

The Equivalence of ∞ -groupoids and Crossed Complexes*

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Introduction

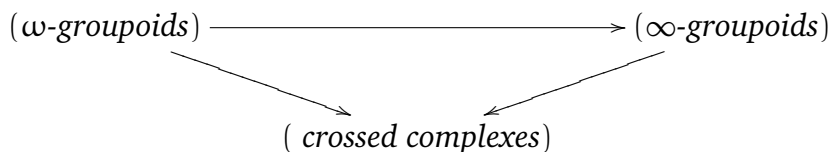
Multiple categories, and in particular n -fold categories, have been considered by various authors [8,11]. The object of this paper is to define ∞ -categories (which are n -categories for all n) and ∞ -groupoids and to prove the equivalence of categories

$$(\infty\text{-groupoids}) \leftrightarrow (\text{crossed complexes.})$$

This result was stated (without definitions) in the previous paper [3] and there placed in a general pattern of equivalences between categories, each of which can be viewed as a higher-dimensional version of the category of groups.

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The interest of the above equivalence arises from the common use of n -categories, particularly in situations describing homotopies, homotopies of homotopies, etc..., and also from the fact that ∞ -groupoids can be regarded as a kind of half-way house between ω -groupoids and crossed complexes. It is easy to construct from any ω -groupoid a subset which has the structure of an ∞ -groupoid and contains the associated crossed complex. In this way we get a diagram of functors



which is commutative up to natural isomorphism.

The reverse equivalence

$$(\infty\text{-groupoids}) \rightarrow (\omega\text{-groupoids})$$

has, however, proved difficult to describe directly.¹

Earlier results which point the way to some of the equivalences described here and in [3] are:

1. the equivalence between 2-categories and double categories with connection described in [10],
2. the equivalences between double groupoids with connections, \mathcal{G} -groupoids (i.e. group objects in the category of groupoids) and crossed modules established in [4,5], and
3. the equivalence between simplicial Abelian groups and chain complexes proved in [7, 9]².

1 ω -categories and ∞ -groupoids.

An n -fold category is a class A together with n mutually compatible category structures $A^i = (A, d_i^0, d_i^1, +_i)$ ($0 \leq i \leq n - 1$) each with A as its class of morphisms (and with d_i^0, d_i^1 giving the initial and final identities for $+_i$). The objects of the category structure A^i are here regarded as members of A , coinciding with the identity morphisms of A^i . The compatibility conditions are:

$$(1.1) \quad d_i^\alpha d_j^\beta = d_j^\beta d_i^\alpha \quad \text{for } i \neq j \quad \text{and} \quad \alpha, \beta \in \{0, 1\};$$

$$(1.2) \quad d_i^\alpha(x +_j y) = d_i^\alpha x +_j d_i^\alpha y \quad \text{for } i \neq j \quad \text{and} \quad \alpha = 0, 1,$$

¹This has been done in the hard category case in F. A. Al-Agl, R. Brown and R. Steiner, "Multiple categories: the equivalence of a globular and a cubical approach", *Advances in Mathematics* 170 (2002) 71-118.

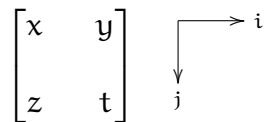
²See also R. Brown and P.J.Higgins, 'Cubical abelian groups with connections are equivalent to chain complexes', *Homology, Homotopy and Applications*, 5(1) (2003) 49-52.

whenever $x, y \in A$ and $x +_j y$ is defined.

(The interchange law) If $i \neq j$, then

$$(1.3) \quad (x +_i y) +_j (z +_i t) = (x +_j z) +_i (y +_j t)$$

whenever $x, y, z, t \in A$ and both sides are defined. As in [1], we denote the two sides of (1.3) by



The category structure of A^i on A is said to be *stronger* than the structure A^j if every object (identity morphism) of A^i is also an object of A^j . An n -fold category A is then called an *n-category* if the category structures A^0, A^1, \dots, A^{n-1} can be arranged in a sequence of increasing (or decreasing) strength. Different authors choose different orders [8, 11]; our exposition will correspond to the order

$$\text{Ob } A^0 \subset \text{Ob } A^1 \subset \dots \subset \text{Ob } A^{n-1}.$$

Adopting this convention, we now define an ∞ -category³ to be a class A with mutually compatible category structures A^i for all integers $i \geq 0$ satisfying

$$(1.4) \quad \text{Ob } A^i \subset \text{Ob } A^{i+1} \quad \text{for all } i \geq 0$$

The ∞ -categories considered in this paper will also satisfy the extra condition

$$(1.5) \quad A = \bigcup_{i \geq 0} \text{Ob}(A^i).$$

An interesting alternative set of axioms for such ∞ -categories, with a more geometric flavour, will be given in Section 2. However, the axioms given above will be used in the later sections for the proof of the main theorem since they make the algebra simpler.

An ∞ -groupoid A is an ∞ -category satisfying condition (1.5) in which each category structure A^i is a groupoid.

Clearly there is a category Cat^∞ of ∞ -categories in which a morphism $f : A \rightarrow B$ is a map preserving all the category structures. The full subcategory of Cat^∞ whose objects are ∞ -groupoids is denoted by \mathcal{H} .

2 The Relation of ∞ -groupoids to ω -groupoids.

In this section we explore a direct route from ω -groupoids to ∞ -groupoids and use it to reformulate the definitions of ω -groupoids and ∞ -categories. This account is intended to

³The terminology has changed since this paper was written. The general term is now ω -category, and one also has to distinguish between cubical and globular theories.

show how ∞ -groupoids fit into the pattern of equivalences established in [1,3]; it will not be needed in later sections.

We recall from [1] that an ω -groupoid G is a cubical set with some extra structures. In particular each G carries n groupoid structures with G_{n-1} as set of objects. The face maps $\partial_i^0, \partial_i^1 : G_n \rightarrow G_n$ give the initial and final objects for the groupoid \oplus_i , and the degeneracy map $\varepsilon_i, \varepsilon_i : G_{n-1} \rightarrow G_n$ embeds G_{n-1} as the set of identity elements of \oplus_i . Adopting the conventions of Section 1, we write

$$\eta_i^\alpha = \varepsilon_i \partial_i^\alpha : G_n \rightarrow G_n$$

and

$$\text{Ob}^i(G_n) = \varepsilon_i G_{n-1} = \{x \in G_n \mid \eta_i^\alpha x = x \text{ for } \alpha = 0, 1\}$$

The axioms for ω -groupoids now ensure that the groupoid structures

$$(G_n, \eta_i^0, \eta_i^1, \oplus_i), i = 1, 2, \dots, n$$

are mutually compatible. Thus for $n \geq 0$, G_n carries the structure of n -fold category (with inverses) and $\varepsilon_j : G_{n-1} \rightarrow G_n$ embeds G_{n-1} as $(n-1)$ -fold subcategory of the $(n-1)$ -fold category obtained from G_n by omitting the j -th category structure.

Now there is an easy procedure for passing from an n -fold category A to an n -category induced on a certain subset S of A . Let

$$A^i = (A, d_i^0, d_i^1, +_i), i = 0, 1, \dots, n-1,$$

be the n category structures on A . Write

$$B^i = \text{Ob}(A^i) \cap \text{Ob}(A^{i+1}) \cap \dots \cap \text{Ob}(A^{n-1}), 0 \leq i \leq n-1,$$

and define

$$S = \{x \in A \mid d_i^\alpha x \in B^i \text{ for } 0 \leq i \leq n-1, \alpha = 0, 1\}.$$

The compatibility conditions (1.1)-(1.3) imply that each B^i is an n -fold subcategory of A and hence that S is also an n -fold subcategory of A , with category structures $s^i = \{S, d_i^0, d_i^1, +_i\}$. But, for $X \in S$, $d_i^\alpha x \in B^i \cap S$, so $\text{Ob}(S^i) \subset B^i \cap S$; conversely, if $y \in B^i \cap S$ then $y \in B^i \subset \text{Ob}(A^i)$, so $d_i^\alpha y = y$. Thus $\text{Ob}(S^i) = B^i \cap S$. Since $B^0 \subset B^1 \subset \dots \subset B^{n-1}$, it follows that S is an n -category.

Before applying this procedure to the n -fold category G_n we renumber the operations to conform with conventions adopted in Section 1. Write

$$+_i \text{ for } \oplus_{n-i} \text{ on } G_n \text{ and } d_i^\alpha \text{ for } \eta_{n-i}^\alpha : G_n \rightarrow G_n.$$

Then G_n is an n -fold category with respect to the structures

$$A^i = (G_n, d_i^0, d_i^1, +_i) \quad i = 0, 1, \dots, n-1.$$

Also

$$\begin{aligned} B^i &= \text{Ob}(A^i) \cap \text{Ob}(A^{i+1}) \cap \dots \cap \text{Ob}(A^{n-1}). \\ &= \varepsilon_{n-i} G_{n-1} \cap \dots \cap \varepsilon_1 G_{n-1} \\ &= \varepsilon_1^{n-i} G_i. \end{aligned}$$

We therefore define

$$\begin{aligned} S_n &= \{x \in G_n \mid d_i^\alpha x \in \varepsilon_1^{n-i} G_i \quad \text{for } 0 \leq i \leq n-1, \alpha = 0, 1\} \\ &= \{x \in G_n \mid \partial_j^\alpha x \in \varepsilon_1^{j-1} G_{n-j} \quad \text{for } 1 \leq j \leq n, \alpha = 0, 1\}, \end{aligned}$$

and deduce that, for each $n \geq 0$, S_n is an n -fold category with respect to the structures $(S_n, d_i^0, d_i^1, +_i)$, $0 \leq i \leq n-1$. These structures are in fact all groupoids. One verifies easily that the family $(S_n)_{n \geq 0}$ admits all the face operators ∂_i^β of G and also the first degeneracy operator ε_1 in each dimension. Since ε_1 embeds G_{n-1} in G_n as $(n-1)$ -fold subcategory omitting \oplus_1 , it embeds S_{n-1} in S_n as $(n-1)$ -subcategory omitting $+_{n-1}$. In other words, it preserves the operations $+_i$, $0 \leq i \leq n-2$ and its image is the set of identities of $+_{n-1}$. It follows that if we define

$$H = \varinjlim (S_0 \xrightarrow{\varepsilon_1} S_1 \xrightarrow{\varepsilon_1} S_2 \hookrightarrow \dots),$$

then the operations $+_i$ (for fixed i) in each dimension combine to give a groupoid structure $H^i = (H, d_i^0, d_i^1, +_i)$ on H . Also $\text{Ob}(H^i)$ is H_i , the image of S_i in H . Thus we have

Proposition 2.1 *If G is an ω -groupoid, then G induces on H the structure of ∞ -groupoid.*

Clearly, the structure on H can also be described in terms of the family $S = (S_n)_{n \geq 0}$. The neatest way to do this is to use the operators

$$\begin{aligned} D_1^\alpha &= (\partial_1^\alpha)^{n-i} = \partial_1^\alpha \partial_2^\alpha \dots \partial_{n-i}^\alpha : G_n \rightarrow G_i, \quad 0 \leq i \leq n-1, \alpha = 0, 1, \\ E_i &= \varepsilon_1^{n-1} : G_i \rightarrow G_n, \quad 0 \leq i \leq n-1. \end{aligned}$$

Since S admits ε_1 and all ∂_i^α , there are induced operators

$$D_i^\alpha : S_n \rightarrow S_i, \quad E_i : S_i \rightarrow S_n, \quad 0 \leq i \leq n-1, \alpha = 0, 1.$$

If $x \in S_n$, we have $\partial_{n-i}^\alpha x = \varepsilon_1^{n-i-1} y$ for some $y \in G_i$ and this y is unique, since ε_1 is an injection. The effect of D_i^α is to pick out this i -dimensional (essential face) y of x , because

$$D_i^\alpha x = \partial_1^\alpha \partial_2^\alpha \dots \partial_{n-i-1}^\alpha (\partial_{n-i}^\alpha x) = (\partial_1^\alpha)^{n-i-1} (\varepsilon_1^{n-i-1} y) = y.$$

If we pass to $H = \varinjlim S_n$ the operators E_i induce the inclusions $H_i \hookrightarrow H$ and the operators D_i^α induce the $d_i^\alpha : H \rightarrow H$, since for $x \in S_n$, we have $d_i^\alpha x = \varepsilon_1^{n-i} y$, where $y = D_i^\alpha x$.

It is easy now to see that the definition of ∞ -category given in Section 1 (including condition (1.5)) is equivalent to the following. A (small) ∞ -category consists of

2.2 A sequence $S = (S_n)_{n \geq 0}$ of sets.

2.3 Two families of functions⁴

$$\begin{aligned} D_i^\alpha : S_n &\rightarrow S_i, & i &= 0, 1, \dots, n-1, \alpha = 0, 1, \\ E_i : S_i &\rightarrow S_n & i &= 0, 1, \dots, n-1, \end{aligned}$$

satisfying the laws

⁴The structure given in 2.2, 2.3 came later to be called a *globular set*.

- (i) $D_i^\alpha D_j^\beta = D_i^\alpha$ for $i < j$, $\alpha, \beta = 0, 1$,
- (ii) $E_j E_i = E_i$ for $i < j$,
- (iii) $D_j^\beta E_i = \begin{cases} D_j^\beta & \text{for } j < i, \\ 1 & \text{for } j = i, \\ E_i & \text{for } j > i. \end{cases}$

2.4 Category structures $+_i$ on S_n ($0 \leq i \leq n - 1$) for each $n \geq 0$ such that $+_i$ has S_i as its set of objects and D_i^0, D_i^1, E_i as its initial, final and identity maps. These category structures must be compatible, that is:

- (i) If $i > j$ and $\alpha = 0, 1$, then

$$D_i^\alpha(x +_j y) = D_i^\alpha x +_j D_i^\alpha y.$$

whenever the left hand side is defined.

- (ii)

$$E_i(x +_j y) = E_i x +_j E_i y$$

in S_n whenever the left hand side is defined.

- (iii) (The interchange law) if $i \neq j$ then

$$(x +_i y) +_j (z +_i t) = (x +_j z) +_i (y +_j t)$$

whenever both sides are defined.

The transition from an ∞ -category A as defined in Section 1 to one of the above type is made by putting $S_n = \text{Ob}(A^n)$ and defining $E_i : S_i \rightarrow (i < n)$ to be the inclusion map and $D_i^\alpha : S_n \rightarrow S_i$ to be the restriction of $D_i^\alpha : A \rightarrow A$.

We note finally that, starting from an ω -groupoid G , the ∞ -groupoid $S = (S_n)_{n \geq 0}$ described above contains the associated crossed complex $C = \gamma G$ defined in [1] by the rule

$$C_n = \{x \in G_n \mid D_i^\alpha x \in \varepsilon_1^{n-1} G_0 \text{ for all } (\alpha, i) \neq (0, 1)\}.$$

The equivalence of categories

$$\gamma : (\omega\text{-groupoids}) \rightarrow (\text{crossed complexes})$$

established in [1] therefore factors through $(\infty\text{-groupoids})$. We shall show below that the factor

$$\alpha : (\infty\text{-groupoids}) \rightarrow (\text{crossed complexes})$$

is an equivalence, with inverse

$$\beta : (\text{crossed complexes}) \rightarrow (\infty\text{-groupoids}).$$

Hence

$$\zeta = \beta\gamma : (\omega\text{-groupoids}) \rightarrow (\infty\text{-groupoids})$$

is an equivalence. By results in [2], any ω -groupoid is the homotopy ω -groupoid $\rho(X_*)$ of a suitable filtered space X_* . Defining the homotopy ∞ -groupoid of X_* to be $\sigma(X_*) = \zeta\rho(X_*)$,⁵ we deduce that any (small) ∞ -groupoid is of the form $\sigma(X_*)$ for some X_* .

3 The Crossed Complex Associated with an ∞ -groupoid.

Let H be an ∞ -groupoid in the sense of Section 1. Then H has groupoid structures $(H, D_i^0, d_i^1, +_i)$ for $i \geq 0$ satisfying the compatibility conditions (1.1), (1.2), (1.3) and the conditions

3.1

$$H_i \subset H_{i+1} \text{ for } i \geq 0; \quad H = \bigcup_{i \geq 0} H_i$$

where $H_i = d_i^0 H = d_i^1 H$ is the set of identities (objects) of the i -th groupoid structure. The conditions (3.1) enable us to define the *dimension* of any $x \in H$ to be the least integer n such that $x \in H_n$; we denote this integer by $\dim x$. It is convenient to picture an n -dimensional element x of H as having two vertices $d_0^\alpha x$, two edges $d_1^\alpha x$ joining these vertices, two faces $d_2^\alpha x$ joining the edges, and so on, with x itself joining the two faces $d_{n-1}^\alpha x \in H_{n-1}$. (The actual dimensions of the faces $d_i^\alpha x$ may of course be smaller than i .)

Some immediate consequences of the definitions are

3.2 (i) For each $x \in H$, $d_j^0 x = d_j^1 x = x$ if $j \geq \dim x$.

(ii) If $i < j$ then $d_i^\alpha d_j^\beta = d_j^\beta d_i^\alpha = d_i^\alpha$ for $\alpha, \beta = 0, 1$.

(iii) If $i < j$ then $d_i^\alpha(x +_i y) = d_i^\alpha x = d_i^\alpha y$ for $\alpha = 0, 1$ whenever $x +_j y$ is defined.

Here (ii) follows from (i), since $\dim(d_i^\alpha x) \leq i < j$, and (iii) follows from (ii) since, for example,

$$d_i^\alpha(x +_j y) = d_i^\alpha d_j^0(x +_j y) = d_i^\alpha d_j^0 x = d_i^\alpha x.$$

We shall show that any ∞ -groupoid H contains a crossed complex $C = \alpha H$, as described in Section 2. First we recall from [1] the axioms for a crossed complex.

A *crossed complex* C (over a groupoid) consists of a sequence

$$\cdots \rightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \xrightleftharpoons[\delta^1]{\delta^0} C_0$$

satisfying the following axioms:

3.3 C_1 is a groupoid over C_0 with δ^0, δ^1 as its initial and final maps. We write $C_1(p, q)$ for the set of arrows from p to q ($p, q \in C_0$) and $C_1(p)$ for the group $C_1(p, p)$.

⁵A paper is in preparation giving a direct homotopical construction of $\sigma(X_*)$ in terms of filter homotopy classes rel vertices of filtered maps of n -globes to X_* .

3.4 For $n \geq 2$, C_n is a family of groups $\{C_n(p)\}_{p \in C_0}$ and for $n \geq 3$ the groups $C_n(p)$ are Abelian.

3.5 The groupoid C_1 operates on the right on each C_n ($n \geq 2$) by an action denoted $(x, a) \mapsto x^a$. Here, if $x \in C_n(p)$ and $a \in C_1(p, q)$ then $x^a \in C_n(q)$. (Thus $C_n(p) \approx C_n(q)$ if p and q lie in the same component of the groupoid C_1 .)

We use additive notation for all groups $C_n(p)$ and for the groupoid C_1 and we use the symbol $0_p \in C_n(p)$, or 0 for all their identity elements.

3.6 For $n \geq 2$, $\delta : C_n \rightarrow C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1 where C_1 acts on the groups $C_1(p)$ by conjugation: $x^a = -a + x + a$.

3.7 $\delta\delta = 0 : C_n \rightarrow C_{n-2}$ for $n \geq 3$ (and $\delta^0\delta = \delta^1\delta : C_2 \rightarrow C_0$ as follows from (3.6).)

3.8 If $c \in C_2$ then δc operates trivially on C_n for $n \geq 3$ and operates on C_2 as conjugation by c , that is,

$$x^{\delta c} = -c + x + c \quad (x, c \in C_2(p)).$$

The category of crossed complexes is denoted by \mathcal{C} .

Given an ∞ -groupoid H , we define $C = \alpha H$ by

$$C_0 = H_0, \quad C_1 = H_1 \text{ and}$$

$$C_n(p) = \{x \in H_n \mid d_{n-1}^1 x = p\} \quad \text{for } p \in C_0, n \geq 2.$$

It follows from (3.2) (ii) that if $x \in C_n(p)$ ($n \geq 2$) then

$$\text{for } 0 \leq i \leq n-2, \quad d_i^\alpha x = d_i^\alpha d_{n-1}^1 x = p.$$

Thus we have the alternative characterisation:

$$C_n(p) = \{x \in H_n \mid d_i^\alpha x = p \quad \text{for } 0 \leq i \leq n-1, \alpha = 0, 1, (i, \alpha) \neq (n-1, 0)\}.$$

For $n \geq 2$, let C_n be the family $\{C_n(p)\}_{p \in C_0}$ and, for $x \in C_n(p)$, define $\delta x = d_{n-1}^0 x$. Then $\delta x \in C_{n-1}(p)$ since

$$d_{n-2}^1 d_{n-1}^0 x = d_{n-2}^1 x = p.$$

This defines $\delta : C_n \rightarrow C_{n-1}$ for $n \geq 2$ and we define

$$\delta^\alpha : C_1 \rightarrow C_0 \quad \text{by } \delta^\alpha = d_0^\alpha \quad (\alpha = 0, 1).$$

Clearly C_1 is a groupoid over C with respect to the composition $+_i$. Also for each $p \in C_0$ and $n \geq 2$, $C_n(p)$ is a group with respect to each of the compositions $+_i$ for $i = 0, 1, \dots, n-2$, with zero element p . If $0 \leq i < j \leq n-2$ and $x, y \in C_n(p)$ then the composites

$$\begin{array}{ccc} \begin{bmatrix} x & p \\ p & y \end{bmatrix} & \begin{bmatrix} p & x \\ y & p \end{bmatrix} & \begin{array}{c} \xrightarrow{j} \\ \downarrow i \end{array} \end{array}$$

are defined. Evaluating them by rows and by columns we find that

$$x +_j y = x +_i y = y +_j x.$$

Thus, for $n \geq 3$, these group structures in $C_n(p)$ all coincide and are Abelian. We write $x + y$ for $x +_0 y$ whenever this is defined in H . Then $C_n(p)$ is a group with respect to $+$. By (1.2), $\delta = d_{n-1}^0 : C_n(p) \rightarrow C_{n-1}(p)$ is a morphism of groups for $n \geq 2$. Also $\delta\delta = 0$ since for $x \in C_n(p)$ and $n \geq 3$,

$$\delta\delta x = d_{n-2}^0 d_{n-1}^0 x = d_{n-2}^0 x = p.$$

Let $x \in C_n(p)$, $n \geq 1$ and let $a \in C_1(p, q)$. We define

$$x^a = -a + x + a.$$

If $n \geq 2$, then

$$\begin{aligned} d_{n-1}^1 x^a &= -d_{n-1}^1 a + d_{n-1}^1 x + d_{n-1}^1 a \quad \text{by (1.2)} \\ &= -a + p + a = q. \end{aligned}$$

If $n = 1$, then $d_0^1 x = d_0^1 a = q$. Thus, in either case, $x^a \in C_n(q)$ and we obtain an action of C_1 on C_n . This action is preserved by δ since for $n \geq 2$,

$$\delta(x^a) = -d_{n-1}^0 a + d_{n-1}^0 x + d_{n-1}^0 a = -a + \delta x + a.$$

Lemma 3.9 *If $n \geq 2$, $x \in C_n(p)$, $u \in H_n$ and $d_0^0 = p$, then*

$$-u + x + u = x^{d_1^0 u}.$$

Proof This follows from evaluating in two ways the composite

$$\begin{array}{ccc} \left[\begin{array}{ccc} d_1^0 u & x & d_1^0 u \\ -u & p & u \end{array} \right] & \begin{array}{c} \xrightarrow{0} \\ \downarrow 1 \end{array} & \end{array}$$

□

From (3.9) we see that if $x, c \in C_2(p)$, then $-c + x + c = x^{\delta c}$, as required in (3.8). Further, if $x \in C_n(p)$, $n \geq 3$ and $c \in C_2(p)$, then the composite

$$\begin{array}{ccc} \left[\begin{array}{ccc} -c & p & c \\ p & x & p \end{array} \right] & \begin{array}{c} \xrightarrow{0} \\ \downarrow 1 \end{array} & \end{array}$$

is also defined, giving $-c + x + c = x$, so in this case (3.9) implies that $x^{\delta c} = x$.

This completes the verification that $C = \{C_n\}_{n \geq 0}$ is a crossed complex, which we denote by αH . We observe that this crossed complex is entirely contained in H , and all its compositions are induced by $+_0$, while its boundary maps are induced by the various d_i^0 . The groups $C_n(q)$, $C_n(p)$ are disjoint if $p \neq q$; the groups $C_m(p)$, $C_n(p)$ have only their zero element p in common if $m \neq n$.

We now aim to show that H can be recovered from the crossed complex $C = \alpha H$ contained in it. The key result for this is

Proposition 3.10 *Let H be an ∞ -groupoid with associated crossed complex $C = \alpha H$. Let $n \geq 1$, $x \in H_n$, $d_0^0 x = p$ and $d_0^1 x = q$. Then x can be written uniquely in the form*

$$(*) \quad x = x_n + x_{n-1} + \cdots + x_1, \text{ where } x_1 \in C_1(p, q), x_i \in C_i(p) \text{ for } i \geq 2 \text{ and } + \text{ stands for } +_0.$$

Further, x_i is given by

$$(**) \quad x_i = d_i^1 x - d_{i-1}^1 x \text{ for } 1 \leq i \leq n.$$

Proof If $(*)$ holds then, for $1 \leq i \leq n$,

$$\begin{aligned} d_i^1 x &= d_i^1 x_n + d_i^1 x_{n-1} + \cdots + d_i^1 x_{i+1} + x_i + x_{i-1} + \cdots + x_1 \\ &= x_i + x_{i-1} + \cdots + x_1 \end{aligned}$$

since $d_i^1 x_j = p$ for $i < j$. The formula for x_i follows, and this proves uniqueness. For existence, let x_i be defined by $(**)$. Then

$$x_n + x_{n-1} + \cdots + x_i = d_n^1 x - d_0^1 x = x - q = x.$$

Also $x_i \in H_i$, and

$$d_{i-1}^1 x_i = d_{i-1}^1 x - d_{i-1}^1 x = d_0^0 d_{i-1}^1 x = p.$$

if $i \geq 2$, that is, $x_i \in C_i(p)$. Similarly, $x_1 \in C_1(p, q)$. □

We now give some basic properties of the decomposition $(*)$ of Proposition (3.10).

3.11 $d_i^1 x = x_i + x_{i-1} + \cdots + x_1, \quad -1 \leq i \leq n.$

3.12 $d_i^0 x = \delta x_{i+1} + x_i + x_{i-1} + \cdots + x_1 = \delta x_{i+1} + d_i^1 x, \quad i \leq i \leq n - 1.$

We have already proved (3.11), and (3.12) is similar. □

3.13 *If $z = x +_0 y$ is defined in H , then*

$$z_i = \begin{cases} x_1 + y_1 & \text{if } i = 1 \\ x_i + y_i^{-x_1} & \text{if } i \geq 2. \end{cases}$$

Proof Clearly

$$z_1 = d_1^1 z = d_1^1 x + d_1^1 y = x_1 + y_1.$$

If $i \geq 2$ then

$$\begin{aligned} z_i &= d_i^1(x +_0 y) - d_{i-1}^1(x +_0 y) = d_i^1 x + d_{i-1}^1 y - d_{i-1}^1 y - d_{i-1}^1 x \\ &= x_i + d_{i-1}^1 x + y_i - d_{i-1}^1 x = x_i + y_i^{-v} \end{aligned}$$

by (3.9), where $v = d_1^0 d_{i-1}^1 x$. If $i = 2$, then

$$v = d_1^0 d_1^1 x = d_1^1 x = x_1$$

If $i \geq 3$, then

$$v = d_1^0 x = \delta x_2 + x_1,$$

But δx_2 acts trivially on C_i for $i \geq 3$, so the result is true in this case also. □

3.14 If $z = x +_j y$ is defined in H , where $j \geq 1$ then

$$z_i = \begin{cases} y_i = x_i & \text{if } i < j \\ y_i & \text{if } i = j \\ x_i + y_i & \text{if } i > j. \end{cases}$$

Proof First note that $d_j^1 x = d_j^0 y$ and hence $d_0^0 x = d_0^0 y = p$, say. If $i < j$, then

$$\begin{aligned} z_i &= d_i^1(x +_j y) - d_{i-1}^1(x +_j y) \\ &= d_i^1 x - d_{i-1}^1 x = d_i^1 y - d_{i-1}^1 y \quad \text{by (3.2)(iii)} \\ &= x_i = y_i. \end{aligned}$$

If $i = j$, then $z = d_i^1 y - d_{i-1}^1 y = y_i$. If $i = j + 1$ then

$$\begin{aligned} z_i &= (d_{j+1}^1 x +_j d_{j+1}^1 y) - d_j^1 y \\ &= \begin{bmatrix} d_{j+1}^1 x & -d_j^1 y \\ d_{j+1}^1 y & -d_j^1 y \end{bmatrix} \begin{array}{l} \xrightarrow{0} \\ \downarrow j \end{array} \end{aligned}$$

But

$$\begin{aligned} d_{j+1}^1 x - d_j^1 y &= x_{j+1} + d_j^0 y \\ &= x_{j+1} + d_j^0 y - d_j^1 y \\ &= x_{j+1} + d_j^0 y_{j+1} \end{aligned}$$

and $d_{j+1}^1 y - d_j^1 y = y_{j+1}$, so

$$\begin{aligned} z_i &= \begin{bmatrix} x_{j+1} & d_j^0 y_{j+1} \\ p & y_{j+1} \end{bmatrix} \begin{array}{l} \xrightarrow{0} \\ \downarrow j \end{array} \\ &= x_{j+1} + y_{j+1}. \end{aligned}$$

If $i \geq j + 2$, then

$$\begin{aligned} z_i &= \begin{bmatrix} d_i^1 x & -d_{i-1}^1 x \\ d_i^1 y & -d_{i-1}^1 y \end{bmatrix} \begin{array}{l} \xrightarrow{0} \\ \downarrow j \end{array} \\ &= x_i +_j y_i = x_i + y_i \end{aligned}$$

□

These results show that the ∞ -groupoid structure of H can be recovered from the crossed complex structure of $C = \alpha H$, a fact which we make more precise in the next section. We observe that all the equations (3.11)-(3.14) and (***) of (3.10) remain valid for values of i and j greater than the dimensions of x, y, z if we adopt the convention that, for $i > \dim x$, $x_i = d_0^0 x$.

4 The Equivalence of Categories.

We have constructed, for any ∞ -groupoid H , a crossed complex αH , and this construction clearly gives a functor $\alpha : \mathcal{H} \rightarrow \mathcal{C}$. We now construct a functor $\beta : \mathcal{C} \rightarrow \mathcal{H}$.

Let C be an arbitrary crossed complex. We form an ∞ -groupoid $K = \beta C$ by imitating the formulas (3.10)-(3.14). Let K be the set of all sequences

$$x = (\dots, x_i, x_{i-1}, \dots, x_1)$$

where $x_1 \in C_1, x_i \in C_i(\delta_0 x_1)$ and $x_i = 0$ for all sufficiently large i . As for polynomials, we shall write

$$x = (x_n, x_{n-1}, \dots, x_1) \quad \text{if } x_i = 0 \text{ for all } i > n.$$

We define maps $d_i^\alpha : K \rightarrow K$ by

$$\begin{aligned} d_0^0 x &= (\dots, 0_p, 0_p, \dots, 0_p), \quad p = \delta^0 x \\ d_0^1 x &= (\dots, 0_q, 0_q, \dots, 0_q), \quad p = \delta^1 x \\ d_i^1 x &= (x_i, x_{i-1}, \dots, x_1), \quad i \geq 1, \\ d_i^0 x &= (\delta x_{i+1} + x_i, x_{i-1}, x_{i-2}, \dots, x_1), \quad i \geq 1. \end{aligned}$$

It is easy to verify the law (1.1). (The crossed module law $\delta\delta = 0$ is needed to prove $d_i^0 d_{i+1}^0 = d_{i+1}^0 d_i^0$). Also, writing $K_i = d_i^1 K = d_i^0 K$ for $i \geq 0$, we have

$$K_i \subset K_{i+1} \quad (i \geq 0) \quad \text{and} \quad \bigcup_{i \geq 0} K_i = K.$$

Suppose now that we are given $x, y \in K$ such that $d_0^1 x = d_0^0 y$, that is $\delta^1 x_1 = \delta^0 y_1$. We define

$$x +_0 y = (\dots, x_n + y_n^{x_1}, \dots, x_2 + y_2^{-x_1}, x_1 + y_1)$$

which is an element of K . Similarly, if $j \geq 1$ and $d_j^1 x = d_j^0 y$, that is $x_i = y_i$, for $i < j$ and $x_j = \delta y_{j+1} + y_j$, we define

$$x +_j y = (\dots, x_n + y_n, \dots, x_{j+1} + y_{j+1}, y_j, y_{j-1}, \dots, y_1)$$

again an element of K . In each case it is easy to see that the composition $+_j$ defines a groupoid structure on K with K_j as its set of identities and d_j^0, d_j^1 as its initial and final maps. The law (1.2) follows trivially from these definitions if $\alpha = 1$ or if $i < j$. If $\alpha = 0$ and $i > j$, it reduces immediately to one of the following equations:

$$\begin{aligned} \delta y_{i+1} + x_i &= x_i + \delta y_{i+1}, \quad i > j \geq 2, \\ \delta(y_{i+1}^{-x_1}) + x_i &= x_i + (\delta y_{i+1})^{-x_1}, \quad i \geq 2, \\ \delta(y_2^{-x_1}) + x_1 &= x_1 + \delta y_2. \end{aligned}$$

These are all easy consequences of the laws for a crossed complex. The interchange law (1.3) is proved in a similar way to complete the verification that $K = \beta C$ is an ∞ -groupoid. The construction is clearly functorial.

Theorem 4.1 *The functors $\alpha : \mathcal{H} \rightarrow \mathcal{C}$ and $\beta : \mathcal{C} \rightarrow \mathcal{H}$ defined above are inverse equivalences.*

Proof. Given an ∞ -groupoid H , the ∞ -groupoid $K = \beta\alpha H$ is naturally isomorphic to H by the map

$$(\dots, x_n, x_{n-1}, \dots, x_1) \mapsto \dots + x_n + x_{n-1} + \dots + x_1$$

(the sum on the right being finite since $x_r = 0$ for large r). This is a consequence of Proposition (3.10) and the relations (3.11)–(3.14).

On the other hand, if C is a crossed complex $H = \beta C$ and $D = \alpha\beta C$, then H consists of elements $x = (x_n, x_{n-1}, \dots, x_1)$ and hence

$$D_n = \{x \in H_n \mid d_{n-1}^1 x \in H_0\}$$

consists of elements $x = (x_n, 0_p, 0_p, \dots, 0_p)$, where $x_n \in C_n(p)$. It is easy to see that the map $C_n \rightarrow D_n$ defined by

$$c \mapsto (c, \underbrace{0_p, 0_p, \dots, 0_p}_{n-1}), \quad c \in C_n(p)$$

gives a natural isomorphism $C \rightarrow \alpha\beta C$. □

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