Groups, groupoids and higher groupoids in algebraic topology

An area of mathematics in which nonabelian structures have proved important is algebraic topology, where the fundamental group $\pi_1(X, a)$ of a space X at a base point a goes back to Poincaré, [Poi96]. One intuitive idea behind this is the notion of paths in a space X with a standard composition and this has led to the fundamental groupoid $\pi_1(X, A)$ on a set of base points, introduced in [Bro67]; its elements are homotopy classes rel end points of paths in X with end points in $A \cap X$, and the composition is the usual one. Because the underlying geometry of a groupoid is that of a directed graph, whereas that of a group is a set with base point, the fundamental groupoid is able to model more geometry than the fundamental group, and this has proved crucial in many applications.

An old problem was to compute the fundamental group and the theorem of this type is known as the Siefert–van Kampen Theorem, recognising work of [Sei31] and van Kampen [Kam33]. Later important work was done by Crowell in [Cro59], formulating the theorem in modern categorical language and giving a clear proof. However this theorem did not calculate the fundamental group of the circle, or more generally of a union of two spaces with non connected intersection. This situation was remedied with the use of the fundamental groupoid on a set of base points since this set could be chosen to have at least one point in each component of the intersection. The most general theorem of this type is as follows:

Generalised Theorem of Seifert-van Kampen, [BRS84] *Suppose X is covered by the union of the interiors of a family $\{U_\lambda : \lambda \in \Lambda\}$ of subsets. If A meets each path component of all 1,2,3-fold intersections of the sets U_λ, then A meets all path components of X and the diagram*

$$\bigsqcup_{(\lambda,\mu)\in\Lambda^2} \pi_1(U_\lambda \cap U_\mu, A) \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \pi_1(U_\lambda, A) \xrightarrow{c} \pi_1(X, A) \qquad (\text{coequaliser}\pi_1)$$

*of morphisms induced by inclusions is a coequaliser in the category **Grpd** of groupoids.*

Here the morphisms a, b, c are induced respectively by the inclusions

$$U_\lambda \cap U_\mu \to U_\lambda, U_\lambda \cap U_\mu \to U_\mu, U_\lambda \to X.$$

Note that the above coequaliser diagram is an algebraic model of the diagram

$$\bigsqcup_{(\lambda,\mu)\in\Lambda^2} U_\lambda \cap U_\mu \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} U_\lambda \xrightarrow{c} X \qquad (\text{coequaliser})$$

which intuitively says that X is obtained from copies of U_λ by gluing along their intersections.

A special case is when A consists of a point a and then we get the version of [Cro59] for fundamental groups.

This version of the S–vKT for groupoids is important even when the cover has only two sets U_1, U_2 when it yields a pushout of groupoids

$$\begin{tikzcd} \pi_1(U_0, A) \arrow[r] \arrow[d] & \pi_1(U_1, A) \arrow[d] \\ \pi_1(U_2, A) \arrow[r] & \pi_1(X, A) \end{tikzcd}$$

provided A meets each path component of U_1, U_2 and $U_0 = U_1 \cap U_2$. A further case is when A consists of a point a and then we get the more traditional version with a pushout of groups, provided U_1, U_2, U_0 are path connected and $a \in U_0$. The problem with any version for fundamental groups is that they do not compute the fundamental group of the circle; this deficit was an initial motivation for the extension to groupoids given in [Bro67], with its development in [Bro68, Bro06]. The extension to groupoids also allowed a range of other applications, such as a proof of the Jordan Curve Theorem, given in the later book. The writing of the first of these books in the 1960s suggested to the author that all of 1-dimensional homotopy theory was better phrased in terms of groupoids rather than groups. This, and an analysis of the proof of the SvKT, suggested the need to explore the use of groupoids in higher homotopy theory.

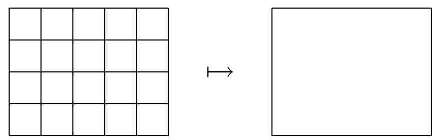
The remarkable fact about even the groupoid pushout form of the Seifert–van Kampen Theorem is that it enables, through a variety of further combinatorial techniques, the explicit computation of a nonabelian invariant, the fundamental group $\pi_1(X, a)$ at some base point a even though the input information *involves two dimensions, namely 0 and 1.* In algebraic topology, the use of such information in two neighbouring dimensions usually involves exact sequences, sometimes with sets with base points, and does *not* give complete information. The success of this groupoid generalisation seems to stem from the fact that groupoids have structure in dimensions 0 and 1, and this enables us to compute groupoids, which are models of homotopy 1-types. In homotopy theory, identifications in low dimensions have profound implications on homotopy invariants in high dimensions, and it seems that in order to model this by gluing information we require algebraic invariants which have structure in a range of dimensions, and which completely model aspects of the homotopy type. Also the input is information not just about the spaces but spaces with structure, in this case a set of base points.

The suggestion is then that other situations involving the analysis of the behaviour of complex hierarchical systems might be able to be analogously modelled, and that this modelling might necessitate a careful choice of the algebraic system. Thus there are many algebraic models of various kinds of homotopy types, but not all of them might fit into this scheme of being able directly to use gluing information.

The successful use of groupoids in 1-dimensional homotopy theory suggested the desirability of investigating the use of groupoids in higher homotopy theory.

One clue was that while a standard argument showed that possible 'higher dimensional groups' yielded only abelian groups, this argument did not apply to groupoids. It turned out that ‘higher dimensional groupoids’, whatever they were, could be much more complicated than groups.

A further aspect was to find a mathematics which allowed higher dimensional ‘algebraic inverses to subdivision’, in the sense that it could represent multiple compositions as in the following diagram:



(multcomp)

in a manner analogous to the use of $(a_1, a_2, \ldots, a_n) \mapsto a_1 a_2 \ldots a_n$ in categories and groupoids, but in dimension 2. Note that going from right to left in the diagram is subdivision, a standard technique in mathematics.

Another crucial aspect of the proof of the Seifert-van Kampen Theorem for groupoids is the use of commutative squares in a groupoid. Even in ordinary category theory we need the 2-dimensional notion of commutative square:

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "a"'] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "b"'] & \cdot \end{tikzcd} \qquad ab = cd \quad (a = cdb^{-1} \text{ in the groupoid case})$$

An easy result is that any composition of commutative squares is commutative. For example, in ordinary equations:

$$ab = cd, ef = bg \text{ implies } aef = abg = cdg.$$

The commutative squares in a category form a *double category*, and this fits with Diagram (multcomp).

There is an obstacle to an analogous construction in the next dimension, and the solution involves a new idea of *double categories or double groupoids with connections*, which we cannot explain in detail here. What we can say is that in groupoid theory, we can stay still, move forward, or turn around and go back. In double groupoid theory, we need in addition to be able to turn left or right! This leads to an entirely new world of 2-dimensional algebra, which is explained for example in [Bro99, BP03, BHS11].

A further subtle point is that to exploit these algebraic ideas in homotopy theory in dimension 2 we found we needed not just spaces but spaces X with subspaces $C \subseteq A \subseteq X$ where C is thought of as a set of base points. In higher dimensions this means that we need to deal with a space and a whole increasing sequence of subspaces, which in dimension 0 is often a set of base points: this structure is called a *filtered space*, and their wide occurrence in mathematics is such that the limitation to applications of this structure still leads to many useful even important applications. With such a structure it is possible to generalise the Seifert-van Kampen Theorem to all dimensions, using cubical methods, and this Higher Homotopy Seifert–van Kampen Theorem, HHSvKT, yields many new results including nonabelian results in dimension 2, and which are independent of and not seemingly obtainable by traditional methods, such as homology. The form of the result is still that of diagram (coequaliser) but with more elaborate structures than π_1. A survey of this theory is given in [Bro09], and a full account is in [BHS11].

A further generalisation of this work involves n-cubes of spaces. The related algebraic structures are known as *cat^n-groups*, introduced in [Lod82], and the equivalent structure of *crossed n-cubes of groups*, of [ES87]. This work is surveyed in [Bro92]. All these structures should be seen as forms of n-fold groupoids, and that the step from 1 to $n > 1$ gives extraordinarily rich algebraic structures.

Thus one sees these methods in terms of 'higher dimensional groupoid theory', developed in the spirit of group theory, so that, in view of the wide importance of group theory in mathematics and science, one seeks for analogies and applications in wider fields than algebraic topology.

In particular, since the main origin of group theory was in symmetry, one seeks for higher order notions of symmetry.

A set can be regarded as an algebraic model of a homotopy 0-type. The symmetries of a set form a group, which is an algebraic model of a pointed homotopy 1-type. The symmetries of a group G should be seen as forming a crossed module, $\chi : G \to \mathrm{Aut}(G)$, given by the inner automorphism map, and crossed modules form an algebraic model of homotopy 2-types: for a recent account of this, see [BHS11].

The situation now gets more complicated, and studies of this are in [Nor90] and [BG89]: one gets a structure called a *crossed square*, which is an algebraic model of homotopy 3-types. Crossed squares are homotopy invariants of a square of pointed spaces, which is a special case of an n-cube of spaces, for which again a van Kampen type theorem is available, [BL86]. The elaborate nonabelian algebraic structures involved have had their riches only lightly explored.

Since representation theory is a crucial aspect of group theory and its applications, this raises the question of what should be representation theory for double and higher groupoids.

Again, groupoids are heavily involved in noncommutative geometry and other related aspects of physics, but it is unknown how to extend these methods to the intrinsically 'more nonabelian' higher groupoids.

Both of these problems may be hard: it took 9 years of experimentation to move successfully from the fundamental groupoid on a set of base points to the fundamental double groupoid of a based pair.

There is an extensive literature on applications of higher forms of groupoids, particularly in areas of high energy physics, and even of special cases such as what are called sometimes called 2-groups. A recent work following many of the ideas of 'algebraic inverse to subdivision', but in a smooth manifold context, is [FMP10], and there is also extensive discussion on the n-category cafe [ncatcaf] and the ncatlab [ncatlab].

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[ncatcaf] n-category café `http://golem.ph.utexas.edu/category/`

[ncatlab] $n$catlab `http://ncatlab.org/nlab/show/HomePage`

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