$(\textit{crossed squares}) \simeq (\textit{cat}^2\textit{-groups})$$

where cat^2-groups are double groupoids internal to the category of groups.

Theorem (Loday, 1982)

There exists a homotopically defined functor

$$\Pi : (\textit{squares of pointed spaces}) \to (\textit{cat}^2\textit{-groups}).$$

Thus pointed, connected weak homotopy 3-types are modelled by special kinds of triple groupoids!
Also true for homotopy n-types! (Loday, 1982).

Theorem (Brown and Loday, 1987)

The above functor Π *preserves certain colimits (with connectivity conditions).*

Let $M, N \triangleleft P$. Define (Brown/Loday, 1987)

$$M \wedge N = (M \otimes N)/\{m \otimes m\} \quad m \in M \cap N$$

Clair Miller:

$$H_2(M) \cong \mathrm{Ker}(\kappa : M \wedge M \to M)$$

Def: (Graham Ellis, 1996(?))

$$H_2(P, M) = \mathrm{Ker}(\kappa : P \wedge M \to P)$$

Exact sequence:

$$H_3(P) \xrightarrow{\eta} H_3(P/M) \to H_2(P, M) \to H_2(P) \to \\ H_2(P/M) \to P/[P, M] \to P^{\mathrm{ab}} \to (P/M)^{\mathrm{ab}}$$

This is the start of work of Ellis to obtain many new results on the classical Schur Multiplier.

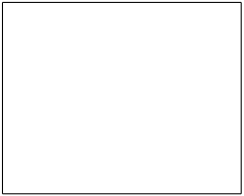
Conclusion

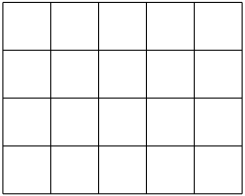
The consideration of multiple groupoids allows for the appearance of new algebraic structures underlying classical homotopy theory,
these structures also throw light on traditional group theory, and
have analogues for other algebraic structures, e.g. Lie algebras.
We have given examples of precise higher dimensional nonabelian methods for local-to-global problems.

Additional comments

Additional points were made on the blackboard and in answer to questions.
1. The usual idea is that we want invariants of topological spaces. However a space has to be specified in some way, by some kind of data, and this data usually has some kind of structure. It can be expected that this structure is reflected somehow in the space. So we should have invariants of structured spaces and these should lead to structured algebraic invariants.

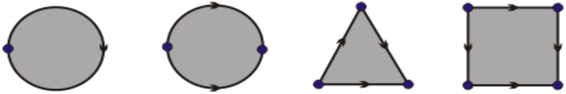
2. Consider the figures:





From left to right gives subdivision.
From right to left should give composition.
What we need for local-to-global problems is:
Algebraic inverses to subdivision.
We know how to cut things up, but how to control algebraically putting them together again?
These figures suggest the advantage of a cubical approach.

3. In moving from 1-dimensional compositions to higher dimensional ones it seems to be necessary to choose the basic geometric objects. But there are an infinite number of compact convex subsets of $\mathbb{R}^n$ for $n \geqslant 2$. With some cell structure they may be seen for $n = 2$ as cell, globe, simplex, cube, etc. It is common to see higher category theory in a globular, or sometimes simplicial context, but we use mainly a cubical approach.



4. The term 'biderivation' may also be used in the context of Lie algebras, and then the bracket of the Lie algebra

$$[\ ,\ ] : L \times L \to L$$

is seen as a biderivation. This leads to a nonabelian tensor product for Lie algebras.
5. In the calculation of $D_{2n} \otimes D_{2n}$ for n even, the elements $y \otimes y, x \otimes x$ represent composition with the Hopf map, and $(x \otimes y)(y \otimes x)$ represents a Whitehead product, when x, y and these tensor products are interpreted in π_2, π_3 of $SK(D_{2n}, 1)$.

6. If $x \in A \cap B \subseteq A, B \subseteq X$ it is natural to consider the space Φ of maps $I^2 \to X$ which map the vertices to x, edges in direction 1 to A, and edges in direction 2 to B. This set also has compositions $\circ_1, \circ_2$ in 2 directions making it a weak double category. However it seems that this structure is not inherited by $\pi_0\Phi$ except under extra conditions, for example that the images of $\pi_2(A, x), \pi_2(B, x)$ in $\pi_2(X, x)$ coincide, which happens of course if $A = B$. However the fundamental group $\pi_1(\Phi, *)$ does inherit these compositions to become a cat^2-group. This and its generalisations are due to J.-L. Loday, 1982.