TEN TOPOLOGIES FOR $X \times Y$

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Introduction

THE study of topologies on $X \times Y$ is motivated by some outstanding deficiencies of the *cartesian*, that is the usual, topology on the product of spaces. (Throughout this paper all spaces will be assumed to be Hausdorff.)

Firstly, the cartesian product of identification maps is not, in general, an identification map. As a consequence certain natural products such as the join and smash product which are formed as identifications of cartesian products, turn out to be non-associative [cf. (**4**)]. Further, the cartesian product of locally uncountable CW-complexes is not a CW-complex, and this is a difficulty in an important application of these complexes: to the singular complex of a space.

Secondly, the cartesian topology does not behave well in considering maps *from*, rather than *into*, $X \times Y$. One example of this, the exponential law for function spaces, will be discussed in detail in a sequel to this paper (**1**) (the difficulty here is, however, not usually traced to the product topology). Another, and similar, example is the use made by Bourbaki in (**2**) of 'fonctions hypocontinues', which are simply bilinear functions $X \times Y \to Z$, continuous in some topology on $X \times Y$ other than the cartesian.

Now there are indefinitely many natural product topologies [cf. § 1]. There are also a surprising number which are relatively close to the cartesian product, have interesting and useful properties, and are in some respects better than the cartesian.

Our main purpose is to prove the theorem:

THEOREM 0.1. *There are on the category of spaces at least ten natural products which are associative and have the same compact subsets as the cartesian product. These products are distinct in general.*

Included among these products is the cartesian product itself, and the *weak product* discussed by Spanier in (**8**). The eight other products we call the *extraordinary products*. They lie between the cartesian and the weak product in the usual partial ordering of topologies on a set.

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As examples where improvements on the cartesian product are obtained, we shall prove the theorems:

THEOREM 4.8. *Two of the extraordinary products have the property: if $f: P \to X$ is an identification map, then for any $Y, f \times 1: P \times Y \to X \times Y$ is an identification map.*

THEOREM 0.2. *The extraordinary products, and the weak product, make the product of* CW-*complexes a* CW-*complex.*

The latter result was well known for the weak product.

Theorem 0.2 is a special case of the result that the extraordinary products coincide with the weak product on the category of k-spaces. Hence the category of k-spaces has a convenient product (namely the weak product) and in (**1**) we shall show the importance of this category for homotopy theory.

It may turn out that the category of k-spaces is adequate and convenient for all purposes of topology. In this case the extraordinary products will lose their interest. At present, their introduction is justified by the applications of some of them [cf. also (**1**)] and by their close interrelationship.

This paper rests on the properties of weak topologies discussed in § 1. § 2 resumes known results on k-spaces and the weak product. § 3 defines the extraordinary products and proves their simpler properties. § 4 is concerned with conditions for the products to coincide, and with the proof of associativity and Theorem 4.8. § 5 uses a result proved in § 6 to show that the products are distinct in general.

1. Weak topologies

Let Σ be a cover of a space X; the elements of Σ are to be taken as subspaces rather than just subsets, of X.

DEFINITION 1.1. *A set $C \subseteq X$ is closed (open) in the 'weak topology with respect to Σ' if and only if $C \cap S$ is closed (open) in S for each $S \in \Sigma$. The set X with this topology is written X_Σ.*

PROPOSITION 1.2. *Each $S \in \Sigma$ retains its own topology in X_Σ: that is, the inclusion $S \to X_\Sigma$ is a homeomorphism into. In particular, if $X \in \Sigma$, then $X = X_\Sigma$.*

PROPOSITION 1.3. *Σ is a cover of X_Σ by subspaces, and $(X_\Sigma)_\Sigma = X_\Sigma$.*

PROPOSITION 1.4. *The topology of X_Σ is larger than that of X: that is, the identity $X_\Sigma \to X$ is continuous.*

PROPOSITION 1.5. *A function $f: X_\Sigma \to Y$ is continuous if and only if $f \mid S$ is continuous for each $S \in \Sigma$.*

These propositions follow immediately from the definition.

For each S of Σ, let

$$P(S) = \{(S, x): x \in S\}, \qquad P(\Sigma) = \bigcup_{S \in \Sigma} P(S).$$

Let $\sigma: P(\Sigma) \to X$ be defined by $\sigma(S, x) = x$, and let $P(\Sigma)$ have the topology that each $P(S)$ is open and closed in $P(\Sigma)$ and $\sigma \mid P(S)$ is a homeomorphism onto S.

PROPOSITION 1.6. *$\sigma: P(\Sigma) \to X_\Sigma$ is an identification map.*

The proof is easy. This proposition says that the weak topology with respect to Σ is exactly the identification topology with respect to σ, and this gives a handy method of proving that a space has the weak topology. [This method is due to Cohen (3).]

Another consequence of Proposition 1.6 is the theorem:

THEOREM 1.7. *Let Σ, Σ' be covers of X, X' respectively. Let $f: X \to X'$ be a function such that for each S of Σ* (i) *there is an S' of Σ' such that $f(S) \subseteq S'$, and* (ii) *$f \mid S$ is continuous. Then $f: X_\Sigma \to X_{\Sigma'}$ is continuous.*

Proof. For each $S \in \Sigma$ we choose an S' such that $f(S) \subseteq S'$, and define $f': P(\Sigma) \to P(\Sigma')$ by $f'(S, x) = (S', fx)$. The following diagram is commutative

$$\begin{array}{ccc} P(\Sigma) & \xrightarrow{f'} & P(\Sigma') \\ \sigma \downarrow & & \downarrow \sigma' \\ X_\Sigma & \xrightarrow{f} & X'_{\Sigma'}. \end{array}$$

Since σ is an identification map, f is continuous.

COROLLARY 1.8. *Let Σ, Σ' be covers of X. If Σ is a refinement of Σ', then the identity $X_\Sigma \to X_{\Sigma'}$, is continuous. If also Σ' is a refinement of Σ, then $X_\Sigma = X_{\Sigma'}$.*

Now let Σ be a cover of X, and let $A \subseteq X$. The *restriction* of Σ to A is the cover of A,

$$\Sigma \mid A = \{S \cap A: S \in \Sigma\}.$$

So we can form the space $A_{\Sigma|A}$. On the other hand, A with its relative topology as a subset of X_Σ determines a space A_Σ, say.

PROPOSITION 1.9. *The identity $A_{\Sigma|A} \to A_\Sigma$ is continuous. If A and each S of Σ is closed in X, then $A_{\Sigma|A} = A_\Sigma$, and A_Σ is a closed subspace of X_Σ.*

Proof. By Theorem 1.7, the inclusion $A_{\Sigma|A} \to X_\Sigma$ is continuous. Hence the identity $A_{\Sigma|A} \to A_\Sigma$ is continuous. Thus the topology of $A_{\Sigma|A}$ is not smaller than that of A_Σ, and we now prove that, under the given assumptions, it is not larger.

Let $C \subseteq A$ be closed in $A_{\Sigma|A}$. Then for each S of Σ, $C \cap S = C \cap S \cap A$ is closed in $S \cap A$ and hence in S. So C is closed in X_Σ, and hence C is closed in A_Σ. Thus $A_{\Sigma|A} = A_\Sigma$.

Similarly $C = A_{\Sigma|A}$ is closed in X_Σ; hence A_Σ is a closed subspace of X_Σ.

DEFINITION 1.10. *Let $\mathscr{X}$ be a category of spaces. A 'natural cover on $\mathscr{X}$' is a function Σ assigning to each space X in $\mathscr{X}$ a cover $\Sigma(X)$ of X and such that, for each map $f\colon X \to Y$ in $\mathscr{X}$ and each S in $\Sigma(X)$,* (i) *there is an S' in $\Sigma(Y)$ such that $f(S) \subseteq S'$;* (ii) *$f \mid S$ is continuous.*

If Σ is a natural cover, we write X_Σ for $X_{\Sigma(X)}$. By Theorem 1.7, if $f\colon X \to Y$ is map in $\mathscr{X}$, then $f\colon X_\Sigma \to Y_\Sigma$ is continuous. This map is sometimes written f_Σ.

Examples 1.11. The following are natural covers on the category of all spaces and continuous maps:

(i) $\Sigma(X) = \{X\}$;
(ii) $\Sigma(X) = \{\{x\}\colon x \in X\}$;
(iii) $\Sigma(X) = \mathscr{C}(X)$, the set of compact subsets of X;
(iv) $\Sigma(X)$ is the set of connected subsets of X.

For many categories $\mathscr{X}$, natural covers on $\mathscr{X}$ can be constructed by the method of the *universal example*. A space U in $\mathscr{X}$ and a set Σ_U of subspaces of U are chosen. For any X, $\Sigma(X)$ consists of the sets $f(S)$ for all S in Σ_U and all maps $f\colon U \to X$. Clearly $\Sigma(X)$ satisfies the strong naturality condition that, if $g\colon X \to Y$ is a map, and $S \in \Sigma(X)$, then $g(S) \in \Sigma(Y)$. However, some conditions on $\mathscr{X}$ are necessary to ensure that $\Sigma(X)$ is a cover of X: it is enough, for example, that all constant maps are in $\mathscr{X}$.

We remark finally that, if $\{\Sigma_i\}_{i\in I}$ is a family of natural covers on $\mathscr{X}$, then their union Σ defined by

$$\Sigma(X) = \bigcup_{i\in I} \Sigma_i(X)$$

is again a natural cover on $\mathscr{X}$.

2. k-spaces

The main purpose of this section is to collect together for ease of reference some known facts on k-spaces.

A *k-space* is a space X such that $X = X_{\mathscr{C}}$: that is, a k-space is a space which has the weak topology with respect to the family of compact subspaces.

PROPOSITION 2.1. (i) *X and $X_{\mathscr{C}}$ have the same compact subsets,* (ii) *$X_{\mathscr{C}}$ is a k-space,* (iii) *$X_{\mathscr{C}}$ has the largest topology with the same compact subsets as X.*

Proof. (i) $X_{\mathscr{C}}$ has a larger topology than that of X, and so

$$\mathscr{C}(X_{\mathscr{C}}) \subseteq \mathscr{C}(X).$$

By Proposition 1.2, $\mathscr{C}(X) \subseteq \mathscr{C}(X_{\mathscr{C}})$.

(ii) This follows from (i) and the definition of weak topologies.

(iii) Let X' have the same compact subsets as X. It is sufficient to prove that these compact subsets are the same spaces in X as in X'. For then the identity $1: X \to X'$ is continuous on compact subspaces, and so $1: X_{\mathscr{C}} \to X'$ is continuous.

The closed subsets of a compact space are the compact subsets. Hence the closed subsets of a compact subspace of X' are the same as the closed subsets of the corresponding compact subspace of X: that is, the compact subsets of X and X' coincide as spaces.

Now $\mathscr{C}$ is a natural cover. So Proposition 2.1 (ii) implies that $\mathscr{C}$ determines a functor from spaces to k-spaces. This functor will be written k.

PROPOSITION 2.2. *If X is locally compact, or satisfies the first axiom of countability, then X is a k-space.*

This is Theorem 7.13 of (**7**).

PROPOSITION 2.3. *Any* CW-*complex is a k-space* (**10**). *If K, L are* CW-*complexes, then $k(K \times_C L)$*† *is a* CW-*complex.*

For proofs, see (**8**) [§ 2].

PROPOSITION 2.4. *A closed subspace of a k-space is a k-space. A quotient space of a k-space is a k-space.*

The first part follows from Proposition 1.9. For a proof of the second part, see (**4**) [(1.81)] (we must assume, of course, that the quotient space is Hausdorff).

Examples of spaces which are not k-spaces are a little hard to find. Ex. 7.J of (**7**) outlines a proof that arbitrary subspaces and arbitrary products of k-spaces are not k-spaces. A stronger result than the latter is

PROPOSITION 2.5. *There are k-spaces K, L such that $K \times_C L$ is not a k-space.*

† $K \times_C L$ denotes the cartesian product space of K and L.

Proof. Dowker in (5) has given an example of CW-complexes K, L such that $K \times_C L$ is not a CW-complex. Proposition 2.3 shows firstly that K, L are k-spaces, and secondly that $K \times_C L \neq k(K \times_C L)$.

This last proposition shows that the cartesian product does not define a product in the category $\mathscr{K}$ of k-spaces. The 'correct' product in $\mathscr{K}$ is in fact the weak product.

DEFINITION 2.6. *The 'weak product' of spaces X, Y is the space*

$$X \times_W Y = k(X \times_C Y).$$

PROPOSITION 2.7. *$X \times_W Y$ has the weak topology with respect to the family $\{A \times_C B\}$, for all compact subsets A of X, B of Y.*

Proof. This follows from Corollary 1.8 since each $A \times_C B$ is compact, and any compact subset of $X \times_C Y$ is contained in the compact product of its projections.

PROPOSITION 2.8. *For any X, Y,*

$$X \times_W Y = k(X) \times_W k(Y).$$

Proof. The identity $1\colon k(X) \times_C k(Y) \to X \times_C Y$ is continuous, and hence so is $1\colon k(X) \times_W k(Y) \to X \times_W Y$.

The identity $1\colon X \times_C Y \to k(X) \times_C k(Y)$ is continuous on $A \times_C B$ for each compact subset A of X, B of Y. By Theorem 1.7 and Proposition 2.7, $1\colon X \times_W Y \to k(X) \times_W k(Y)$ is continuous. Hence

$$X \times_W Y = k(X) \times_W k(Y).$$

PROPOSITION 2.9. *The weak product is associative and commutative, and the projections $X \times_W Y \to X$, $X \times_W Y \to Y$ are continuous.*

Proof. The only non-trivial part is associativity. By Proposition 2.8,

$$X \times_W (Y \times_W Z) = X \times_W (Y \times_C Z) = k(X \times_C Y \times_C Z),$$

and similarly $(X \times_W Y) \times_W Z = k(X \times_C Y \times_C Z)$.

We may ask for conditions on X, Y that $X \times_W Y = X \times_C Y$. In one direction there is an easy result:

PROPOSITION 2.10. *If $X \times_W Y = X \times_C Y$, then X and Y are k-spaces.*

Proof. Let $y \in Y$. Then X is homeomorphic to $X \times_C \{y\}$, which is a closed subspace of the k-space $X \times_C Y$. So X is a k-space. Similarly Y is a k-space.

Only incomplete results are known in the other direction. The following theorem seems to cover all known cases:

THEOREM 2.11. *$X \times_W Y = X \times_C Y$ if* (i) *X and Y satisfy the first axiom of countability, or* (ii) *X and Y are k-spaces one of which is locally compact, or* (iii) *X and Y are* CW*-complexes one of which is locally finite, or* (iv) *X and Y are locally countable* CW*-complexes.*

In case (i), $X \times_C Y$ also satisfies the first axiom of countability, and so is a k-space by Proposition 2.2. Case (iii) is a special case of (ii) which is due to Cohen (**3**). In case (iv) it was proved by Dowker [(**6**) § 8] that $X \times_C Y$ is a CW-complex. Hence $X \times_C Y = X \times_W Y$ by Proposition 2.3.

3. Product topologies

Let $\mathscr{X}$ be the category whose objects are the cartesian products $X \times_C Y$ for all spaces X, Y, and whose maps are $f \times_C g$ for all continuous maps f, g. Let Σ be a natural cover on $\mathscr{X}$. Then Σ determines a *product* $X \times_\Sigma Y$ of spaces by the rule

$$X \times_\Sigma Y = (X \times_C Y)_\Sigma.$$

This product is natural in the sense that, if $f\colon X \to X'$, $g\colon Y \to Y'$ are continuous, then the product map $f \times_\Sigma g\colon X \times_\Sigma Y \to X' \times_\Sigma Y'$ is continuous.

The *opposite* of Σ is the natural cover Σ^* on $\mathscr{X}$ such that, for any X, Y,

$$\Sigma^*(X \times_C Y) = \{T(U)\colon U \in \Sigma(Y \times_C X)\},$$

where $T\colon Y \times_C X \to X \times_C Y$ is the natural map. Then Σ^* defines a product $X \times_{\Sigma^*} Y$, the *opposite* of the product $X \times_\Sigma Y$.

A product $X \times_\Sigma Y$ is *commutative* if for all X, Y, $X \times_\Sigma Y = X \times_{\Sigma^*} Y$. It is *associative* if for all X, Y, Z the natural map

$$(X \times_\Sigma Y) \times_\Sigma Z \to X \times_\Sigma (Y \times_\Sigma Z)$$

is a homeomorphism.

In Table 3.1 we define ten natural covers on X. We abbreviate $\Sigma(X \times_C Y)$ to Σ, $X \times_C Y$ to $X \times Y$, and the sets $\{x\}$, $\{y\}$ to x, y respectively. It is to be understood that x, y range over the elements, and A, B over the compact subsets, of X, Y respectively.

TABLE 3.1

$W = \{A \times B\}$	$C = \{X \times Y\}$
$P^* = \{A \times B,\ x \times Y\}$	$P = \{A \times B,\ X \times y\}$
$R^* = \{A \times Y\}$	$R = \{X \times B\}$
$S^* = \{A \times Y,\ X \times y\}$	$S = \{X \times B,\ x \times Y\}$
$Q = \{A \times B,\ x \times Y,\ X \times y\}$	$T = \{A \times Y,\ X \times B\}$

These natural covers define ten natural products, which, by Corollary 1.8, are related by the following diagram of continuous identity maps:

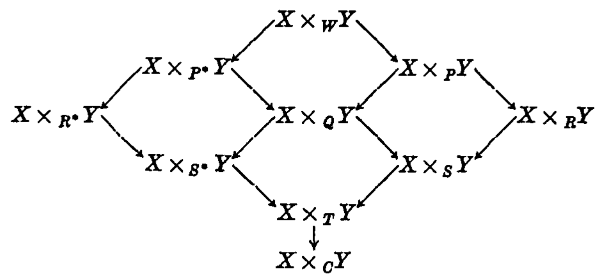

DIAGRAM 3.2

The products $X \times_W Y$, $X \times_C Y$ are, of course, the weak and cartesian products respectively. The eight other products we call the *extraordinary products*.

There are other natural covers on the category $\mathscr{X}$, but the author has not found any which determine products that are associative, that have the same compact subsets as the cartesian product, and that are distinct from the products of Diagram 3.2.

The products $X \times_W Y$, $X \times_Q Y$, $X \times_T Y$, $X \times_C Y$ are commutative. The other six products consist of three pairs of a product and its opposite, but the proof that the products $X \times_P Y$, $X \times_S Y$ are not commutative is non-trivial [cf. §§ 5, 6].

We now prove the easy part of Theorem 0.1, namely the proposition:

PROPOSITION 3.3. *The extraordinary products and the weak product have the same compact subsets as the cartesian product.*

Proof. By Proposition 2.1 the weak and the cartesian product have the same compact subsets. Since the extraordinary products lie between these two, the result follows.

PROPOSITION 3.4. *The extraordinary products and the weak products satisfy* (i) *the projections* $X \times Y \to X$, $X \times Y \to Y$ *are continuous,* (ii) *if* X', Y' *are closed subspaces of* X, Y *respectively, then* $X' \times Y'$ *is a closed subspace of* $X \times Y$.

Proof. The first part is clear from Diagram 3.2 since the projections are continuous for the cartesian product.

The second part follows from Proposition 1.9 and the fact that the natural covers Σ which determine the products all satisfy

$$\Sigma(X' \times_C Y') = \Sigma(X \times_C Y)|(X' \times_C Y').$$

The restriction to *closed* subspaces is essential here.

We now give simple conditions for some of the products to coincide.

PROPOSITION 3.5. *The* 10 *products of* (3.2) *coincide if and only if* $X \times_C Y$ *is a k-space.*

The proof is clear. Conditions for $X \times_C Y$ to be a k-space were given in Theorem 2.11.

THEOREM 3.6. $X \times_C Y = X \times_T Y = X \times_S Y = X \times_R Y$ *if* Y *is locally compact.* $X \times_C Y = X \times_T Y = X \times_{S^*} Y = X \times_{R^*} Y$ *if* X *is locally compact.*

Proof. We prove only the first statement, from which the second follows. From Diagram 3.2, it is sufficient to prove that

$$X \times_C Y = X \times_R Y$$

if Y is locally compact.

We remark first that, if Y is compact, then $X \times_C Y = X \times_R Y$ by definition of $X \times_R Y$ and Proposition 1.2. In general we know that the topology of $X \times_R Y$ is not smaller than that of $X \times_C Y$, and we now prove that it is not larger if Y is locally compact.

Let U be an open subset of $X \times_R Y$, and let $(x, y) \in U$; let V be a compact neighbourhood of y. By the definition of $X \times_R Y$, $U \cap (X \times_C V)$ is open in $X \times_C V$. Since V is a neighbourhood of y, there are open sets U_x in X, U_y in Y such that

$$(x, y) \in U_x \times U_y \subseteq U \cap (X \times_C V) \subseteq U$$

Hence U is open in $X \times_C Y$.

4. Identification maps

As before, when no confusion will arise, we abbreviate $X \times_C Y$ to $X \times Y$. We recall that all spaces are assumed to be Hausdorff.

The results of this section all depend on the following lemma, which is a special case of Lemma 4 of (**9**):

LEMMA 4.1. *If* $f: P \to X$ *is an identification map, and* B *is compact, then* $f \times_C 1: P \times_C B \to X \times_C B$ *is an identification map.*

The first use we make of this lemma is to prove the theorems:

THEOREM 4.2. *If* Y *is a k-space, then*

(i) $X \times_T Y = X \times_S Y = X \times_R Y$,

(ii) $X \times_{S^*} Y = X \times_Q Y = X \times_P Y$,

(iii) $X \times_{R^*} Y = X \times_{P^*} Y = X \times_W Y$.

THEOREM 4.3. *If* X *is a k-space, then*

(i) $X \times_T Y = X \times_{S^*} Y = X \times_{R^*} Y$,

(ii) $X \times_S Y = X \times_Q Y = X \times_{P^*} Y$,

(iii) $X \times_R Y = X \times_P Y = X \times_W Y$, THEOREM 4.4. *If X and Y are k-spaces, then the extraordinary and the weak products coincide.*

Theorem 4.3 results from Theorem 4.2 by replacing each product by its opposite. These two theorems together imply Theorem 4.4 of which Theorem 0.2 of the Introduction is a special case.

In the proof of Theorem 4.2 we need the following lemma whose proof is trivial:

LEMMA 4.5. *Let X, Y be the union of disjoint subspaces X_α, Y_α each open and closed in X, Y respectively. Let $\sigma\colon X \to Y$ be a function such that* (i) $\sigma(X_\alpha) = Y_\alpha$, (ii) $\sigma \mid X_\alpha\colon X_\alpha \to Y_\alpha$ *is an identification map. Then σ is an identification map.*

Proof of Theorem 4.2 (i). It is sufficient to prove $X \times_T Y = X \times_R Y$. We recall that T is the cover $\{A \times Y, X \times B\}$, and R is the cover $\{X \times B\}$, for all compact subsets $A \subseteq X$, $B \subseteq Y$. We abbreviate symbols such as $P(\{A \times B\})$, where P is as in § 1, to $P\{A \times B\}$.

Since any compact space can be embedded as a closed subspace of a non-compact space, we may assume X is non-compact. This ensures that the spaces $P\{X \times B\}$ and $P\{A \times B\}$ are disjoint. In the diagram

$$\begin{array}{c} P\{X \times B\} \cup P\{A \times B\} \\ {\scriptstyle \sigma_1}\downarrow \qquad\qquad \downarrow{\scriptstyle \sigma_2} \\ P\{X \times B\} \cup P\{A \times Y\} \\ \downarrow{\scriptstyle \sigma} \\ X \times_T Y, \end{array}$$

σ is the identification map of Proposition 1.6, σ_1 is the identity, and σ_2 is defined by $\sigma_2(A \times B, z) = (A \times Y, z)$ $(z \in A \times B)$.

It is sufficient to prove that σ_2 is an identification map. For then, by Lemma 4.5, $\sigma_1 \cup \sigma_2$ is an identification map, as is $\sigma(\sigma_1 \cup \sigma_2)$. Hence $X \times_T Y$ has the weak topology with respect to the family $\{X \times B, A \times B\}$, which refines, and is a refinement of, $R = \{X \times B\}$. So

$$X \times_T Y = X \times_R Y.$$

Since Y is a k-space, there is an identification map $\sigma_3\colon P\{B\} \to Y$. For each fixed compact subset $A_0 \subseteq X$, $\sigma_2 | P\{A_0 \times B\}$ may be identified with $1 \times \sigma_3\colon A_0 \times P\{B\} \to A_0 \times Y$. Hence, by Lemma 4.1,

$$\sigma_2 \mid P\{A_0 \times B\}\colon P\{A_0 \times B\} \to P\{A_0 \times Y\}$$

is an identification map. By Lemma 4.5, σ_2 is an identification map.

Proof of Theorem 4.2 (ii). We give the proof in less detail than the preceding.

It is sufficient to prove $X \times_{S^*} Y = X \times_P Y$. Now $S^* = \{A \times Y, X \times \{y\}\}$.

Since Y is a k-space, the method of the preceding proof shows that S^* determines the same weak topology as $\{A \times B, X \times \{y\}\} = P$. Hence $X \times_{S^*} Y = X \times_P Y$.

Proof of Theorem 4.2 (iii). Since Y is a k-space, $R^* = \{A \times Y\}$ determines the same weak topology as $\{A \times B\} = W$: that is,

$$X \times_{R^*} Y = X \times_W Y.$$

THEOREM 4.6. *The extraordinary products are associative.*

Proof. I give the proof for $\times_Q$, and leave the similar proofs for the other products to the reader.

In the following, let x, y, z range over the elements, and A, B, C over the compact subsets, of X, Y, Z respectively. Let L range over the compact subsets of $Y \times_C Z$. By Proposition 3.3, the compact subsets of $Y \times_C Z$ are the compact subsets of $Y \times_Q Z$, and so $X \times_Q (Y \times_Q Z)$ has the weak topology with respect to

$$\Sigma = \{A \times L,\ x \times (Y \times_Q Z),\ X \times y \times z\}.$$

By Corollary 1.8, Σ determines the same weak topology as

$$\Sigma_1 = \{A \times B \times C,\ x \times (Y \times_Q Z),\ X \times y \times z\}.$$

The method of proof of Theorem 4.2 shows that Σ_1 determines the same weak topology as

$$\begin{aligned}\Sigma_2 &= \{A \times B \times C,\ x \times y \times Z,\ x \times Y \times z,\ x \times B \times C,\ X \times y \times z\} \\ &= \{A \times B \times C,\ x \times y \times Z,\ x \times Y \times z,\ X \times y \times z\}.\end{aligned}$$

A similar computation shows that $(X \times_Q Y) \times_Q Z$ also has the weak topology with respect to Σ_2.

There are other associativity relations among the products. I state two of these, and leave the reader to investigate the situation:

THEOREM 4.7. *For all* X, Y, Z, *the natural maps*

$$X \times_{R^*} (Y \times_R Z) \to (X \times_{R^*} Y) \times_R Z,$$
$$X \times_{P^*} (Y \times_S Z) \to (X \times_{P^*} Y) \times_S Z,$$

are homeomorphisms.

The proof is omitted.

THEOREM 4.8. *Let* $f\colon P \to X$ *be an identification map. For any* Y, $f \times_R 1\colon P \times_R Y \to X \times_R Y$, $f \times_S 1\colon P \times_S Y \to X \times_S Y$ *are identification maps.*

Proof. We prove first that $h = f \times_R 1$ is an identification map. Let $U \subseteq P \times_R Y$ be open and saturated with respect to h. We must prove that $h(U)$ is open in $X \times_R Y$, i.e. that $h(U) \cap (X \times_C B)$ is open in $X \times_C B$ for each compact $B \subseteq Y$.

Let $B \subseteq Y$. Since $h = f\times 1$,

$$h(U) \cap (X\times B) = h(U \cap (P\times B)).$$

Let $h_B = h \mid P\times B\colon P\times_C B \to X\times_C B$. Then

$$\begin{aligned} h_B^{-1}h_B(U \cap (P\times B)) &= h_B^{-1}(h(U) \cap (X\times B)) \\ &= h^{-1}h(U) \cap (P\times B) \\ &= U \cap (P\times B). \end{aligned}$$

Thus $U \cap (P\times B)$ is saturated with respect to h_B.

Let $B \subseteq Y$ be compact. Then h_B is an identification map by Lemma 4.1. Also $U \cap (P\times_C B)$ is open in $P\times_C B$ and saturated with respect to h_B. Then

$$h_B(U \cap (P\times_C B)) = h(U) \cap (X\times_C B)$$

is open in $X\times_C B$. Therefore $h(U)$ is open in $X\times_R Y$.

To prove $h' = f\times_S 1$ an identification map, we start with $U \subseteq P\times_S Y$ open and saturated with respect to h'. Then U is open in $P\times_R Y$ and saturated with respect to h. So $h(U) = h'(U)$ is open in $X\times_R Y$. We show that $h'(U)$ is also open in $X\times_S Y$.

For any $p \in P$,

$$h' \mid p\times_C Y\colon p\times_C Y \to fp\times_C Y$$

is a homeomorphism. Now $U \cap (p\times_C Y)$ is open in $p\times_C Y$; so $h'(U \cap (p\times_C Y))$ is open in $fp\times_C Y$. For any $x \in X$, $h'(U) \cap (x\times_C Y)$ is the union of the sets $h'(U \cap (p\times_C Y))$ for all $p \in P$ such that $f(p) = x$. Hence $h'(U) \cap (x\times_C Y)$ is open in $x\times_C Y$. So $h'(U)$ is open in $X\times_S Y$, and h' is an identification map.

Let $f\colon P \to X$ and Y be as in Theorem 4.8. By combining Theorem 4.8 with Theorems 3.5, 3.6, 4.2, and 4.3 we can obtain conditions that $f\times 1$ be an identification map for various natural products. For example, if Y is locally compact, then $f\times_C 1$ is an identification map [a result due to Cohen (4)]. I give only two other results, the first because it is needed in (1) and the second because it seems to be a new result on the cartesian product.

COROLLARY 4.9. *Let $f\colon P \to X$, $g\colon Q \to Y$ be identification maps, and let P and Q be k-spaces. Then $f\times_W g\colon P\times_W Q \to X\times_W Y$ is an identification map.*

COROLLARY 4.10. *Let $f\colon P \to X$, $g\colon Q \to Y$ be identification maps, and let P, Q, X, Y satisfy the first axiom of countability. Then $f\times_C g\colon P\times_C Q \to X\times_C Y$ is an identification map.*

Proof of Corollary 4.9. By Proposition 2.4, X and Y are k-spaces. By Theorem 4.4, $P\times_W Q = P\times_R Q$, $X\times_W Q = X\times_R Q$, and so, by

Theorem 4.8, $f \times_W 1 \colon P \times_W Q \to X \times_W Q$ is an identification map. Similarly $1 \times_W g \colon X \times_W Q \to X \times_W Y$ is an identification map. Hence $f \times_W g = (1 \times_W g) \circ (f \times_W 1)$ is an identification map.

Proof of Corollary 4.10. This follows immediately from Corollary 4.9 since the given conditions on P, Q, X, Y imply $P \times_W Q = P \times_C Q$, $X \times_W Y = X \times_C Y$.

We do not know if $f \times_W g$ is an identification map for arbitrary P, Q. The fact that $f \times_C g$ is in general not an identification map is a consequence of Proposition 2.5.

Theorem 4.8 suggests the following question: Are there natural products such that the product of identification maps is an identification map? Two trivial examples of such products are those with the discrete, and with the indiscrete, topologies.

Another such product is $X \times_D Y$, which has the weak topology with respect to $D = \{x \times Y, X \times y\}$. This product is associative and commutative, but does not have the same compact subsets as the cartesian product (the *diagonal* in $X \times_D X$ has the discrete topology).

There are other natural products: for example that formed by the cover $\{x \times B, A \times y\}$ for x in X, y in Y, A in $\mathscr{C}(X)$, B in $\mathscr{C}(Y)$. This product is commutative but does not have the same compact subsets as the cartesian product. We do not know if this product is associative.

5. The products are distinct

In § 6 we shall prove the theorem:

THEOREM 5.1. *There are spaces X, Y such that X is compact and $X \times_S Y \neq X \times_T Y$.*

Here, we use Theorem 5.1 to complete the proof of Theorem 0.1. We prove the theorem:

THEOREM 5.2. *The ten products of Diagram* 3.2 *are distinct in general.*

The proof consists of a sequence of propositions:

PROPOSITION 5.3. *There are spaces X, Y such that $X \times_C Y \neq X \times_T Y$.*

Proof. By Proposition 2.5, there are k-spaces X, Y such that $X \times_C Y$ is not a k-space. However, if X, Y are k-spaces, so is $X \times_T Y$.

PROPOSITION 5.4. *If any one of $X \times_T Y$, $X \times_Q Y$, $X \times_S Y$, $X \times_{S^*} Y$ is a k-space, then X and Y are k-spaces.*

Proof. These products all coincide with $X \times_C Y$ if X or Y has only one element. So, if one of these products is a k-space, then $X \times_C \{y\}$, $\{x\} \times_C Y$ $(x \in X,\ y \in Y)$ are also k-spaces. Hence X and Y are k-spaces.

PROPOSITION 5.5. *The three sets of spaces*

$$S_1 = \{X \times_T Y, X \times_{S^*} Y, X \times_{R^*} Y\},$$
$$S_2 = \{X \times_S Y, X \times_Q Y, X \times_{P^*} Y\},$$
$$S_3 = \{X \times_R Y, X \times_P Y, X \times_W Y\}$$

are disjoint in general.

Proof. Let X, Y be as in Theorem 5.1. Since X is compact,

$$X \times_T Y = X \times_{S^*} Y = X \times_{R^*} Y,$$
$$X \times_S Y = X \times_Q Y = X \times_{P^*} Y,$$
$$X \times_R Y = X \times_P Y = X \times_W Y.$$

Now Y is not a k-space since $X \times_S Y \neq X \times_T Y$, and so, by Proposition 5.4, the elements of S_1 and S_2 are not k-spaces. But $X \times_W Y$ is a k-space; so $S_1 \cap S_3 = S_2 \cap S_3 = \varnothing$. Also $S_1 \cap S_2 = \varnothing$ since $X \times_S Y \neq X \times_T Y$.

PROPOSITION 5.6. *The three sets of spaces*

$$\{X \times_T Y, X \times_S Y, X \times_R Y\},$$
$$\{X \times_{S^*} Y, X \times_Q Y, X \times_P Y\},$$
$$\{X \times_{R^*} Y, X \times_{P^*} Y, X \times_W Y\}$$

are disjoint in general.

This follows from Proposition 5.5 on replacing each product by its opposite.

Theorem 5.2 follows immediately from Propositions 5.3, 5.5, 5.6.

Remark. These results leave open a number of questions. For example, if $X \times_T Y = X \times_S Y$ for all X, does it follow that Y must be a k-space?

6. Proof of Theorem 5.1

In the example of spaces X, Y such that $X \times_S Y \neq X \times_T Y$, X is the 1-point compactification of a countable discrete space, and Y is the function space R^J (with the compact-open topology), where R is the real line and J is an uncountable discrete space. This example was suggested by Ex. 7.J (*b*) of (7).

We note that the compact-open topology on R^J is the same as the cartesian product topology, in which R^J is regarded as the product of copies of R.

First we prove the lemma:

LEMMA 6.1. *The set* $F^n \subseteq R^J$ *of functions taking only values* 0, n *and having at most* n *zeros, is closed.*

Proof. Any g in the closure of F^n also can take only values 0, n; if g has zeros at $i_1, \ldots, i_m$, then g has a neighbourhood

$$U = \{f : f \in R^J, -\tfrac{1}{2} \leqslant f(i_r) \leqslant \tfrac{1}{2}, 1 \leqslant r \leqslant m\}$$

which does not meet F^n unless $m \leqslant n$. Hence $g \in F^n$, as asserted.

Let $X = Z^+ \cup \{\omega\}$ be the 1-point compactification of the discrete space Z^+ of positive integers; since X is compact, $X \times_T R^J = X \times R^J$ by Theorem 3.6. So Theorem 5.1 will follow if we show that

$$X \times_S R^J \neq X \times R^J.$$

Let $Q \subseteq X \times R^J$ be the set

$$Q = \bigcup_{n \in Z^+} \{n\} \times F^n,$$

we prove the lemma:

LEMMA 6.2. *Q is* (i) *closed in $X \times_S R^J$, and* (ii) *not closed in $X \times R^J$. In fact, in $X \times R^J$, $\bar{Q} = Q \cup \{(\omega, \mathbf{0})\}$, where $\mathbf{0}: J \to R$ is the zero function.*

The last statement is used to simplify the proof of (i).

Proof of Lemma 6.2 (ii). We first prove that $(\omega, \mathbf{0}) \in \bar{Q}$. For any finite set $A \subseteq J$, let $\alpha_{n,A}$ be the function whose values are 0, n and whose zeros are the points of A; for $n \geqslant$ cardinal A, $\alpha_{n,A} \in F^n$, and $(n, \alpha_{n,A}) \in Q$. Now any neighbourhood of $(\omega, \mathbf{0})$ in $X \times R^J$ contains a product set $U \times V$, where U is a neighbourhood of ω and V is a basic neighbourhood of 0 of the type

$$V = \{f : f \in R^J, f(a) \in V_a \text{ for } a \in A\}$$

for some finite non-empty set $A \subseteq J$ and some neighbourhoods V_a of 0 in R. Then U contains integers $n \geqslant$ cardinal A, and, for such an n,

$$(n, \alpha_{n,A}) \in Q \cap (U \times V).$$

Thus $(\omega, \mathbf{0}) \in \bar{Q} - Q$.

Next we show that this is the only point in $\bar{Q} - Q$, which clearly cannot contain any (n, f) $(n \in Z^+)$ since

$$Q \cap (\{n\} \times R^J) = \{n\} \times F^n$$

is closed in $\{n\} \times R^J$ by Lemma 6.1. Also it is easily seen that for any $f \neq \mathbf{0}$ in R^J such that either

(i) f takes a value in $R - (Z^+ \cup \{0\})$, or

(ii) f takes more than one value in Z^+,

the point (ω, f) has a neighbourhood disjoint from Q. Finally, if $f \neq \mathbf{0}$ takes the values 0, n only, then ω has a neighbourhood not containing n, and f has a neighbourhood disjoint from F^m for any $m \neq n$, so that

(ω, f) again has a neighbourhood disjoint from Q. This completes the proof of Lemma 6.2 except for 6.2 (i).

COROLLARY 6.3. *The closure of Q in $X \times_S R^J$ is contained in*

$$Q \cup \{(\omega, \mathbf{0})\}.$$

For the natural map $X \times_S R^J \to X \times R^J$ is continuous, and so $Q \cup \{(\omega, \mathbf{0})\}$ is closed in $X \times_S R^J$.

Now a set $C \subseteq X \times_S R^J$ is closed if and only if C meets each set of the types $\{x\} \times R^J$, $X \times B$ in a closed set (in the cartesian topology), where $x \in X$ and B is compact in R^J. Clearly

$$Q \cap (\{\omega\} \times R^J) = \varnothing, \qquad Q \cap (\{n\} \times R^J) = \{n\} \times F^n$$

are closed [Lemma 6.1]. Therefore Lemma 6.2 (i) will follow at once from the lemma:

LEMMA 6.4. *$Q \cap (X \times B)$ is closed in $X \times B$ for any compact set $B \subseteq R^J$.*

Proof. By Corollary 6.3, we have only to show that $(\omega, \mathbf{0})$ is not in the closure of such an intersection; there is nothing more to prove if $\mathbf{0} \notin B$, and so we suppose hereafter that $\mathbf{0} \in B$. Furthermore, since the projections of B are compact, B is contained in the compact product of its projections (from Tychonoff's theorem); so without loss we may suppose that

$$B = \{f : f \in R^J, f(i) \in B_i\},$$

where each B_i is a compact neighbourhood of $\mathbf{0}$.

Let $J_n = \{i : i \in J, n \leqslant \text{some member of } B_i\}$. Then $\bigcap_n J_n = \varnothing$ since each B_i is compact, and so $J = \bigcup_n (J - J_n)$. Hence, since J is uncountable, not all of $J - J_n$ are finite. Let N be selected such that $J - J_N$ is infinite and let $A \subseteq J - J_N$ be a subset of cardinal N. Define a basic neighbourhood V of $\mathbf{0}$ in R^J by

$$V = \{f : f \in R^J, -\tfrac{1}{2} \leqslant f(a) \leqslant \tfrac{1}{2}, a \in A\}.$$

We assert that the neighbourhood $X \times V$ of $(\omega, \mathbf{0})$ does not meet $Q \cap (X \times B)$.

Suppose to the contrary that, for some (n, g),

$$(n, g) \in Q \cap (X \times B) \cap (X \times V).$$

Then $g \in F^n$ has value n except for at most n zeros; therefore, since $J - J_N$ is infinite, $g(i) = n$ for some $i \in J - J_N$. Since $g \in B$, we have $g(i) \in B_i$ and so, by the definition of J_N, $n < N$. But this is impossible, for $g \in V$ and so is zero on A which has cardinal N; yet $g \in F^n$, and so g has at most n zeros. This completes the proof of Corollary 6.3, and so of Lemma 6.2 and Theorem 5.1.

COROLLARY 6.5. *The natural map* $X \times_S R^J \to R^J \times_S X$ *is bijective and continuous, but not a homeomorphism.*

For $R^J \times_S X = R^J \times X$ since X is compact; thus the natural map $X \times R^J \to R^J \times_S X$ is a homeomorphism; however, by Lemma 6.2, the natural map $X \times {}_S R^J \to X \times R^J$ is continuous but not a homeomorphism.

REFERENCES

1. R. Brown, 'Function spaces and product topologies', *Quart. J. of Math.* (to appear).
2. N. Bourbaki, *Espaces vectoriels topologiques*, (Paris 1955), ch. iv.
3. D. E. Cohen, 'Spaces with weak topology', *Quart. J. of Math.* (Oxford) (2) 5 (1954) 77–80.
4. —— 'Products and carrier theory', *Proc. London Math. Soc.* (3) 7 (1957) 219–48.
5. C. H. Dowker, 'Topology of metric complexes', *American J. of Math.* 74 (1952) 555–77.
6. I. M. James, 'Reduced product spaces', *Ann. of Math.* 62 (1955) 170–97.
7. J. L. Kelley, *General topology* (New York, 1956).
8. E. H. Spanier, 'Infinite symmetric products, function spaces and duality', *Ann. of Math.* 69 (1959) 142–98.
9. J. H. C. Whitehead, 'Note on a theorem due to Borsuk', *Bull. American Math. Soc.* 54 (1948) 1125–32.
10. —— 'Combinatorial homotopy I', ibid. 55 (1949) 213–45.

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