

# The Equivalence of $\infty$ -groupoids and Crossed Complexes\*

Ronald Brown and Philip J Higgins

## Contents

Introduction	374
1 $\omega$ -categories and $\infty$ -groupoids.	375
2 The Relation of $\infty$ -groupoids to $\omega$ -groupoids.	376
3 The Crossed Complex Associated with an $\infty$ -groupoid.	380
4 The Equivalence of Categories.	385
References	386

## Introduction

Multiple categories, and in particular  $n$ -fold categories, have been considered by various authors [8,11]. The object of this paper is to define  $\infty$ -categories (which are  $n$ -categories for all  $n$ ) and  $\infty$ -groupoids and to prove the equivalence of categories

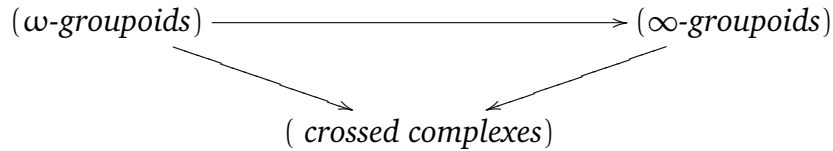
$$(\infty\text{-groupoids}) \leftrightarrow (\text{crossed complexes.})$$

This result was stated (without definitions) in the previous paper [3] and there placed in a general pattern of equivalences between categories, each of which can be viewed as a higher-dimensional version of the category of groups.

---

\*This is a slightly edited version of the paper published in CAHIERS DE TOPOLOGIE ET GÉOMETRIE DIFFÉRENTIELLE Vol. XXII-4 (1981) 371-386. 3<sup>e</sup> COLLOQUE SUR LES CATÉGORIES DEDIE A CHARLES EHRESMANN Amiens, juillet 1980.

The interest of the above equivalence arises from the common use of  $n$ -categories, particularly in situations describing homotopies, homotopies of homotopies, etc..., and also from the fact that  $\infty$ -groupoids can be regarded as a kind of half-way house between  $\omega$ -groupoids and crossed complexes. It is easy to construct from any  $\omega$ -groupoid a subset which has the structure of an  $\infty$ -groupoid and contains the associated crossed complex. In this way we get a diagram of functors



which is commutative up to natural isomorphism.

The reverse equivalence

$$(\infty\text{-groupoids}) \rightarrow (\omega\text{-groupoids})$$

has, however, proved difficult to describe directly.<sup>1</sup>

Earlier results which point the way to some of the equivalences described here and in [3] are:

1. the equivalence between 2-categories and double categories with connection described in [10],
2. the equivalences between double groupoids with connections,  $\mathcal{G}$ -groupoids (i.e. group objects in the category of groupoids) and crossed modules established in [4,5], and
3. the equivalence between simplicial Abelian groups and chain complexes proved in [7, 9]<sup>2</sup>.

## 1 $\omega$ -categories and $\infty$ -groupoids.

An  $n$ -fold category is a class  $A$  together with  $n$  mutually compatible category structures  $A^i = (A, d_i^0, d_i^1, +_i)$  ( $0 \leq i \leq n - 1$ ) each with  $A$  as its class of morphisms (and with  $d_i^0, d_i^1$  giving the initial and final identities for  $+_i$ ). The objects of the category structure  $A^i$  are here regarded as members of  $A$ , coinciding with the identity morphisms of  $A^i$ . The compatibility conditions are:

$$(1.1) \quad d_i^\alpha d_j^\beta = d_j^\beta d_i^\alpha \quad \text{for } i \neq j \quad \text{and} \quad \alpha, \beta \in \{0, 1\};$$

$$(1.2) \quad d_i^\alpha(x +_j y) = d_i^\alpha x +_j d_i^\alpha y \quad \text{for } i \neq j \quad \text{and} \quad \alpha = 0, 1,$$

<sup>1</sup>This has been done in the hard category case in F. A. Al-Agl, R. Brown and R. Steiner, "Multiple categories: the equivalence of a globular and a cubical approach", *Advances in Mathematics* 170 (2002) 71-118.

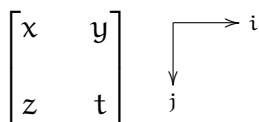
<sup>2</sup>See also R. Brown and P.J.Higgins, 'Cubical abelian groups with connections are equivalent to chain complexes', *Homology, Homotopy and Applications*, 5(1) (2003) 49-52.

whenever  $x, y \in A$  and  $x +_j y$  is defined.

(The interchange law) If  $i \neq j$ , then

$$(1.3) \quad (x +_i y) +_j (z +_i t) = (x +_j z) +_i (y +_j t)$$

whenever  $x, y, z, t \in A$  and both sides are defined. As in [1], we denote the two sides of (1.3) by



The category structure of  $A^i$  on  $A$  is said to be *stronger* than the structure  $A^j$  if every object (identity morphism) of  $A^i$  is also an object of  $A^j$ . An  $n$ -fold category  $A$  is then called an *n-category* if the category structures  $A^0, A^1, \dots, A^{n-1}$  can be arranged in a sequence of increasing (or decreasing) strength. Different authors choose different orders [8, 11]; our exposition will correspond to the order

$$\text{Ob } A^0 \subset \text{Ob } A^1 \subset \dots \subset \text{Ob } A^{n-1}.$$

Adopting this convention, we now define an  $\infty$ -category<sup>3</sup> to be a class  $A$  with mutually compatible category structures  $A^i$  for all integers  $i \geq 0$  satisfying

$$(1.4) \quad \text{Ob } A^i \subset \text{Ob } A^{i+1} \quad \text{for all } i \geq 0$$

The  $\infty$ -categories considered in this paper will also satisfy the extra condition

$$(1.5) \quad A = \bigcup_{i \geq 0} \text{Ob}(A^i).$$

An interesting alternative set of axioms for such  $\infty$ -categories, with a more geometric flavour, will be given in Section 2. However, the axioms given above will be used in the later sections for the proof of the main theorem since they make the algebra simpler.

An  $\infty$ -groupoid  $A$  is an  $\infty$ -category satisfying condition (1.5) in which each category structure  $A^i$  is a groupoid.

Clearly there is a category  $\text{Cat}^\infty$  of  $\infty$ -categories in which a morphism  $f : A \rightarrow B$  is a map preserving all the category structures. The full subcategory of  $\text{Cat}^\infty$  whose objects are  $\infty$ -groupoids is denoted by  $\mathcal{H}$ .

## 2 The Relation of $\infty$ -groupoids to $\omega$ -groupoids.

In this section we explore a direct route from  $\omega$ -groupoids to  $\infty$ -groupoids and use it to reformulate the definitions of  $\omega$ -groupoids and  $\infty$ -categories. This account is intended to

---

<sup>3</sup>The terminology has changed since this paper was written. The general term is now  $\omega$ -category, and one also has to distinguish between cubical and globular theories.

show how  $\infty$ -groupoids fit into the pattern of equivalences established in [1,3]; it will not be needed in later sections.

We recall from [1] that an  $\omega$ -groupoid  $G$  is a cubical set with some extra structures. In particular each  $G$  carries  $n$  groupoid structures with  $G_{n-1}$  as set of objects. The face maps  $\partial_i^0, \partial_i^1 : G_n \rightarrow G_n$  give the initial and final objects for the groupoid  $\oplus_i$ , and the degeneracy map  $\varepsilon_i, \varepsilon_i : G_{n-1} \rightarrow G_n$  embeds  $G_{n-1}$  as the set of identity elements of  $\oplus_i$ . Adopting the conventions of Section 1, we write

$$\eta_i^\alpha = \varepsilon_i \partial_i^\alpha : G_n \rightarrow G_n$$

and

$$\text{Ob}^i(G_n) = \varepsilon_i G_{n-1} = \{x \in G_n \mid \eta_i^\alpha x = x \text{ for } \alpha = 0, 1\}$$

The axioms for  $\omega$ -groupoids now ensure that the groupoid structures

$$(G_n, \eta_i^0, \eta_i^1, \oplus_i), i = 1, 2, \dots, n$$

are mutually compatible. Thus for  $n \geq 0$ ,  $G_n$  carries the structure of  $n$ -fold category (with inverses) and  $\varepsilon_j : G_{n-1} \rightarrow G_n$  embeds  $G_{n-1}$  as  $(n-1)$ -fold subcategory of the  $(n-1)$ -fold category obtained from  $G_n$  by omitting the  $j$ -th category structure.

Now there is an easy procedure for passing from an  $n$ -fold category  $A$  to an  $n$ -category induced on a certain subset  $S$  of  $A$ . Let

$$A^i = (A, d_i^0, d_i^1, +_i), i = 0, 1, \dots, n-1,$$

be the  $n$  category structures on  $A$ . Write

$$B^i = \text{Ob}(A^i) \cap \text{Ob}(A^{i+1}) \cap \dots \cap \text{Ob}(A^{n-1}), 0 \leq i \leq n-1,$$

and define

$$S = \{x \in A \mid d_i^\alpha x \in B^i \text{ for } 0 \leq i \leq n-1, \alpha = 0, 1\}.$$

The compatibility conditions (1.1)-(1.3) imply that each  $B^i$  is an  $n$ -fold subcategory of  $A$  and hence that  $S$  is also an  $n$ -fold subcategory of  $A$ , with category structures  $s^i = \{S, d_i^0, d_i^1, +_i\}$ . But, for  $X \in S$ ,  $d_i^\alpha x \in B^i \cap S$ , so  $\text{Ob}(S^i) \subset B^i \cap S$ ; conversely, if  $y \in B^i \cap S$  then  $y \in B^i \subset \text{Ob}(A^i)$ , so  $d_i^\alpha y = y$ . Thus  $\text{Ob}(S^i) = B^i \cap S$ . Since  $B^0 \subset B^1 \subset \dots \subset B^{n-1}$ , it follows that  $S$  is an  $n$ -category.

Before applying this procedure to the  $n$ -fold category  $G_n$  we renumber the operations to conform with conventions adopted in Section 1. Write

$$+_i \text{ for } \oplus_{n-i} \text{ on } G_n \text{ and } d_i^\alpha \text{ for } \eta_{n-i}^\alpha : G_n \rightarrow G_n.$$

Then  $G_n$  is an  $n$ -fold category with respect to the structures

$$A^i = (G_n, d_i^0, d_i^1, +_i) \quad i = 0, 1, \dots, n-1.$$

Also

$$\begin{aligned} B^i &= \text{Ob}(A^i) \cap \text{Ob}(A^{i+1}) \cap \dots \cap \text{Ob}(A^{n-1}). \\ &= \varepsilon_{n-i} G_{n-1} \cap \dots \cap \varepsilon_1 G_{n-1} \\ &= \varepsilon_1^{n-i} G_i. \end{aligned}$$

We therefore define

$$\begin{aligned} S_n &= \{x \in G_n \mid d_i^\alpha x \in \varepsilon_1^{n-i} G_i \quad \text{for } 0 \leq i \leq n-1, \alpha = 0, 1\} \\ &= \{x \in G_n \mid \partial_j^\alpha x \in \varepsilon_1^{j-1} G_{n-j} \quad \text{for } 1 \leq j \leq n, \alpha = 0, 1\}, \end{aligned}$$

and deduce that, for each  $n \geq 0$ ,  $S_n$  is an  $n$ -fold category with respect to the structures  $(S_n, d_i^0, d_i^1, +_i)$ ,  $0 \leq i \leq n-1$ . These structures are in fact all groupoids. One verifies easily that the family  $(S_n)_{n \geq 0}$  admits all the face operators  $\partial_i^\beta$  of  $G$  and also the first degeneracy operator  $\varepsilon_1$  in each dimension. Since  $\varepsilon_1$  embeds  $G_{n-1}$  in  $G_n$  as  $(n-1)$ -fold subcategory omitting  $\oplus_1$ , it embeds  $S_{n-1}$  in  $S_n$  as  $(n-1)$ -subcategory omitting  $+_{n-1}$ . In other words, it preserves the operations  $+_i$ ,  $0 \leq i \leq n-2$  and its image is the set of identities of  $+_{n-1}$ . It follows that if we define

$$H = \varinjlim (S_0 \xrightarrow{\varepsilon_1} S_1 \xrightarrow{\varepsilon_1} S_2 \hookrightarrow \dots),$$

then the operations  $+_i$  (for fixed  $i$ ) in each dimension combine to give a groupoid structure  $H^i = (H, d_i^0, d_i^1, +_i)$  on  $H$ . Also  $\text{Ob}(H^i)$  is  $H_i$ , the image of  $S_i$  in  $H$ . Thus we have

**Proposition 2.1** *If  $G$  is an  $\omega$ -groupoid, then  $G$  induces on  $H$  the structure of  $\infty$ -groupoid.*

Clearly, the structure on  $H$  can also be described in terms of the family  $S = (S_n)_{n \geq 0}$ . The neatest way to do this is to use the operators

$$\begin{aligned} D_1^\alpha &= (\partial_1^\alpha)^{n-i} = \partial_1^\alpha \partial_2^\alpha \dots \partial_{n-i}^\alpha : G_n \rightarrow G_i, \quad 0 \leq i \leq n-1, \alpha = 0, 1, \\ E_i &= \varepsilon_1^{n-1} : G_i \rightarrow G_n, \quad 0 \leq i \leq n-1. \end{aligned}$$

Since  $S$  admits  $\varepsilon_1$  and all  $\partial_i^\alpha$ , there are induced operators

$$D_i^\alpha : S_n \rightarrow S_i, \quad E_i : S_i \rightarrow S_n, \quad 0 \leq i \leq n-1, \alpha = 0, 1.$$

If  $x \in S_n$ , we have  $\partial_{n-i}^\alpha x = \varepsilon_1^{n-i-1} y$  for some  $y \in G_i$  and this  $y$  is unique, since  $\varepsilon_1$  is an injection. The effect of  $D_i^\alpha$  is to pick out this  $i$ -dimensional (essential face)  $y$  of  $x$ , because

$$D_i^\alpha x = \partial_1^\alpha \partial_2^\alpha \dots \partial_{n-i-1}^\alpha (\partial_{n-i}^\alpha x) = (\partial_1^\alpha)^{n-i-1} (\varepsilon_1^{n-i-1} y) = y.$$

If we pass to  $H = \varinjlim S_n$  the operators  $E_i$  induce the inclusions  $H_i \hookrightarrow H$  and the operators  $D_i^\alpha$  induce the  $d_i^\alpha : H \rightarrow H$ , since for  $x \in S_n$ , we have  $d_i^\alpha x = \varepsilon_1^{n-i} y$ , where  $y = D_i^\alpha x$ .

It is easy now to see that the definition of  $\infty$ -category given in Section 1 (including condition (1.5)) is equivalent to the following. A (small)  $\infty$ -category consists of

**2.2** A sequence  $S = (S_n)_{n \geq 0}$  of sets.

**2.3** Two families of functions<sup>4</sup>

$$\begin{aligned} D_i^\alpha : S_n &\rightarrow S_i, & i &= 0, 1, \dots, n-1, \alpha = 0, 1, \\ E_i : S_i &\rightarrow S_n & i &= 0, 1, \dots, n-1, \end{aligned}$$

satisfying the laws

---

<sup>4</sup>The structure given in 2.2, 2.3 came later to be called a *globular set*.

(i)  $D_i^\alpha D_j^\beta = D_i^\alpha$  for  $i < j$ ,  $\alpha, \beta = 0, 1$ ,

(ii)  $E_j E_i = E_i$  for  $i < j$ ,

(iii)  $D_j^\beta E_i = \begin{cases} D_j^\beta & \text{for } j < i, \\ 1 & \text{for } j = i, \\ E_i & \text{for } j > i. \end{cases}$

**2.4** Category structures  $+_i$  on  $S_n$  ( $0 \leq i \leq n - 1$ ) for each  $n \geq 0$  such that  $+_i$  has  $S_i$  as its set of objects and  $D_i^0, D_i^1, E_i$  as its initial, final and identity maps. These category structures must be compatible, that is:

(i) If  $i > j$  and  $\alpha = 0, 1$ , then

$$D_i^\alpha(x +_j y) = D_i^\alpha x +_j D_i^\alpha y.$$

whenever the left hand side is defined.

(ii)

$$E_i(x +_j y) = E_i x +_j E_i y$$

in  $S_n$  whenever the left hand side is defined.

(iii) (The interchange law) if  $i \neq j$  then

$$(x +_i y) +_j (z +_i t) = (x +_j z) +_i (y +_j t)$$

whenever both sides are defined.

The transition from an  $\infty$ -category  $A$  as defined in Section 1 to one of the above type is made by putting  $S_n = \text{Ob}(A^n)$  and defining  $E_i : S_i \rightarrow (i < n)$  to be the inclusion map and  $D_i^\alpha : S_n \rightarrow S_i$  to be the restriction of  $D_i^\alpha : A \rightarrow A$ .

We note finally that, starting from an  $\omega$ -groupoid  $G$ , the  $\infty$ -groupoid  $S = (S_n)_{n \geq 0}$  described above contains the associated crossed complex  $C = \gamma G$  defined in [1] by the rule

$$C_n = \{x \in G_n \mid D_i^\alpha x \in \varepsilon_1^{n-1} G_0 \text{ for all } (\alpha, i) \neq (0, 1)\}.$$

The equivalence of categories

$$\gamma : (\omega\text{-groupoids}) \rightarrow (\text{crossed complexes})$$

established in [1] therefore factors through  $(\infty\text{-groupoids})$ . We shall show below that the factor

$$\alpha : (\infty\text{-groupoids}) \rightarrow (\text{crossed complexes})$$

is an equivalence, with inverse

$$\beta : (\text{crossed complexes}) \rightarrow (\infty\text{-groupoids}).$$

Hence

$$\zeta = \beta\gamma : (\omega\text{-groupoids}) \rightarrow (\infty\text{-groupoids})$$

is an equivalence. By results in [2], any  $\omega$ -groupoid is the homotopy  $\omega$ -groupoid  $\rho(X_*)$  of a suitable filtered space  $X_*$ . Defining the homotopy  $\infty$ -groupoid of  $X_*$  to be  $\sigma(X_*) = \zeta\rho(X_*)$ ,<sup>5</sup> we deduce that any (small)  $\infty$ -groupoid is of the form  $\sigma(X_*)$  for some  $X_*$ .

### 3 The Crossed Complex Associated with an $\infty$ -groupoid.

Let  $H$  be an  $\infty$ -groupoid in the sense of Section 1. Then  $H$  has groupoid structures  $(H, D_i^0, d_i^1, +_i)$  for  $i \geq 0$  satisfying the compatibility conditions (1.1), (1.2), (1.3) and the conditions

**3.1**

$$H_i \subset H_{i+1} \text{ for } i \geq 0; \quad H = \bigcup_{i \geq 0} H_i$$

where  $H_i = d_i^0 H = d_i^1 H$  is the set of identities (objects) of the  $i$ -th groupoid structure. The conditions (3.1) enable us to define the *dimension* of any  $x \in H$  to be the least integer  $n$  such that  $x \in H_n$ ; we denote this integer by  $\dim x$ . It is convenient to picture an  $n$ -dimensional element  $x$  of  $H$  as having two vertices  $d_0^\alpha x$ , two edges  $d_1^\alpha x$  joining these vertices, two faces  $d_2^\alpha x$  joining the edges, and so on, with  $x$  itself joining the two faces  $d_{n-1}^\alpha x \in H_{n-1}$ . (The actual dimensions of the faces  $d_i^\alpha x$  may of course be smaller than  $i$ .)

Some immediate consequences of the definitions are

**3.2** (i) For each  $x \in H$ ,  $d_j^0 x = d_j^1 x = x$  if  $j \geq \dim x$ .

(ii) If  $i < j$  then  $d_i^\alpha d_j^\beta = d_j^\beta d_i^\alpha = d_i^\alpha$  for  $\alpha, \beta = 0, 1$ .

(iii) If  $i < j$  then  $d_i^\alpha(x +_i y) = d_i^\alpha x = d_i^\alpha y$  for  $\alpha = 0, 1$  whenever  $x +_j y$  is defined.

Here (ii) follows from (i), since  $\dim(d_i^\alpha x) \leq i < j$ , and (iii) follows from (ii) since, for example,

$$d_i^\alpha(x +_j y) = d_i^\alpha d_j^0(x +_j y) = d_i^\alpha d_j^0 x = d_i^\alpha x.$$

We shall show that any  $\infty$ -groupoid  $H$  contains a crossed complex  $C = \alpha H$ , as described in Section 2. First we recall from [1] the axioms for a crossed complex.

A *crossed complex*  $C$  (over a groupoid) consists of a sequence

$$\cdots \rightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \begin{matrix} \xrightarrow{\delta^0} \\ \xrightarrow{\delta^1} \end{matrix} C_0$$

satisfying the following axioms:

**3.3**  $C_1$  is a groupoid over  $C_0$  with  $\delta^0, \delta^1$  as its initial and final maps. We write  $C_1(p, q)$  for the set of arrows from  $p$  to  $q$  ( $p, q \in C_0$ ) and  $C_1(p)$  for the group  $C_1(p, p)$ .

---

<sup>5</sup>A paper is in preparation giving a direct homotopical construction of  $\sigma(X_*)$  in terms of filter homotopy classes rel vertices of filtered maps of  $n$ -globes to  $X_*$ .

**3.4** For  $n \geq 2$ ,  $C_n$  is a family of groups  $\{C_n(p)\}_{p \in C_0}$  and for  $n \geq 3$  the groups  $C_n(p)$  are Abelian.

**3.5** The groupoid  $C_1$  operates on the right on each  $C_n$  ( $n \geq 2$ ) by an action denoted  $(x, a) \mapsto x^a$ . Here, if  $x \in C_n(p)$  and  $a \in C_1(p, q)$  then  $x^a \in C_n(q)$ . (Thus  $C_n(p) \approx C_n(q)$  if  $p$  and  $q$  lie in the same component of the groupoid  $C_1$ .)

We use additive notation for all groups  $C_n(p)$  and for the groupoid  $C_1$  and we use the symbol  $0_p \in C_n(p)$ , or  $0$  for all their identity elements.

**3.6** For  $n \geq 2$ ,  $\delta : C_n \rightarrow C_{n-1}$  is a morphism of groupoids over  $C_0$  and preserves the action of  $C_1$  where  $C_1$  acts on the groups  $C_1(p)$  by conjugation:  $x^a = -a + x + a$ .

**3.7**  $\delta\delta = 0 : C_n \rightarrow C_{n-2}$  for  $n \geq 3$  (and  $\delta^0\delta = \delta^1\delta : C_2 \rightarrow C_0$  as follows from (3.6).)

**3.8** If  $c \in C_2$  then  $\delta c$  operates trivially on  $C_n$  for  $n \geq 3$  and operates on  $C_2$  as conjugation by  $c$ , that is,

$$x^{\delta c} = -c + x + c \quad (x, c \in C_2(p)).$$

The category of crossed complexes is denoted by  $\mathcal{C}$ .

Given an  $\infty$ -groupoid  $H$ , we define  $C = \alpha H$  by

$$C_0 = H_0, \quad C_1 = H_1 \text{ and}$$

$$C_n(p) = \{x \in H_n \mid d_{n-1}^1 x = p\} \quad \text{for } p \in C_0, n \geq 2.$$

It follows from (3.2) (ii) that if  $x \in C_n(p)$  ( $n \geq 2$ ) then

$$\text{for } 0 \leq i \leq n - 2, \quad d_i^\alpha x = d_i^\alpha d_{n-1}^1 x = p.$$

Thus we have the alternative characterisation:

$$C_n(p) = \{x \in H_n \mid d_i^\alpha x = p \quad \text{for } 0 \leq i \leq n - 1, \alpha = 0, 1, (i, \alpha) \neq (n - 1, 0)\}.$$

For  $n \geq 2$ , let  $C_n$  be the family  $\{C_n(p)\}_{p \in C_0}$  and, for  $x \in C_n(p)$ , define  $\delta x = d_{n-1}^0 x$ . Then  $\delta x \in C_{n-1}(p)$  since

$$d_{n-2}^1 d_{n-1}^0 x = d_{n-2}^1 x = p.$$

This defines  $\delta : C_n \rightarrow C_{n-1}$  for  $n \geq 2$  and we define

$$\delta^\alpha : C_1 \rightarrow C_0 \quad \text{by } \delta^\alpha = d_0^\alpha \quad (\alpha = 0, 1).$$

Clearly  $C_1$  is a groupoid over  $C$  with respect to the composition  $+_i$ . Also for each  $p \in C_0$  and  $n \geq 2$ ,  $C_n(p)$  is a group with respect to each of the compositions  $+_i$  for  $i = 0, 1, \dots, n - 2$ , with zero element  $p$ . If  $0 \leq i < j \leq n - 2$  and  $x, y \in C_n(p)$  then the composites

$$\begin{array}{ccc} \begin{bmatrix} x & p \\ p & y \end{bmatrix} & \begin{bmatrix} p & x \\ y & p \end{bmatrix} & \begin{array}{c} \xrightarrow{j} \\ \downarrow i \end{array} \end{array}$$



are defined. Evaluating them by rows and by columns we find that

$$x +_j y = x +_i y = y +_j x.$$

Thus, for  $n \geq 3$ , these group structures in  $C_n(p)$  all coincide and are Abelian. We write  $x + y$  for  $x +_0 y$  whenever this is defined in  $H$ . Then  $C_n(p)$  is a group with respect to  $+$ . By (1.2),  $\delta = d_{n-1}^0 : C_n(p) \rightarrow C_{n-1}(p)$  is a morphism of groups for  $n \geq 2$ . Also  $\delta\delta = 0$  since for  $x \in C_n(p)$  and  $n \geq 3$ ,

$$\delta\delta x = d_{n-2}^0 d_{n-1}^0 x = d_{n-2}^0 x = p.$$

Let  $x \in C_n(p)$ ,  $n \geq 1$  and let  $a \in C_1(p, q)$ . We define

$$x^a = -a + x + a.$$

If  $n \geq 2$ , then

$$\begin{aligned} d_{n-1}^1 x^a &= -d_{n-1}^1 a + d_{n-1}^1 x + d_{n-1}^1 a \quad \text{by (1.2)} \\ &= -a + p + a = q. \end{aligned}$$

If  $n = 1$ , then  $d_0^1 x = d_0^1 a = q$ . Thus, in either case,  $x^a \in C_n(q)$  and we obtain an action of  $C_1$  on  $C_n$ . This action is preserved by  $\delta$  since for  $n \geq 2$ ,

$$\delta(x^a) = -d_{n-1}^0 a + d_{n-1}^0 x + d_{n-1}^0 a = -a + \delta x + a.$$

**Lemma 3.9** *If  $n \geq 2$ ,  $x \in C_n(p)$ ,  $u \in H_n$  and  $d_0^0 = p$ , then*

$$-u + x + u = x^{d_1^0 u}.$$

**Proof** This follows from evaluating in two ways the composite

$$\begin{array}{ccc} \left[ \begin{array}{ccc} d_1^0 u & x & d_1^0 u \\ -u & p & u \end{array} \right] & \begin{array}{l} \xrightarrow{0} \\ \downarrow 1 \end{array} & \end{array}$$

□

From (3.9) we see that if  $x, c \in C_2(p)$ , then  $-c + x + c = x^{\delta c}$ , as required in (3.8). Further, if  $x \in C_n(p)$ ,  $n \geq 3$  and  $c \in C_2(p)$ , then the composite

$$\begin{array}{ccc} \left[ \begin{array}{ccc} -c & p & c \\ p & x & p \end{array} \right] & \begin{array}{l} \xrightarrow{0} \\ \downarrow 1 \end{array} & \end{array}$$

is also defined, giving  $-c + x + c = x$ , so in this case (3.9) implies that  $x^{\delta c} = x$ .

This completes the verification that  $C = \{C_n\}_{n \geq 0}$  is a crossed complex, which we denote by  $\alpha H$ . We observe that this crossed complex is entirely contained in  $H$ , and all its compositions are induced by  $+_0$ , while its boundary maps are induced by the various  $d_i^0$ . The groups  $C_n(q)$ ,  $C_n(p)$  are disjoint if  $p \neq q$ ; the groups  $C_m(p)$ ,  $C_n(p)$  have only their zero element  $p$  in common if  $m \neq n$ .

We now aim to show that  $H$  can be recovered from the crossed complex  $C = \alpha H$  contained in it. The key result for this is

**Proposition 3.10** *Let  $H$  be an  $\infty$ -groupoid with associated crossed complex  $C = \alpha H$ . Let  $n \geq 1$ ,  $x \in H_n$ ,  $d_0^0 x = p$  and  $d_0^1 x = q$ . Then  $x$  can be written uniquely in the form*

$$(*) \quad x = x_n + x_{n-1} + \cdots + x_1, \text{ where } x_1 \in C_1(p, q), x_i \in C_i(p) \text{ for } i \geq 2 \text{ and } + \text{ stands for } +_0.$$

Further,  $x_i$  is given by

$$(**) \quad x_i = d_i^1 x - d_{i-1}^1 x \text{ for } 1 \leq i \leq n.$$

**Proof** If  $(*)$  holds then, for  $1 \leq i \leq n$ ,

$$\begin{aligned} d_i^1 x &= d_i^1 x_n + d_i^1 x_{n-1} + \cdots + d_i^1 x_{i+1} + x_i + x_{i-1} + \cdots + x_1 \\ &= x_i + x_{i-1} + \cdots + x_1 \end{aligned}$$

since  $d_i^1 x_j = p$  for  $i < j$ . The formula for  $x_i$  follows, and this proves uniqueness. For existence, let  $x_i$  be defined by  $(**)$ . Then

$$x_n + x_{n-1} + \cdots + x_i = d_n^1 x - d_0^1 x = x - q = x.$$

Also  $x_i \in H_i$ , and

$$d_{i-1}^1 x_i = d_{i-1}^1 x - d_{i-1}^1 x = d_0^0 d_{i-1}^1 x = p.$$

if  $i \geq 2$ , that is,  $x_i \in C_i(p)$ . Similarly,  $x_1 \in C_1(p, q)$ . □

We now give some basic properties of the decomposition  $(*)$  of Proposition (3.10).

$$3.11 \quad d_i^1 x = x_i + x_{i-1} + \cdots + x_1, \quad -1 \leq i \leq n.$$

$$3.12 \quad d_i^0 x = \delta x_{i+1} + x_i + x_{i-1} + \cdots + x_1 = \delta x_{i+1} + d_i^1 x, \quad i \leq i \leq n - 1.$$

We have already proved (3.11), and (3.12) is similar. □

**3.13** *If  $z = x +_0 y$  is defined in  $H$ , then*

$$z_i = \begin{cases} x_1 + y_1 & \text{if } i = 1 \\ x_i + y_i^{-x_1} & \text{if } i \geq 2. \end{cases}$$

**Proof** Clearly

$$z_1 = d_1^1 z = d_1^1 x + d_1^1 y = x_1 + y_1.$$

If  $i \geq 2$  then

$$\begin{aligned} z_i &= d_i^1(x +_0 y) - d_{i-1}^1(x +_0 y) = d_i^1 x + d_{i-1}^1 y - d_{i-1}^1 y - d_{i-1}^1 x \\ &= x_i + d_{i-1}^1 x + y_i - d_{i-1}^1 x = x_i + y_i^{-v} \end{aligned}$$

by (3.9), where  $v = d_1^0 d_{i-1}^1 x$ . If  $i = 2$ , then

$$v = d_1^0 d_1^1 x = d_1^1 x = x_1$$

If  $i \geq 3$ , then

$$v = d_1^0 x = \delta x_2 + x_1,$$

But  $\delta x_2$  acts trivially on  $C_i$  for  $i \geq 3$ , so the result is true in this case also. □

**3.14** If  $z = x +_j y$  is defined in  $H$ , where  $j \geq 1$  then

$$z_i = \begin{cases} y_i = x_i & \text{if } i < j \\ y_i & \text{if } i = j \\ x_i + y_i & \text{if } i > j. \end{cases}$$

**Proof** First note that  $d_j^1 x = d_j^0 y$  and hence  $d_0^0 x = d_0^0 y = p$ , say. If  $i < j$ , then

$$\begin{aligned} z_i &= d_i^1(x +_j y) - d_{i-1}^1(x +_j y) \\ &= d_i^1 x - d_{i-1}^1 x = d_i^1 y - d_{i-1}^1 y \quad \text{by (3.2)(iii)} \\ &= x_i = y_i. \end{aligned}$$

If  $i = j$ , then  $z = d_i^1 y - d_{i-1}^1 y = y_i$ . If  $i = j + 1$  then

$$\begin{aligned} z_i &= (d_{j+1}^1 x +_j d_{j+1}^1 y) - d_j^1 y \\ &= \begin{bmatrix} d_{j+1}^1 x & -d_j^1 y \\ d_{j+1}^1 y & -d_j^1 y \end{bmatrix} \begin{array}{l} \xrightarrow{0} \\ \downarrow j \end{array} \end{aligned}$$

But

$$\begin{aligned} d_{j+1}^1 x - d_j^1 y &= x_{j+1} + d_j^0 y \\ &= x_{j+1} + d_j^0 y - d_j^1 y \\ &= x_{j+1} + d_j^0 y_{j+1} \end{aligned}$$

and  $d_{j+1}^1 y - d_j^1 y = y_{j+1}$ , so

$$\begin{aligned} z_i &= \begin{bmatrix} x_{j+1} & d_j^0 y_{j+1} \\ p & y_{j+1} \end{bmatrix} \begin{array}{l} \xrightarrow{0} \\ \downarrow j \end{array} \\ &= x_{j+1} + y_{j+1}. \end{aligned}$$

If  $i \geq j + 2$ , then

$$\begin{aligned} z_i &= \begin{bmatrix} d_i^1 x & -d_{i-1}^1 x \\ d_i^1 y & -d_{i-1}^1 y \end{bmatrix} \begin{array}{l} \xrightarrow{0} \\ \downarrow j \end{array} \\ &= x_i +_j y_i = x_i + y_i \end{aligned}$$

□

These results show that the  $\infty$ -groupoid structure of  $H$  can be recovered from the crossed complex structure of  $C = \alpha H$ , a fact which we make more precise in the next section. We observe that all the equations (3.11)-(3.14) and (\*\*\*) of (3.10) remain valid for values of  $i$  and  $j$  greater than the dimensions of  $x, y, z$  if we adopt the convention that, for  $i > \dim x$ ,  $x_i = d_0^0 x$ .

### 4 The Equivalence of Categories.

We have constructed, for any  $\infty$ -groupoid  $H$ , a crossed complex  $\alpha H$ , and this construction clearly gives a functor  $\alpha : \mathcal{H} \rightarrow \mathcal{C}$ . We now construct a functor  $\beta : \mathcal{C} \rightarrow \mathcal{H}$ .

Let  $C$  be an arbitrary crossed complex. We form an  $\infty$ -groupoid  $K = \beta C$  by imitating the formulas (3.10)-(3.14). Let  $K$  be the set of all sequences

$$x = (\dots, x_i, x_{i-1}, \dots, x_1)$$

where  $x_1 \in C_1, x_i \in C_i(\delta_0 x_1)$  and  $x_i = 0$  for all sufficiently large  $i$ . As for polynomials, we shall write

$$x = (x_n, x_{n-1}, \dots, x_1) \quad \text{if } x_i = 0 \text{ for all } i > n.$$

We define maps  $d_i^\alpha : K \rightarrow K$  by

$$\begin{aligned} d_0^0 x &= (\dots, 0_p, 0_p, \dots, 0_p), \quad p = \delta^0 x \\ d_0^1 x &= (\dots, 0_q, 0_q, \dots, 0_q), \quad p = \delta^1 x \\ d_i^1 x &= (x_i, x_{i-1}, \dots, x_1), \quad i \geq 1, \\ d_i^0 x &= (\delta x_{i+1} + x_i, x_{i-1}, x_{i-2}, \dots, x_1), \quad i \geq 1. \end{aligned}$$

It is easy to verify the law (1.1). (The crossed module law  $\delta\delta = 0$  is needed to prove  $d_i^0 d_{i+1}^0 = d_{i+1}^0 d_i^0$ ). Also, writing  $K_i = d_i^1 K = d_i^0 K$  for  $i \geq 0$ , we have

$$K_i \subset K_{i+1} \quad (i \geq 0) \quad \text{and} \quad \bigcup_{i \geq 0} K_i = K.$$

Suppose now that we are given  $x, y \in K$  such that  $d_0^1 x = d_0^0 y$ , that is  $\delta^1 x_1 = \delta^0 y_1$ . We define

$$x +_0 y = (\dots, x_n + y_n^{x_1}, \dots, x_2 + y_2^{-x_1}, x_1 + y_1)$$

which is an element of  $K$ . Similarly, if  $j \geq 1$  and  $d_j^1 x = d_j^0 y$ , that is  $x_i = y_i$ , for  $i < j$  and  $x_j = \delta y_{j+1} + y_j$ , we define

$$x +_j y = (\dots, x_n + y_n, \dots, x_{j+1} + y_{j+1}, y_j, y_{j-1}, \dots, y_1)$$

again an element of  $K$ . In each case it is easy to see that the composition  $+_j$  defines a groupoid structure on  $K$  with  $K_j$  as its set of identities and  $d_j^0 d_j^1$  as its initial and final maps. The law (1.2) follows trivially from these definitions if  $\alpha = 1$  or if  $i < j$ . If  $\alpha = 0$  and  $i > j$ , it reduces immediately to one of the following equations:

$$\begin{aligned} \delta y_{i+1} + x_i &= x_i + \delta y_{i+1}, \quad i > j \geq 2, \\ \delta(y_{i+1}^{-x_1}) + x_i &= x_i + (\delta y_{i+1})^{-x_1}, \quad i \geq 2, \\ \delta(y_2^{-x_1}) + x_1 &= x_1 + \delta y_2. \end{aligned}$$

These are all easy consequences of the laws for a crossed complex. The interchange law (1.3) is proved in a similar way to complete the verification that  $K = \beta C$  is an  $\infty$ -groupoid. The construction is clearly functorial.

**Theorem 4.1** *The functors  $\alpha : \mathcal{H} \rightarrow \mathcal{C}$  and  $\beta : \mathcal{C} \rightarrow \mathcal{H}$  defined above are inverse equivalences.*

**Proof.** Given an  $\infty$ -groupoid  $H$ , the  $\infty$ -groupoid  $K = \beta\alpha H$  is naturally isomorphic to  $H$  by the map

$$(\dots, x_n, x_{n-1}, \dots, x_1) \mapsto \dots + x_n + x_{n-1} + \dots + x_1$$

(the sum on the right being finite since  $x_r = 0$  for large  $r$ ). This is a consequence of Proposition (3.10) and the relations (3.11)–(3.14).

On the other hand, if  $C$  is a crossed complex  $H = \beta C$  and  $D = \alpha\beta C$ , then  $H$  consists of elements  $x = (x_n, x_{n-1}, \dots, x_1)$  and hence

$$D_n = \{x \in H_n \mid d_{n-1}^1 x \in H_0\}$$

consists of elements  $x = (x_n, 0_p, 0_p, \dots, 0_p)$ , where  $x_n \in C_n(p)$ . It is easy to see that the map  $C_n \rightarrow D_n$  defined by

$$c \mapsto (c, \underbrace{0_p, 0_p, \dots, 0_p}_{n-1}), \quad c \in C_n(p)$$

gives a natural isomorphism  $C \rightarrow \alpha\beta C$ . □

## References.

1. R. BROWN & P. J. HIGGINS, On the algebra of cubes, *J. Pure & Appl. Algebra*, 21 (1981), 233-260.
2. R. BROWN & P. J. HIGGINS, Colimit theorems for relative homotopy groups, *J. Pure & Appl. Algebra* 22 (1981), 11-41.
3. R. BROWN & P. J. HIGGINS, The equivalence of  $\omega$ -groupoids and cubical T-complexes, *Cah. Top. Géom. Diff.* 22 (1981), 349–370.
4. R. BROWN & C.B. SPENCER,  $\mathcal{G}$ -groupoids, crossed modules and the fundamental groupoid of a topological group, *Proc. Kon. Akad. v. Wet.* 79 (1976), 296–302.
5. R. BROWN & C.B. SPENCER, Double groupoids and crossed modules *Cahiers Top. et Géom. Diff.* XVII-4 (1976), 343–362.
6. M. K. DAKIN, Kan complexes and multiple groupoid structures, Ph.D. Thesis University of Wales, 1977.
7. A. DOLD, Homology of symmetric products and other functors of complexes, *Ann. of Math.* 68 (1958), 54–80.
8. A. & C. EHRESMANN, Multiple functors II, III, *Cahiers Top. et Géom. Diff.* XX (1978), 295–333, 387–443.

9. D.M.KAN, Functors involving c.s.s. complexes, *Trans. Amer. Math. Soc.* 87 (1958), 330–346.
10. C. B. SPENCER, An abstract setting for homotopy pushouts and pullbacks *Cahiers Top. et Géom. Diff.* XVIII-4 (1977), 409–430.
11. O. WYLER, *Multiple functar categories*, Mimeographed Notes, Carnegie-Mellon University, 1972.

R. BROWN:

School of Mathematics and Computer Science  
University College of N. Wales  
BANGOR LL57 2UW, U.K.

P.J. HIGGINS:

Department of Mathematics  
Science Laboratories  
South Road DURHAM DH1 3LE. U.K.